

Even number game

Rules of the game:

- ▶ player 1 chooses a set A of k numbers
- ▶ player 2 chooses an even number between $\min A$ and $\max A$
- ▶ both choices are without repetition.

If player 2 runs out of moves, player 1 wins; otherwise player 2 wins.

Who wins? This is open for $k > 3$.

Positional games are key in compressibility of enumerations.

Initial segment complexity of c.e. sets

Let A be a c.e. set.

▶ $C(A \upharpoonright_n | n) \stackrel{\pm}{\leq} \log n$ and $C(A \upharpoonright_n) \stackrel{\pm}{\leq} 2 \log n$. (Barzdins 1968)

▶ $\exists^\infty n, C(A \upharpoonright_n) \stackrel{\pm}{\leq} \log n$ and $\exists^\infty n, K(A \upharpoonright_n) \stackrel{\pm}{\leq} 2 \log n$. (Solovay 1975)

▶ A is computable iff $C(A \upharpoonright_n) \stackrel{\pm}{\leq} \log n$ (Chaitin 1976)

▶ A is linearly-complete iff $C(A \upharpoonright_n) \stackrel{\pm}{\geq} \log n$ (Barmpalias et.al. 2013)

▶ \exists c.e. A with $\forall n C(A \upharpoonright_n) \stackrel{\pm}{\geq} \log n$. (Barzdins 1968)

▶ \exists c.e. A with $\exists^\infty n, 2 \log n \stackrel{\pm}{\leq} C(A \upharpoonright_n)$. (Kummer 1996)

A is linearly-complete iff \forall c.e. $W C(W \upharpoonright_n | A \upharpoonright_n) = O(1)$

iff A is the standard c.e. representation of Ω as a c.e. set.

More on the complexity of c.e. sets

- ▶ x computes a DNC function iff $\exists f \leq_T x \forall n : C(f(n)) \geq n$
- ▶ x truth-table computes a DNC function iff $\exists f \leq_{tt} x \forall n : C(f(n)) \geq n$
- ▶ x computes a PA real iff it truth-table computes f such that

$$\forall n \forall \sigma \in 2^n (f(n) \in 2^n \wedge C(f(n)) \geq C(\sigma))$$

- ▶ x computes a $0'$ iff it computes f such that

$$\forall n \forall \sigma \in 2^n (f(n) \in 2^n \wedge K(f(n)) \geq K(\sigma))$$

Reference. B. Kjos-Hanssen, W. Merkle, and F. Stephan.

Kolmogorov complexity and the recursion theorem. Trans. AMS 363, 2011.

Compressions of enumerations

Given an effective enumeration of $A \subseteq \mathbb{N}$, obtain a compression of it:

- ▶ an effective enumeration of another set D which:
 - ▶ essentially containing the information in A , but in a compact form
 - ▶ ‘essentially’ means indifference to finitely many errors to finite errors.
-
- ▶ $D \upharpoonright_{\ell_n}$ is mapped to n -bits with bounded Hamming-distance from $A \upharpoonright_n$.
 - ▶ Since A, D are c.e. this means: $C(A \upharpoonright_n \mid D \upharpoonright_{\ell_n}) = O(1)$

By D being more compact than A we mean that

- ▶ $\ell_n \ll n$, or at least,
- ▶ $\ell_n = n$ and $|D \upharpoonright_{\ell_n}|$ is considerably smaller than $|A \upharpoonright_n|$.

Compression with gain

Given c.e. A, D , we say that D is a

- ▶ **compression** of A if $|D \upharpoonright_n| \leq |A \upharpoonright_n|/2 \wedge C(A \upharpoonright_n | D \upharpoonright_n) = O(1)$
- ▶ **strong compression of A** if $C(A \upharpoonright_n | D \upharpoonright_{\lfloor n/2 \rfloor}) = O(1)$
- ▶ replacing $n/2$ by ϵn for $\epsilon \in (0, 1)$, we define **ϵ -compression**.

By iteration:

- ▶ If every c.e. set has a compression, it also has an ϵ -compression.
- ▶ the same is true of strong compression.

Theorem. Every c.e. A has a strong compression (obtained effectively).

Coding is simple but produces extra information **gain**,
which is not recoverable from the source.

