Irreducibility and enumerability in betting strategies

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Restricted martingales literature (abbreviated)

(2009) Muchnik. Algorithmic randomness and splitting of supermartingales. Prob. Inf. Transm.

(2012) Bienvenu, Stephan, Teutsch. How powerful are integer-valued martingales? Theory Comput. Syst.

(2012) Chalcraft et. al. How to build a probability-free casino. Inf. Comp.

(2014) Teutsch. A savings paradox for integer-valued strategies. Int. J. Game T.

(2015) Barmpalias, Downey, McInerney. Integer valued betting strategies and Turing degrees. J. Comput. System Sci.

(2015) Peretz, Bavly. How to gamble against all odds. Games Econ. Behav.

(2015) Peretz. Effective martingales with restricted wagers. Inf. Comp.

(2016) Barmpalias, Lewis-Pye, Teutsch. Lower bounds via betting strategies with restricted wagers. Inf. Comp.

(2017) Pardo. Randomness of restricted value martingales, selection rules, and graph sequences. PhD Dissertation. Penn State U.

(2019) Masulkar, Nandakumar, Ng. Martingales and Restricted Ratio Betting.

(2020) Barmpalias, Fang, Lewis-Pye. Monotonous betting strategies. Inf. Comp.

(2020) Barmpalias, Fang. Granularity and the possibility of saving. Inf. Comp.

(2021) Liu. A computable analysis of majorizing martingales. Bull. LMS

Applications of restricted betting

- Computation from random oracles

> Barmpalias, Lewis-Pye, Teutsch. Lower bounds via betting strategies with restricted wagers. Inf. Comp. (2016)

- Reverse math and measure

> Liu. A computable analysis of majorizing martingales. Bull. LMS (2021)

answering questions from

> Greenberg, Miller, Nies. Highness properties close to PA-completeness. Isr. J. Math. (2021)

related to the reverse mathematics in

> Barmpalias, Wang. Pathwise-randomness and models of second order arithmetic Arxiv 2104.12066 (2021)

Modelling betting strategies

A stage in a game is determined by the series of previous outcomes.

A strategy is a function that maps any finite sequences of outcomes to the amount the player bets at that particular stage.

If $\mathrm{M}(\sigma)$ is the capital at σ and we bet b on 0, then $\mathrm{b} \leq \mathrm{M}(\sigma)$ and

$$\mathrm{M}(\sigma * 0) = \mathrm{M}(\sigma) + \mathrm{b} \quad \text{and} \quad \mathrm{M}(\sigma * 1) = \mathrm{M}(\sigma) - \mathrm{b} \quad \text{so}$$

$$\mathrm{M}(\sigma) = \frac{\mathrm{M}(\sigma * 0) + \mathrm{M}(\sigma * 1)}{2}.$$

Conversely given M we can define the bet: $\mathrm{b}_{\mathrm{M}}(\sigma * \mathrm{i}) = \mathrm{M}(\sigma * \mathrm{i}) - \mathrm{M}(\sigma)$.

If I consider M_{s}, the capital at stage s, as a random variable in a fair game, then $(\mathrm{M}_{\mathrm{s}})$ is a martingale, relative to the outcome process $(\mathrm{I}_{\mathrm{s}})$:

$$\mathbb{E}(\mathrm{M}_{\mathrm{s}+1} \mid \mathrm{I}_{\mathrm{s}}) = \mathrm{M}_{\mathrm{s}}$$

Hence the probability that $\mathrm{M}_{\mathrm{s}} > 100 \cdot \mathrm{M}_0$ at some stage s is $< 1/100$.

Success of strategies

The standard success measure is the condition:

$$\limsup_n M(X \upharpoonright_n) = \infty$$

while a savings method shows that this is equivalent to

$$\lim_n M(X \upharpoonright_n) = \infty$$

Exponential success means that:

$$\limsup_n \frac{M(X \upharpoonright_n)}{\alpha^n} = \infty \qquad \text{for some } \alpha > 1.$$

Realistic considerations:

- infinitely divisible currency
- time to success
- computational restrictions
- inflation

Types of restricted betting

- Restricted wagers: granularity
  - Integer-valued
  - Fine and coarse granularity
- Restrictions on favorable outcomes
  - Monotonous, single-sided
  - predictable favorable outcomes, determined by an effective rule
- Restrictions on betting stages (Muchnik 2009)
- Effectivity: computability or computable enumerability
  - of capital, wagers, wager-ratio, favorable outcomes
  - countable mixtures of effective strategies

Main question

- Are there as powerful as the unrestricted strategies ?
- Do the associated randomness notions coincide ?
- Can randomness be defined using a subclass of strategies ?

Kastermans (2009): Can 1-randomness be defined by l.c.e. supermartingales whose favorable outcome is determined by a partial computable rule (kastergales)?

Hitchcock (2009): Can 1-randomness be defined by kastergales where the wager is approximable from below?

Our work gives a negative answer to these questions, and similar questions.

Theorem: there exists a real x where some l.c.e. supermartingale succeeds but no kastergale succeeds on x.

