Compression of enumerations and gain

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Even number game

Rules of the game:

- player 1 chooses a set A of k numbers
- player 2 chooses an even number between $\min A$ and $\max A$
- both choices are without repetition.

If player 2 runs out of moves, player 1 wins; otherwise player 2 wins.

Who wins? This is open for $k > 3$.

Positional games are key in compressibility of enumerations.

Initial segment complexity of c.e. sets

Let A be a c.e. set.

- $C(A \restriction_n \mid n) \stackrel{+}{\leq} \log n$ and $C(A \restriction_n) \stackrel{+}{\leq} 2\log n$. (Barzdins 1968)
- $\exists^\infty n,\ C(A \restriction_n) \stackrel{+}{\leq} \log n$ and $\exists^\infty n,\ K(A \restriction_n) \stackrel{+}{\leq} 2\log n$. (Solovay 1975)

- A is computable iff $C(A \restriction_n) \stackrel{+}{\leq} \log n$ (Chaitin 1976)
- A is linearly-complete iff $C(A \restriction_n) \stackrel{+}{\geq} \log n$ (Barmpalias et.al. 2013)

- $\exists$ c.e. A with $\forall n\ C(A \restriction_n) \stackrel{+}{\geq} \log n$. (Barzdins 1968)
- $\exists$ c.e. A with $\exists^\infty n,\ 2\log n \stackrel{+}{\leq} C(A \restriction_n)$. (Kummer 1996)

A is linearly-complete iff $\forall$c.e. W $C(W \restriction_n \mid A \restriction_n) = O(1)$

iff A is the standard c.e. representation of Ω as a c.e. set.

More on the complexity of c.e. sets

- x computes a DNC function iff $\exists f \leq_T x \; \forall n : C(f(n)) \geq n$
- x truth-table computes a DNC function iff $\exists f \leq_{tt} x \; \forall n : C(f(n)) \geq n$
- x computes a PA real iff if truth-table computes f such that

$$\forall n \; \forall \sigma \in 2^n \left(f(n) \in 2^n \;\; \wedge \;\; C(f(n)) \geq C(\sigma) \right)$$

- x computes a $0'$ iff it computes f such that

$$\forall n \; \forall \sigma \in 2^n \left(f(n) \in 2^n \;\; \wedge \;\; K(f(n)) \geq K(\sigma) \right)$$

Reference. B. Kjos-Hanssen, W. Merkle, and F. Stephan.

Kolmogorov complexity and the recursion theorem. Trans. AMS 363, 2011.

Compressions of enumerations

Given an effective enumeration of $A \subseteq \mathbb{N}$, obtain a compression of it:

- an effective enumeration of another set D which:
- essentially containing the information in A, but in a compact form
- 'essentially' means indifference to finitely many errors to finite errors.

- $D \upharpoonright_{\ell_n}$ is mapped to n-bits with bounded Hamming-distance from $A \upharpoonright_n$.
- Since A, D are c.e. this means: $C(A \upharpoonright_n \mid D \upharpoonright_{\ell_n}) = O(1)$

By D being more compact than A we mean that

- $\ell_n \ll n$, or at least,
- $\ell_n = n$ and $|D \upharpoonright_n|$ is considerably smaller than $|A \upharpoonright_n|$.

Compression with gain

Given c.e. A, D, we say that D is a

- compression of A if $|D\restriction_n| \leq |A\restriction_n|/2 \;\wedge\; C(A\restriction_n | D\restriction_n) = O(1)$
- strong compression of A if $C(A\restriction_n | D\restriction_{\lfloor n/2 \rfloor}) = O(1)$
- replacing $n/2$ by ϵn for $\epsilon \in (0,1)$, we define ϵ-compression.

By iteration:

- If every c.e. set has a compression, it also has an ϵ-compression.
- the same is true of strong compression.

Theorem. Every c.e. A has a strong compression (obtained effectively).

Coding is simple but produces extra information gain,
which is not recoverable from the source.

Compression with gain

We say that a compression is gainless if it has finite gain, where the

- gain of a compression D of A is $C(D\upharpoonright_n | A\upharpoonright_n)$
- gain of a strong compression D of A is $C(D\upharpoonright_{\lfloor n/2 \rfloor} | A\upharpoonright_n)$.

The halting problem has a gainless strong ϵ-compression, for arbitrary $\epsilon \in (0,1)$.

