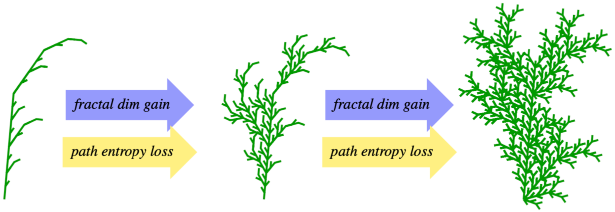
Path-random trees and models of arithmetic



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Algorithmic randomness appears in different forms:

- infinite bit-sequences (reals)
- arrays, trees, and structures.

Consider the task of:

- changing forms without sacrificing algorithmic complexity
- transforming Bernoulli distribution into uniform random (von Neumann)
- general cases, including non-computable distributions, have been explored.

We study the hardness of transformations

- of closed sets of random points (trees with incompressible paths)
- with respect to dimensionality: branching and accumulation points.

Tree versions of König's lemma

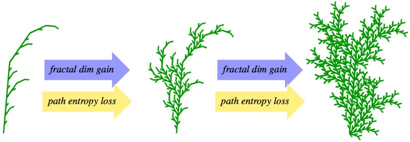
Motivated by the separation of:

WKL :	Every infinite tree has a path
P^+ :	Every positive tree has a positive perfect subtree
P :	Every positive tree has a perfect subtree
P^- :	Every positive tree has an infinite countable family of paths
WWKL :	Every positive tree has a path

Provable in RCA_0: $\mathrm{WKL} \to \mathrm{P}^+ \to \mathrm{P} \to \mathrm{P}^- \to \mathrm{WWKL}$

A tree T is

- pruned if each $\sigma \in T$ has at least one proper extension in T
- proper if it has infinitely many paths
- positive the measure of its paths is positive
- path-incompressible if it is pruned and has finite deficiency
- path-random if it is pruned and all of its paths are random



Can randomness can be effectively manipulated

- with respect to topological or density characteristics
- branching frequency, accumulation points etc.

Theorem (Patey)

There exists a perfect path-random tree which does not compute any complete extension of Peano Arithmetic.

(Every positive tree contains one)

Theorem (with Wang, 2021)

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Independently obtained by Greenberg, Miller, Nies (2021).

We ask, within the path-incompressible trees:

- can every tree compute a proper tree?
- can every perfect tree compute a positive tree?
- can every proper tree compute a perfect tree?
- can every sparse perfect tree be effectively transformed into a denser tree?

Lemma (B., Wang and Hirschfeldt et. al.)

Every perfect path-random tree computes a perfect path-incompressible tree.

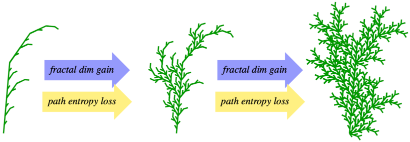
These questions are tree-analogues of

- problems of randomness extraction such as effectively increasing the Hausdorff dimension of reals (Miller, Bienvenu,…)
- except that the increase is now on the structural density of the tree, without loss in the algorithmic complexity of its paths.

Takeaway message

Path-random (proper) trees are somewhat rare:

- not easily produced as random members with respect to a distribution
- there are precise restrictions on their branching density
- not your typical random tree (Galton–Watson processes etc.)



Branching density

Given increasing $\ell = (\ell_n)$, we say that a tree T is ℓ-perfect if each node of length ℓ_n in T has at least two extensions of length ℓ_{n+1}.

Theorem

If $\ell = (\ell_n)$ is computable and increasing the following are equivalent:

(i) $\exists$ an ℓ-perfect path-random tree

(ii) $\exists$ an ℓ-perfect path-incompressible tree

(iii) $\sum_n 2^{-(\ell_{n+1}-\ell_n)} < \infty$.

Effective densification

Can a sparse perfect path-incompressible tree be effectively transformed into a denser path-incompressible tree?

Theorem (Informal)

Any sparse computably-perfect path-incompressible tree can be effectively transformed into an n^2-perfect path-incompressible tree, almost surely.