Compression with gain

We say that a compression is **gainless** if it has finite **gain**, where the

- ▶ **gain** of a compression D of A is $C(D \upharpoonright_n | A \upharpoonright_n)$
- ▶ **gain** of a strong compression D of A is $C(D \upharpoonright_{\lfloor n/2 \rfloor} | A \upharpoonright_n)$.

The halting problem has a gainless strong ϵ -compression, for arbitrary $\epsilon \in (0, 1)$.

- ▶ gainless compressions are harder.
- ▶ but essential for density in relative Kolmogorov complexity of c.e. sets.

Theorem. Any c.e. A has a gainless compression $D \subseteq A$ obtained effectively.

Question. We do not know if every c.e. set has a gainless strong compression.

Comparing initial segment complexity of c.e. sets

Downey, Hirschfeldt, Laforte (2004) defined:

$$A \leq_{rK} B \iff C(A \upharpoonright_n \mid B \upharpoonright_n) = O(1)$$

$$A \leq_K B \iff \forall n, K(A \upharpoonright_n) \stackrel{\pm}{\leq} K(B \upharpoonright_n)$$

$$A \leq_C B \iff \forall n, C(A \upharpoonright_n) \stackrel{\pm}{\leq} C(B \upharpoonright_n)$$

- ▶ bottom degree in \leq_{rK}, \leq_C is the computable sets. (Chaitin 1976)
- ▶ the bottom degree in \leq_K includes noncomputable sets. (Solovay 1975)
- ▶ \leq_{rK} implies Turing reducibility. (Downey, Hirschfeldt, LaForte 2004)
- ▶ there is a maximum c.e. degree in $\leq_{rK}, \leq_C, \leq_K$. (Barmpalias et.al. 2013)

Density of c.e. sets with respect to $\leq_{rK}, \leq_K, \leq_C$ is open.

Density of computably enumerable sets

Theorem (Density in $\leq_{rK}, \leq_K, \leq_C$).

Density holds amongst the c.e. sets that have a gainless strong compression.

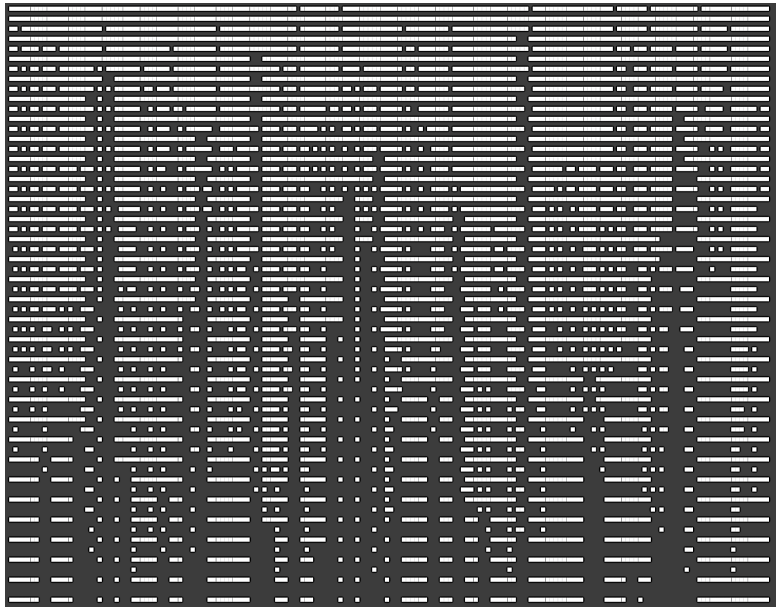
Density with respect to the slightly stronger \leq_{cl}, \leq_{ibT} fails.

Reference. Algorithmic randomness and measures of complexity.

G. Barmalias. Bulletin of Symbolic logic (2013)

Our paper. Compressions of enumerations. <http://arxiv.org/abs/2304.03030>

G. Barmalias, Xiaoyan Zhang, and Bohua Zhan.



Even number game:

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Who has a winning strategy?