Main challenges to solving this problem

- supermartingales have correlated differences (martingale case is easier)

  Solution: force the supermartingale to approximate a martingale

- achieving $\limsup_n M(x \upharpoonright_n) \neq \infty$ is harder than $\lim_n M(x \upharpoonright_n) \neq \infty$

  Solution: an extra complication in the strategy for the construction

- there is no universality: mixtures of kastergales are not kastergale

  Solution: work with supermartingale vectors M and $\|M(x \upharpoonright_n)\|$

Core case – the Game

- Given single-sided left-c.e. martingale M with $M(\lambda) < 1$
- construct non-random x such that $\liminf_n M(x \upharpoonright_n) < 1$

Solution is based on a supermartingale game and variance analysis

Given n consider a game between Alice and Baby, where

- Alice enumerates $A \subseteq 2^n$ and seeks to keep $\mu(A)$ small
- Baby controls a left-c.e. supermartingale vector $M = (M_0, \ldots, M_{k-1})$
- Alice seeks to enumerate σ into A such that $\|M(\sigma)\|$ is small.
- Baby attempts to keep $\|M(\lambda)\|$ sufficiently small

Winning condition $W(t)$ is a function of $\mu(A_t), \|M_t(\lambda)\|, \|M(\sigma)\|$ for $\sigma \in A_t$;

Alice wins if $\exists t\ W(t)$, in which case $\mu(A_t)$ is called the cost.

Game we need to win: static

The static game (c, d, ϵ) at level n with

- floor $c > 0$ and ceiling $d > c$
- cost-bound ϵ

is determined by the winning condition that $\exists t$ such that:

$$\Big(\|M_t(\lambda)\| \geq c \quad \vee \; \exists \sigma \in A : \; \|M_t(\sigma)\| < d \Big) \; \wedge \; \mu(A_t) < \epsilon$$

The normalized static game corresponds to $d = 1$ and is denoted by (c, ϵ).

Need a winning strategy for:

- fixed $\epsilon_0 < 1$
- any $0 < c < d < 1$ and some $n = n(c, d)$

Then we can build:

- the required x by initial segments
- along with a Martin-Löf test that captures x.

Reduction to normal form and dynamic game

- Scaling: hardness for Alice is the size of $d - c$;

  by scaling the martingale, solving all (c, d, ϵ) reduces to solving all normalized (c, ϵ)

- Nesting strategies:

  if Alice wins $(c, \epsilon)_n$ and $(c', \epsilon')_{n'}$ then Alice wins $(c \cdot c', \epsilon \cdot \epsilon')_{n+n'}$

- Nesting reduces (c, ϵ) to solving all normal forms: $(1 - \delta, 1 - a\delta)$

Dynamic a-game at level n:

Determined by the rule that for each t, $\forall \sigma \in A_t : \; M_t(\sigma) \geq 1$ and:

$$\exists t \left(a \cdot \left(1 - \|M_t(\lambda)\|\right) \leq 1 - \mu(A_t) \;\wedge\; \mu(A_t) < 1 \right).$$

as the winning condition.

Solving the dynamic implies $\forall a > 0 \exists \delta_0 > 0 \forall \delta \in (0, \delta_0)$ Alice wins $(1 - \delta, 1 - a\delta)$.

Winning the dynamic game

Main idea:

- if M is restricted and Baby persists avoiding the winning condition, she builds up large variance on certain stopping times of the game.
- reaching high capital without playing on certain stages raises the local of M.
- this contradicts the martingale property of M.

Denote the average of M on a prefix-free set A of strings by:

$$\int_{\sigma \in A} M(\sigma) := \int_A M := \sum_{\sigma \in A} M(\sigma) \cdot \mu(\sigma).$$

$$\mathbb{E}(M \mid B) = \frac{1}{\mu(B)} \cdot \int_B M$$

$$\mathrm{Var}(M \mid B) = \frac{1}{\mu(B)} \cdot \int_B \left(\| M - \mathbb{E}(M \mid B) \|_2 \right)^2 .$$

Stopping times – filtrations – quadatic variation

A sequence of prefix-free sets $F_i \subseteq 2^{\leq n}, i \leq m$ is called a filtration of $2^{\leq n}$ if

$$[F_i] = 2^\omega \qquad \text{and} \qquad F_i \subseteq [F_{i-1}].$$

Given a filtration (F_i) we let $F_\rho := [\rho] \cap F_i$ for each i and $\rho \in F_{i-1}$.

Predictable quadratic variation $\langle M \rangle$:

$$\langle M \rangle_t := \sum_{i<t} \operatorname{Var}(M_{i+1} \mid F_i).$$

Fact:

- if M is a martingale then $M^2 - \langle M \rangle$ is a martingale.
- hence if M is bounded, the expectation of $\langle M \rangle$ is small.

Winning the end-game

$$\mathbb{E}\left(\langle M\rangle_t\right) = \sum_{i<t} \int_{\rho\in F_i} \mathrm{Var}\left(M \mid F_\rho\right)$$

- Start enumerating n-bit strings in a certain order into A.
- if Baby persists, each ρ will see a stage where $\mathrm{Var}\left(M \mid F_\rho\right)$ grows.
- Baby needs to quit if M is a martingale, as $\mathbb{E}\left(\langle M\rangle_t\right)$ can't grow too much.
- Alice wins with $\mu(A) < 1$.

Thanks for listening!