Randomness below complete theories of arithmetic

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Formal arithmetic (Peano Arithmetic)

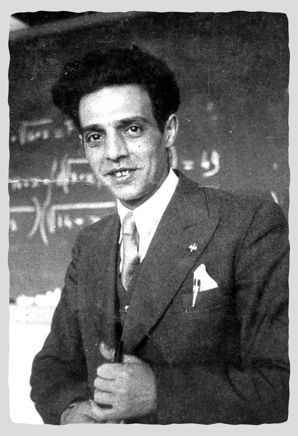
Based on Peano's axioms and their variations

Axioms for PA

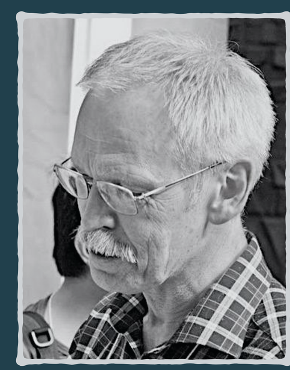
P1 $\forall x(sx \neq 0)$
P2 $\forall x \forall y(sx = sy \supset x = y)$ s is 1-1 function
P3 $\forall x(x + 0 = x)$
P4 $\forall x \forall y(x + sy = s(x + y))$ } define $+$
P5 $\forall x(x \cdot 0 = 0)$
P6 $\forall x \forall y(x \cdot sy = (x \cdot y) + x)$ } define $\cdot$

Induction Scheme: Let $Ind(A(x))$ be the sentence

$$\forall y_1 \cdots \forall y_k[(A(0) \wedge \forall x(A(x) \supset A(sx))) \supset \forall x A(x)]$$



Randomization in Arithmetic

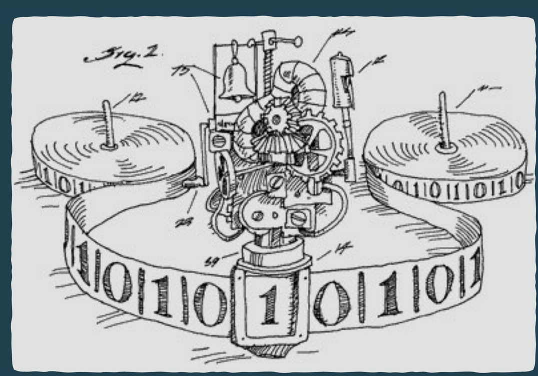


- completions of arithmetic can be obtained probabilistically
- but the probability of a successful outcome is as small as it can be
- the same is true for completions of finite segments of arithmetic
- algorithmically random binaries do not encode complete theories of arithmetic (unless they compute the halting problem)
- Complete theories of arithmetic are deep in the sense of Bennett: it is hard to obtain them probabilistically.

Language Coding & Computation

Theories of arithmetic can be written in binary, via coding of formulas into integers.

Then proofs can be viewed as computations.

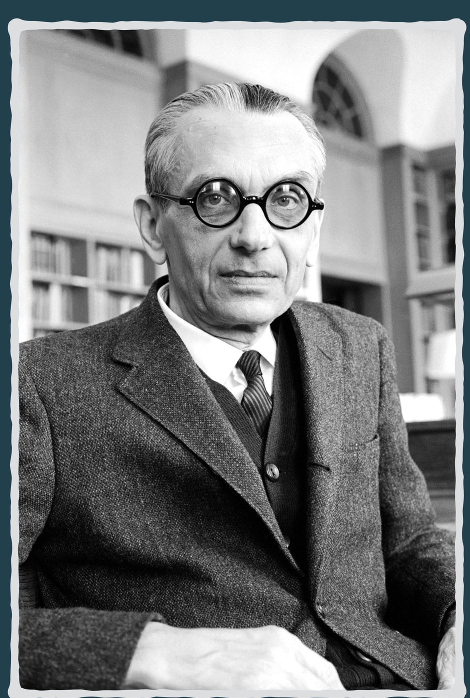


- Kucera pioneered the study of interactions between algorithmic randomness and extensions of formal arithmetic
- Stephan & Levin demonstrated the weakness of randomization for completing arithmetic and binary predicates
- Barmpalias, Lewis-Pye, Ng: every complete theory of arithmetic is computable from two algorithmically random bit-sequences

Incompleteness in arithmetic

Discovered by Gödel: there is no computable binary predicate consistently evaluating the truth of each arithmetical sentence.

This is all that was known about computing complete theories of arithmetic with the use of random binaries...