Our paper. Compressions of enumerations. <http://arxiv.org/abs/2304.03030>
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Postscript – PA and randoms

- ▶ Interesting relationships
- ▶ complete PA probabilistically
- ▶ Bennett's logical depth
- ▶ pioneering work by Kucera
- ▶ later Stephan, Levin, Miller, ...
- ▶ randoms hardly compute PA

Randomness below complete theories of arithmetic

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DEGREE SYSTEM | TRUTH DEPTH | FAILURE CODING | RANDOM

Formal arithmetic (Peano Arithmetic)

Based on Peano's axioms and their variations

AXIOMS FOR PA

P1 $\forall x(x \neq 0)$
 P2 $\forall x(y(x \rightarrow 0) \rightarrow x = 0)$ $\forall x \exists y (x = y + 1)$ function
 P3 $\forall x(x = 0 \rightarrow x = 1)$
 P4 $\forall x(y(x = 0 \rightarrow x = y) \rightarrow 0)$ add- =
 P5 $\forall x(x = 0 \rightarrow x = 1)$ add- =
 P6 $\forall x(x \rightarrow 0 = 1) \rightarrow x = 1$ add- =

INDUCTION SCHEM: Let $\mathcal{A}(x, A(x))$ be the sentence
 $\forall x (\mathcal{A}(x) \rightarrow A(x)) \wedge \forall x(A(x) \rightarrow A(x+1)) \rightarrow A(0)$

Randomization in Arithmetic

- ▶ completions of arithmetic can be obtained probabilistically
- ▶ but the probability of a successful outcome is as small as it can be
- ▶ the same is true for completions of finite segments of arithmetic
- ▶ algorithmically random binaries do not encode complete theories of arithmetic (unless they compute the halting problem)
- ▶ Complete theories of arithmetic are *deep* in the sense of Bennett: it is hard to obtain them probabilistically.

Language Coding & Computation

Theories of arithmetic can be written in binary, via coding of formulas into integers.

Then proofs can be viewed as computations.

Incompleteness in arithmetic

Discovered by Gödel: there is no computable binary predicate consistently evaluating the truth of each arithmetical sentence.

True but unprovable statements

(a) Kruskal's tree theorem: the finite trees over a well-quasi-ordered set of labels is well-quasi-ordered under homeomorphic embedding.

(b) Ramsey: there are monochromatic cliques in any coloring of sufficiently large complete graphs.

(c) Graph minor theorem: the undirected graphs, partially ordered by the graph minor relationship, form a well-quasi-ordering.

Complete extensions of Arithmetic

There are arithmetical sentences whose validity cannot be decided from the axioms. A complete extension is obtained by adding such sentences (or their negation) to the theory, while maintaining consistency.

Extensions of arithmetic may not be consistent with each other: some may include a certain undecidable sentence, while others may include its negation.

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KURT GÖDEL

ON FORMALLY UNDECIDABLE PROPOSITIONS OF PRINCIPALS MATHEMATICS AND RELATED SYSTEMS

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Complexity of True Arithmetic

The computational complexity of the true sentences of arithmetic is overwhelming: it is as hard as infinitely many iterations of the halting problem.

However many complete extensions of arithmetic, though inconsistent, are much easier to compute than many known problems, such as the halting problem.

Outcome and Methodology

- ▶ Coding deep (Bennett's logical depth) information into unstructured (random) binary sequences
- ▶ Random coding technology was not up to this task
- ▶ Randomizing the method of Kucera, we obtained the required coding method to give a positive answer.

RANDOM ORACLE COMPUTATION
 DEGREE SYSTEM | TRUTH DEPTH | FAILURE CODING | RANDOM
 EXTENSION COMPLETION PARTITION
 CLASS TRUTH DEGREE
 REDUCTION COMPUTABILITY PROBABILITY SYSTEM

