Upper bounds for ideals in the Turing degrees

George Barmpalias and André Nies

Victoria University of Wellington/Auckland University

Asian Logic Conference,
IMS Singapore, June 2009

The leading question

We study ideals in the c.e. Turing degrees.
The leading question is the following.

The leading question

We study ideals in the c.e. Turing degrees.
The leading question is the following.

Let I be a proper ideal with a certain type of effective presentation.

The leading question

We study ideals in the c.e. Turing degrees.
The leading question is the following.

Let I be a proper ideal with a certain type of effective presentation.

What can we say about upper bounds of I in the c.e. degrees?

The ideal lattice of an usl U

- Let $(U, \leq \vee)$ be an uppersemilattice (usl).
- A set $I \subseteq U$ is an ideal if I is closed downwards and under the join operation $\vee$.
- An upper bound of an ideal I is a degree $\mathbf{b}$ such that $I \subseteq [\mathbf{0}, \mathbf{b}]$.

The ideal lattice of an usl U

- Let $(U, \leq \vee)$ be an uppersemilattice (usl).
- A set $I \subseteq U$ is an ideal if I is closed downwards and under the join operation $\vee$.
- An upper bound of an ideal I is a degree $\mathbf{b}$ such that $I \subseteq [\mathbf{0}, \mathbf{b}]$.

Some Facts:

- The set of ideals of U is a lattice, where the meet of I, J is the intersection, and the join of I, J is the ideal generated by $I \cup J$.
- An ideal I is called proper if $I \neq U$.
- Each $u \in U$ determines the ideal $\{x\colon\ x \leq u\}$, called a principal ideal.

Effective presentations of ideals

There are two interrelated approaches to effectively presenting an ideal I in the c.e. degrees.

(a) Require that I is generated by a uniformly c.e. sequence (possibly with further conditions).
We say that I is uniformly generated.

(b) Describe the index set $\Theta I = \{e \colon \text{ the degree of } W_e \text{ is in } I\}$ within the arithmetical hierarchy.
If ΘI is Σ^0_k etc. we say that I is a Σ^0_k ideal.

Effective presentations of ideals

There are two interrelated approaches to effectively presenting an ideal I in the c.e. degrees.

(a) Require that I is generated by a uniformly c.e. sequence (possibly with further conditions).
We say that I is uniformly generated.

(b) Describe the index set $\Theta I = \{e\colon \text{ the degree of } W_e \text{ is in } I\}$ within the arithmetical hierarchy.
If ΘI is Σ^0_k etc. we say that I is a Σ^0_k ideal.

Basic Facts:

- The class of uniformly generated ideals is closed under join of ideals.

Effective presentations of ideals

There are two interrelated approaches to effectively presenting an ideal I in the c.e. degrees.

(a) Require that I is generated by a uniformly c.e. sequence (possibly with further conditions).
We say that I is uniformly generated.

(b) Describe the index set $\Theta I = \{e\colon \text{ the degree of } W_e \text{ is in } I\}$ within the arithmetical hierarchy.
If ΘI is Σ^0_k etc. we say that I is a Σ^0_k ideal.

Basic Facts:

- The class of uniformly generated ideals is closed under join of ideals.
- Each principal ideal is Σ^0_4.

Effective presentations of ideals

There are two interrelated approaches to effectively presenting an ideal I in the c.e. degrees.

(a) Require that I is generated by a uniformly c.e. sequence (possibly with further conditions).
We say that I is uniformly generated.

(b) Describe the index set $\Theta I = \{e\colon \text{ the degree of } W_e \text{ is in } I\}$ within the arithmetical hierarchy.
If ΘI is Σ^0_k etc. we say that I is a Σ^0_k ideal.

Basic Facts:

- The class of uniformly generated ideals is closed under join of ideals.
- Each principal ideal is Σ^0_4.
- For $k \geq 4$, the Σ^0_k ideals form a lattice.

Classes of ideals

For ideals, we have the implications

$$\Sigma^0_3 \Longrightarrow \text{uniformly generated} \Longrightarrow \Sigma^0_4.$$

Classes of ideals

For ideals, we have the implications

$$\Sigma^0_3 \Longrightarrow \text{uniformly generated} \Longrightarrow \Sigma^0_4.$$

It is not hard to show that the converse implications fail.

Definability and global properties

- Several natural ideals came out of the intrusion of randomness-related concepts into computability: K-trivial, strongly jump traceable, ...

Definability and global properties

- Several natural ideals came out of the intrusion of randomness-related concepts into computability: K-trivial, strongly jump traceable, ...
- Earlier investigations of ideals focussed on their definability, and on the global properties of ideal lattices.