Theorem

Let $\ell = (\ell_n)$, $m = (m_n)$ be computable and increasing such that

$$\ell_{n+1} - \ell_n \geq m_{n+1} - m_n \quad \textit{and} \quad \sum_n 2^{-(m_{n+1} - m_n - n)} < \infty.$$

There exists a truth-table map $\Phi : \mathcal{T}_\ell \to \mathcal{T}_m$ such that for $T \in \mathcal{T}_\ell$:

- if T is path-incompressible, so is $\Phi(T)$
- with probability 1, T is ℓ-perfect and $\Phi(T)$ is m-perfect.

Theorem (with Wang, 2021)

If z is random and computes or enumerates a path-incompressible tree of unbounded width, then $z \geq_T \emptyset'$.

Hirschfeldt, Jockusch, and Schupp (2021) obtained a similar statement for perfect trees and 2-randoms.

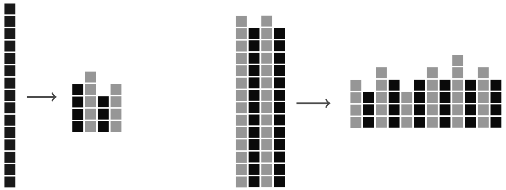
Corollary

The set of paths through the array:

$$x := x_0 \oplus x_1 \oplus x_2 \oplus \dots$$

for any $x \not\geq_T \emptyset'$ has infinite randomness deficiency.

Finitary consequences



- Effectively splitting a random source into k many random sources without a significant increase in the randomness deficiency is about as hard as computing the k-bit halting problem.

- Levin (2013): randomly guessing a completion of the k-bit segment of PA is about as improbable as randomly guessing the k-bit halting problem.

(also, Bienvenu, Porter, Deep classes)

Distribution for path-randomness?

Corollary

If F is a computable space of trees, for every computable measure ν on F the class of proper pruned path-incompressible members of F is ν-null.

Question. Can you construct a measure ν on the space of trees such that all ν-random trees are path-incompressible?

- The difficulty is not only on the computability of ν.
- it is possible to define ν as a mixture of neutral measures (fixed points)
- such measures are not representable robustly in 2^ω

The question remains.

Theorem (with Wang, 2021)

- There exists a perfect path-random tree which does not compute any path-random positive tree.
- Every positive tree contains a perfect path-random tree T such that no T-c.e. positive tree is path-random.

Forcing with sets of sets of positive measure.

Involving hitting sets and Poisson point processes.

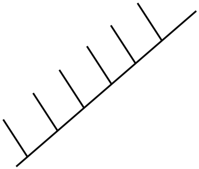
Methodology

- families of hitting-sets that intersect, and missing-sets that avoid the inverse images of the maps
- intuition from Fell (hit-or-miss) topology on the space of closed sets
- probability of a closed set in terms of the measure of its hitting-sets
- Choquet capacities and measures.

Skeletal trees

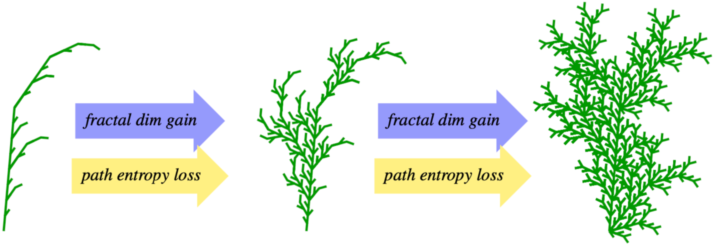
Theorem (with Zhang)

There exists a proper path-incompressible tree which does not compute any perfect path-random tree with computable oracle-use.



Conjecture: If a path-incompressible tree with at most one non-isolated path computes a perfect path-random tree, it also computes the halting problem.

Open problems



- build a model for $\mathrm{P}^- + \neg\mathrm{P}$
- build measures ν such that ν-randomness gives the above separations.

Thanks for listening!