Join	$\forall \mathbf{x} < \mathbf{a} \exists \mathbf{y} < \mathbf{a} (\mathbf{x} \neq \mathbf{0} \rightarrow (\mathbf{x} \vee \mathbf{y} = \mathbf{a}))$
Meet	$\forall \mathbf{x} < \mathbf{a} \exists \mathbf{y} < \mathbf{a} ((\mathbf{y} \neq \mathbf{0}) \& \mathbf{x} \wedge \mathbf{y} = \mathbf{0})$
Cupping	$\forall \mathbf{y} > \mathbf{a} \exists \mathbf{x} < \mathbf{y} (\mathbf{a} \vee \mathbf{x} = \mathbf{y})$
Complementation	$\forall \mathbf{x} < \mathbf{a} \exists \mathbf{y} < \mathbf{a} (\mathbf{x} \neq \mathbf{0} \rightarrow (\mathbf{x} \wedge \mathbf{y} = \mathbf{0} \& \mathbf{x} \vee \mathbf{y} = \mathbf{a}))$
Top of a diamond	$\exists \mathbf{x} \exists \mathbf{y} (\mathbf{x} \neq \mathbf{0} \& \mathbf{x} \vee \mathbf{y} = \mathbf{a} \& \mathbf{x} \wedge \mathbf{y} = \mathbf{0})$
Being a minimal degree	$\mathbf{a} > \mathbf{0} \& (\mathbf{0}, \mathbf{a}) = \emptyset$
Bounding a minimal degree	$\exists \mathbf{x} \leq \mathbf{a} [\mathbf{x} > \mathbf{0} \& (\mathbf{0}, \mathbf{x}) = \emptyset]$
Being a minimal cover	$\exists \mathbf{x} < \mathbf{a} [(\mathbf{x}, \mathbf{a}) = \emptyset]$
Having a minimal cover	$\exists \mathbf{y} > \mathbf{a} [(\mathbf{a}, \mathbf{y}) = \emptyset]$
Being a strong minimal cover	$\exists \mathbf{x} < \mathbf{a} [(\mathbf{0}, \mathbf{a}) = (\mathbf{0}, \mathbf{x})]$
Having a strong minimal cover	$\exists \mathbf{y} > \mathbf{a} [(\mathbf{0}, \mathbf{a}) = (\mathbf{0}, \mathbf{y})]$

We show that the PA degrees have the join property with random degrees.

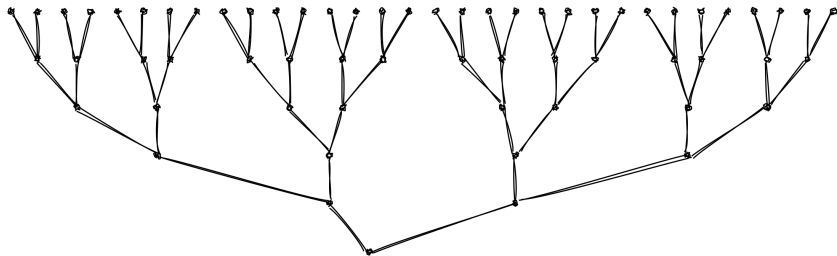
The PA degrees do not, in general, satisfy the join property (Lewis-Pye, 2012).

PA and randoms – state of the art

- ▶ low PA degrees do not have the **join property** (Lewis-Pye, 2012)
- ▶ 2-random degrees have the join property (Barnpalias et al., 2014)
- ▶ PA degrees have the **cupping property** (Kucera, 1985)
- ▶ 2-random degrees do not have the cupping property (Barnp. et al., 2014)
- ▶ any pair of DNC degrees below $0'$ fails to be a **minimal pair** (Kucera, 1988)
- ▶ every PA degree bounds a minimal pair (Jockusch and Soare, 1971).

Randomizing Kucera-Gács coding

Given PA real z and random $x \leq_T z$, find random $y \leq_T z$ such that $x \oplus y \equiv_T z$.



Let z can work with a **partition** of extendible branches.

Theorem (with Wang Wei).

If z is PA and $x <_T z$ is random, there exists random $y <_T z$ such that $x \oplus y \equiv_T z$.
The same is true for truth-table and weak truth-table reducibility.

Corollary. Let DOM denote the class of $0'$ -dominated reals.

- (a) For each random $x \in \text{DOM}$, there exists random $y \in \text{DOM}$ such that $x \oplus y \in \text{DOM} \cap \text{PA}$.
- (b) For each random x , there exists random y such that $x \oplus y$ is deep.

Our paper. Randomness below complete theories of arithmetic.

George Barmpalias and Wei Wang. Information and Computation 290 (2023)

Question. what about the efficiency of computation and oracle-use growth?

Thanks for listening!