Definability and global properties

- Several natural ideals came out of the intrusion of randomness-related concepts into computability: K-trivial, strongly jump traceable, ...
- Earlier investigations of ideals focussed on their definability, and on the global properties of ideal lattices.
- A few proper ideals are known to be first-order definable without parameters in the c.e. degrees: the cappable degrees, and its subideal, the non-cuppable degrees.

Definability and global properties

- Several natural ideals came out of the intrusion of randomness-related concepts into computability: K-trivial, strongly jump traceable, ...
- Earlier investigations of ideals focussed on their definability, and on the global properties of ideal lattices.
- A few proper ideals are known to be first-order definable without parameters in the c.e. degrees: the cappable degrees, and its subideal, the non-cuppable degrees.
- Nies (2001) showed that a definable set generates a definable ideal.

Definability and global properties

- Several natural ideals came out of the intrusion of randomness-related concepts into computability: K-trivial, strongly jump traceable, ...
- Earlier investigations of ideals focussed on their definability, and on the global properties of ideal lattices.
- A few proper ideals are known to be first-order definable without parameters in the c.e. degrees: the cappable degrees, and its subideal, the non-cuppable degrees.
- Nies (2001) showed that a definable set generates a definable ideal.
- Applying this, Yang Yue and Yu Liang found a few more examples of definable ideals: for instance, the ideal generated by the non-bounding degrees.

Back to the leading question

- By the Thickness Lemma every proper u.g. ideal has an incomplete upper bound.
- The Π^0_4 ideal of cappable degrees has no incomplete upper bound.
- What can we say about upper bounds of a proper Σ^0_3 ideal?
- How about a proper Σ^0_4 ideal?

Bounds for proper Σ^0_3 ideals

Theorem

Each proper Σ^0_3 ideal $\mathbb{I}$ in the c.e. degrees has a low$_2$ upper bound

Bounds for proper Σ^0_3 ideals

Theorem

Each proper Σ^0_3 ideal $\mathbb{I}$ in the c.e. degrees has a low$_2$ upper bound

- We first prove that each uniformly c.e. subsequence of a proper Σ^0_3 ideal is uniformly low$_2$.

Bounds for proper Σ^0_3 ideals

Theorem

Each proper Σ^0_3 ideal $\mathbb{I}$ in the c.e. degrees has a low$_2$ upper bound

- We first prove that each uniformly c.e. subsequence of a proper Σ^0_3 ideal is uniformly low$_2$.
- This uniform low$_2$-ness allows us to code all of $\mathbb{I}$ into an upper bound, while keeping this bound low$_2$.

Bounds for proper Σ^0_3 ideals

Theorem

Each proper Σ^0_3 ideal $\mathbb{I}$ in the c.e. degrees has a low$_2$ upper bound

- We first prove that each uniformly c.e. subsequence of a proper Σ^0_3 ideal is uniformly low$_2$.
- This uniform low$_2$-ness allows us to code all of $\mathbb{I}$ into an upper bound, while keeping this bound low$_2$.
- We have a $\emptyset''$ construction with a tree of strategies to read a low$_2$-ness index of the upper bound off the true path.

Corollary

There is a low$_2$ c.e. degree above all the K-trivials.

Bounds for proper Σ^0_4 ideals

Theorem

Each proper Σ^0_4 ideal $\mathbb{I}$ in the c.e. degrees has an incomplete upper bound.

Bounds for proper Σ^0_4 ideals

Theorem

Each proper Σ^0_4 ideal $\mathbb{I}$ in the c.e. degrees has an incomplete upper bound.

- The proof uses that there is a high c.e. set H of non-cuppable degree (Harrington and Miller 1981).

Bounds for proper Σ^0_4 ideals

Theorem

Each proper Σ^0_4 ideal $\mathbb{I}$ in the c.e. degrees has an incomplete upper bound.

- The proof uses that there is a high c.e. set H of non-cuppable degree (Harrington and Miller 1981).
- We may assume that the degree of H is in $\mathbb{I}$.

Bounds for proper Σ^0_4 ideals

Theorem

Each proper Σ^0_4 ideal $\mathbb{I}$ in the c.e. degrees has an incomplete upper bound.

- The proof uses that there is a high c.e. set H of non-cuppable degree (Harrington and Miller 1981).
- We may assume that the degree of H is in $\mathbb{I}$.
- The construction now works because $\mathbb{I}$ is only $\Sigma^0_3(H)$.

Bounds for proper Σ^0_4 ideals

Theorem

Each proper Σ^0_4 ideal $\mathbb{I}$ in the c.e. degrees has an incomplete upper bound.

- The proof uses that there is a high c.e. set H of non-cuppable degree (Harrington and Miller 1981).
- We may assume that the degree of H is in $\mathbb{I}$.
- The construction now works because $\mathbb{I}$ is only $\Sigma^0_3(H)$.
- It is a $\emptyset''$ construction, but no explicit tree of strategies is needed. It suffices to use hat computations.