- gainless compressions are harder.
- but essential for density in relative Kolmogorov complexity of c.e. sets.

Theorem. Any c.e. A has a gainless compression $D \subseteq A$ obtained effectively.

Question. We do not know if every c.e. set has a gainless strong compression.

Comparing initial segment complexity of c.e. sets

Downey, Hirschfeldt, Laforte (2004) defined:

$$A \leq_{rK} B \iff C(A\upharpoonright_n \mid B\upharpoonright_n) = O(1)$$

$$A \leq_{K} B \iff \forall n,\ K(A\upharpoonright_n) \stackrel{+}{\leq} K(B\upharpoonright_n)$$

$$A \leq_{C} B \iff \forall n,\ C(A\upharpoonright_n) \stackrel{+}{\leq} C(B\upharpoonright_n)$$

- bottom degree in $\leq_{rK}, \leq_C$ is the computable sets. (Chaitin 1976)
- the bottom degree in $\leq_K$ includes noncomputable sets. (Solovay 1975)
- $\leq_{rK}$ implies Turing reducibility. (Downey, Hirschfeldt, LaForte 2004)
- there is a maximum c.e. degree in $\leq_{rK}, \leq_C, \leq_K$. (Barmpalias et.al. 2013)

Density of c.e. sets with respect to $\leq_{rK}, \leq_K, \leq_C$ is open.

Density of computably enumerable sets

Theorem (Density in $\leq_{rK}, \leq_{K}, \leq_{C}$).

Density holds amongst the c.e. sets that have a gainless strong compression.

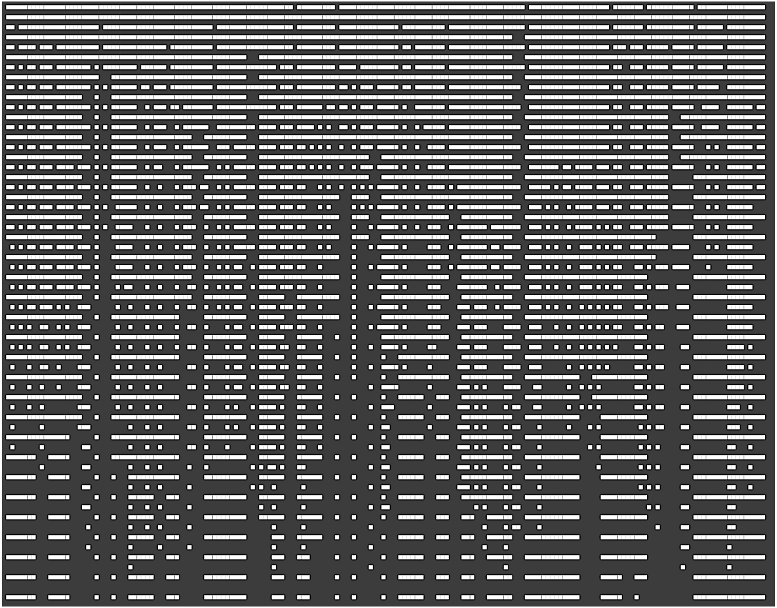
Density with respect to the slightly stronger $\leq_{cl}, \leq$ ibT fails.

Reference. Algorithmic randomness and measures of complexity.

G. Barmpalias. Bulletin of Symbolic logic (2013)

Our paper. Compressions of enumerations. http://arxiv.org/abs/2304.03030

G. Barmpalias, Xiaoyan Zhang, and Bohua Zhan.



Even number game:

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- player 2 chooses an even number between $\min A$ and $\max A$
- both choices are without repetition.

If player 2 runs out of moves, player 1 wins; otherwise player 2 wins.

Who has a winning strategy?

Our paper. Compressions of enumerations. http://arxiv.org/abs/2304.03030
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Postcript – PA and randoms

- Interesting relationships
- complete PA probabilistically
- Bennett's logical depth
- pioneering work by Kucera
- later Stephan, Levin, Miller, ...
- randoms hardly compute PA