True but unprovable statements

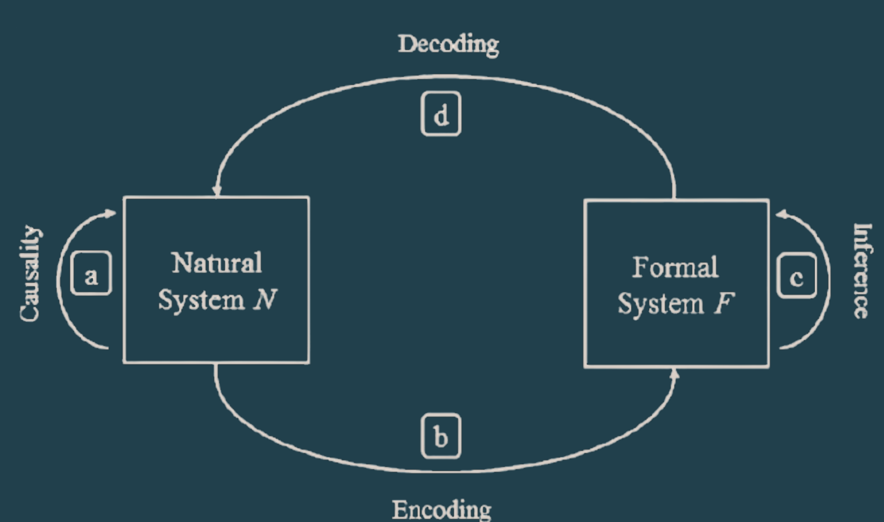
(a) Kruskal's tree theorem: the finite trees over a well-quasi-ordered set of labels is well-quasi-ordered under homeomorphic embedding.

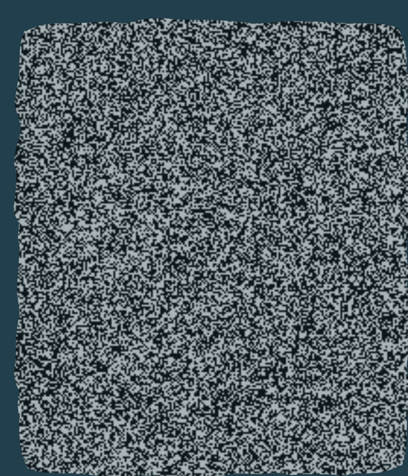
(b) Ramsey: there are monochromatic cliques in any coloring of sufficiently large complete graphs.

(c) Graph minor theorem: the undirected graphs, partially ordered by the graph minor relationship, form a well-quasi-ordering

Research problem

Given a complete theory of arithmetic and any random binary that it computes, is there another random binary that computes the theory with the help of the first one?

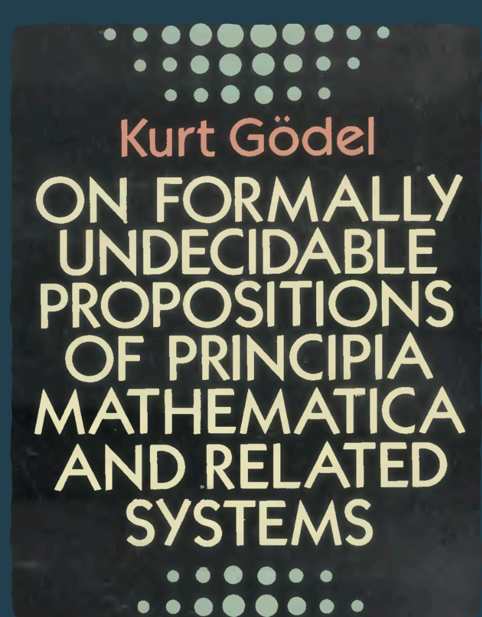




Complete extensions of Arithmetic

There are arithmetical sentences whose validity cannot be decided from the axioms. A complete extension is obtained by adding such sentences (or their negation) to the theory, while maintaining consistency.

Extensions of arithmetic may not be consistent with each other: some may include a certain undecidable sentence, while others may include its negation.



Outcome and Methodology

- Coding deep (Bennett's logical depth) information into unstructured (random) binary sequences
- Random coding technology was not up to this task
- Randomizing the method of Kucera, we obtained the required coding method to give a positive answer.

Complexity of True Arithmetic

The computational complexity of the true sentences of arithmetic is overwhelming: it is as hard as infinitely many iterations of the halting problem.

However many complete extensions of arithmetic, though incomputable, are much easier to compute than many known problems, such as the halting problem.