Bounds for proper Σ^0_4 ideals

Theorem

Each proper Σ^0_4 ideal $\mathbb{I}$ in the c.e. degrees has an incomplete upper bound.

- The proof uses that there is a high c.e. set H of non-cuppable degree (Harrington and Miller 1981).
- We may assume that the degree of H is in $\mathbb{I}$.
- The construction now works because $\mathbb{I}$ is only $\Sigma^0_3(H)$.
- It is a $\emptyset''$ construction, but no explicit tree of strategies is needed. It suffices to use hat computations.

Prime ideals

Definition

A proper subset F of an usl U with least element 0 is called a *strong filter* if it is upward closed and below every pair elements of F there is a further element of F.
An ideal $I \subseteq U$ called *prime ideal* if its complement $U - I$ is a strong filter.

The cappable degrees form a Π^0_4 prime ideal.

Prime ideals

Definition

A proper subset F of an usl U with least element 0 is called a *strong filter* if it is upward closed and below every pair elements of F there is a further element of F.
An ideal $I \subseteq U$ called *prime ideal* if its complement $U - I$ is a strong filter.

The cappable degrees form a Π^0_4 prime ideal.
We can now answer a question of Calhoun (1990).

Prime ideals

Definition

A proper subset F of an usl U with least element 0 is called a *strong filter* if it is upward closed and below every pair elements of F there is a further element of F.
An ideal $I \subseteq U$ called *prime ideal* if its complement $U - I$ is a strong filter.

The cappable degrees form a Π^0_4 prime ideal.
We can now answer a question of Calhoun (1990).

Corollary

No proper Σ^0_4 ideal is prime.

Prime ideals

Definition

A proper subset F of an usl U with least element 0 is called a *strong filter* if it is upward closed and below every pair elements of F there is a further element of F.
An ideal $I \subseteq U$ called *prime ideal* if its complement $U - I$ is a strong filter.

The cappable degrees form a Π^0_4 prime ideal.
We can now answer a question of Calhoun (1990).

Corollary

No proper Σ^0_4 ideal is prime.

For, there is a minimal pair of degrees, none of which are below the upper bound of the ideal (Welch 1981).

Density of partial orders of ideals

Each principal ideal $[\mathbf{0}, \mathbf{b}]$, where $\mathbf{b} \neq \mathbf{0}$, has a maximal subideal that is $\Delta^0_4(\mathbf{b})$.

Density of partial orders of ideals

Each principal ideal $[\mathbf{0}, \mathbf{b}]$, where $\mathbf{b} \neq \mathbf{0}$, has a maximal subideal that is $\Delta^0_4(\mathbf{b})$.
Choosing $\mathbf{b}$ low, this shows that the lattice of Σ^0_4 ideals is not dense.

Density of partial orders of ideals

Each principal ideal $[\mathbf{0}, \mathbf{b}]$, where $\mathbf{b} \neq \mathbf{0}$, has a maximal subideal that is $\Delta^0_4(\mathbf{b})$.
Choosing $\mathbf{b}$ low, this shows that the lattice of Σ^0_4 ideals is not dense.
In contrast, we have:

Theorem

The partial order of Σ^0_3 ideals in the c.e. degrees is dense.

Density of partial orders of ideals

Each principal ideal $[\mathbf{0}, \mathbf{b}]$, where $\mathbf{b} \neq \mathbf{0}$, has a maximal subideal that is $\Delta^0_4(\mathbf{b})$.
Choosing $\mathbf{b}$ low, this shows that the lattice of Σ^0_4 ideals is not dense.
In contrast, we have:

Theorem

The partial order of Σ^0_3 ideals in the c.e. degrees is dense.

In fact if $\mathbb{I}$ is a proper Σ^0_3 ideal in the c.e. degrees, then each degree $\mathbf{d} \notin \mathbb{I}$ splits in the quotient usl.

Some open questions on ideals

- Is every Σ^0_4 ideal $\mathbb{I}$ the intersection of the principal ideals it is contained in? (This would strengthen our result that $\mathbb{I}$ has an incomplete upper bound.)
- For $k \geq 4$, is the class of principal ideals definable in the lattice of Σ^0_k ideals?
- Let $\mathbf{K}$ be the ideal of K-trivial degrees. Are there c.e. degrees $\mathbf{a}, \mathbf{b}$ such that $\mathbf{K} = [\mathbf{0}, \mathbf{a}] \cap [\mathbf{0}, \mathbf{b}]$?

Some references

- A. Nies, Parameter definable subsets of the recursively enumerable degrees, JML, 2002.
- Yang/Yu, On the definable ideal generated by the non-bounding degrees, JSL 2005
- G. Barmpalias and A. Nies, Upper bounds on ideals in the Turing degrees, to appear.