Join	$\forall \mathbf{x} < \mathbf{a}\ \exists \mathbf{y} < \mathbf{a}\ (\mathbf{x} \neq \mathbf{0} \rightarrow (\mathbf{x} \vee \mathbf{y} = \mathbf{a}))$
Meet	$\forall \mathbf{x} < \mathbf{a}\ \exists \mathbf{y} < \mathbf{a}\ ((\mathbf{y} \neq \mathbf{0})\ \&\ \mathbf{x} \wedge \mathbf{y} = \mathbf{0})$
Cupping	$\forall \mathbf{y} > \mathbf{a}\ \exists \mathbf{x} < \mathbf{y}\ (\mathbf{a} \vee \mathbf{x} = \mathbf{y})$
Complementation	$\forall \mathbf{x} < \mathbf{a}\ \exists \mathbf{y} < \mathbf{a}\ (\mathbf{x} \neq \mathbf{0} \rightarrow (\mathbf{x} \wedge \mathbf{y} = \mathbf{0}\ \&\ \mathbf{x} \vee \mathbf{y} = \mathbf{a}))$
Top of a diamond	$\exists \mathbf{x}\ \exists \mathbf{y}\ (\mathbf{x} \neq \mathbf{0}\ \&\ \mathbf{x} \vee \mathbf{y} = \mathbf{a}\ \&\ \mathbf{x} \wedge \mathbf{y} = \mathbf{0})$
Being a minimal degree	$\mathbf{a} > \mathbf{0}\ \&\ (\mathbf{0}, \mathbf{a}) = \emptyset$
Bounding a minimal degree	$\exists \mathbf{x} \leq \mathbf{a}\ [\mathbf{x} > \mathbf{0}\ \&\ (\mathbf{0}, \mathbf{x}) = \emptyset]$
Being a minimal cover	$\exists \mathbf{x} < \mathbf{a}\ [(\mathbf{x}, \mathbf{a}) = \emptyset]$
Having a minimal cover	$\exists \mathbf{y} > \mathbf{a}\ [(\mathbf{a}, \mathbf{y}) = \emptyset]$
Being a strong minimal cover	$\exists \mathbf{x} < \mathbf{a}\ [(\mathbf{0}, \mathbf{a}) = (\mathbf{0}, \mathbf{x}]]$
Having a strong minimal cover	$\exists \mathbf{y} > \mathbf{a}\ [(\mathbf{0}, \mathbf{a}] = (\mathbf{0}, \mathbf{y})]$

We show that the PA degrees have the join property with random degrees.

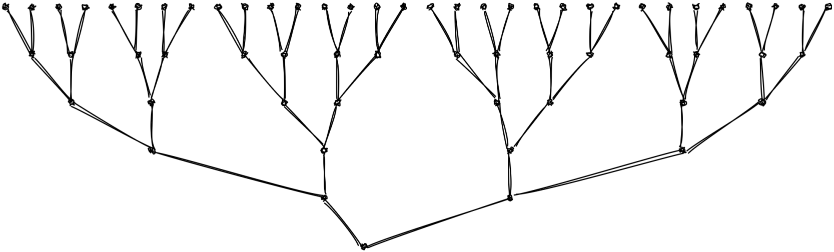
The PA degrees do not, in general, satisfy the join property (Lewis-Pye, 2012).

PA and randoms – state of the art

- low PA degrees do not have the join property (Lewis-Pye, 2012)
- 2-random degrees have the join property (Barmpalias et al., 2014)
- PA degrees have the cupping property (Kucera, 1985)
- 2-random degrees do not have the cupping property (Barmp. et al., 2014)
- any pair of DNC degrees below $0'$ fails to be a minimal pair (Kucera, 1988)
- every PA degree bounds a minimal pair (Jockusch and Soare, 1971).

Randomizing Kucera-Gács coding

Given PA real z and random $x <_T z$, find random $y <_T z$ such that $x \oplus y \equiv_T z$.



Let z can work with a partition of extendible branches.

Theorem (with Wang Wei).

If z is PA and $x <_T z$ is random, there exists random $y <_T z$ such that $x \oplus y \equiv_T z$. The same is true for truth-table and weak truth-table reducibility.

Corollary. Let DOM denote the class of $0'$-dominated reals.

(a) For each random $x \in \mathrm{DOM}$, there exists random $y \in \mathrm{DOM}$ such that $x \oplus y \in \mathrm{DOM} \cap \mathrm{PA}$.

(b) For each random x, there exists random y such that $x \oplus y$ is deep.

Our paper. Randomness below complete theories of arithmetic.

George Barmpalias and Wei Wang. Information and Computation 290 (2023)

Question. what about the efficiency of computation and oracle-use growth?

Thanks for listening!