RELATIVE RANDOMNESS AND CARDINALITY

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ABSTRACT. A set $B \subseteq \mathbb{N}$ is called low for Martin-Löf random if every Martin-Löf random set is also Martin-Löf random relative to B. We show that a Δ_2^0 set B is low for Martin-Löf random iff the class of oracles which compress less efficiently than B, namely the class

$$\mathcal{C}^B = \{A \mid \forall n \; K^B(n) \leq^+ K^A(n)\}$$

is countable (where K denotes the prefix free complexity and $\leq^+$ denotes inequality modulo a constant). It follows that Δ_2^0 is the largest arithmetical class with this property and if $\mathcal{C}^B$ is uncountable, it contains a perfect Π_1^0 set of reals. The proof introduces a new method for constructing non-trivial reals below a Δ_2^0 set which is not low for Martin-Löf random.

1. INTRODUCTION

One of the most popular approaches to the definition of an algorithmically random sequence is the so-called *measure-theoretic* paradigm. According to this doctrine, a random sequence should have certain stochastic properties. For example it should have about as many 0s as 1s. This approach can be traced at least back to Von Mises' work [vM19]. Church [Chu40] built on these ideas and his main contribution to this area was to make the connection with computability theory. Later, Martin-Löf [ML66] gave a definitive mathematical definition of a random sequence by specifying a canonical countable family of null sets and calling a sequence random when it does not belong to any member of this family. This family is the collection of effectively null sets, i.e. sets of the form $\cap_j U_j$ where (U_j) is a uniform sequence of Σ_1^0 classes such that $\mu(U_j) < 2^{-j-1}$. The idea behind this definition is that each member of the canonical family represents a stochastic test which looks for special properties of sequences. The sequences which have algorithmically special features will belong to a member of this family, hence they will not be random; and vice-versa. These sequences are called Martin-Löf random.

Kolmogorov [Kol65] proposed that a string (a finite sequence) is random if it does not have a short description. That is, any program that generates it is more or less as long as the sequence itself. Levin [Lev73] and Chaitin [Cha75] required that the underlying machine which gives descriptions must

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be prefix-free, i.e. it does not use two programs, one of which is an extension of the other, to describe different strings. They also proposed that an infinite sequence A is random if its initial segments have maximal descriptive complexity (modulo a constant) with respect to prefix-free machines, i.e. $n \leq^+ K(A \upharpoonright n)$ for every $n \in \mathbb{N}$, where $\leq^+$ means that the inequality $\leq$ holds for all $n \in \mathbb{N}$ provided that we add an integer constant on one side of it, and $K(\sigma)$ is the *prefix-free complexity* of a string σ: the length of the shortest string τ in the domain of the universal[1] prefix-free machine M such that $M(\tau) = \sigma$. In other words, a sequence is random if its initial segments are *incompressible*. Let us call such sequences Kolmogorov-Levin-Chaitin random.

The following result of Schnorr showed that these two approaches are equivalent, thus demonstrating the robustness of this mathematical concept of randomness.

Theorem 1.1 (Schnorr, see Chaitin [Cha75])**.** *A sequence is Martin-Löf random iff it is Kolmogorov-Levin-Chaitin random.*

For an exposition of the basic concepts and results of algorithmic randomness we refer to [DHNT06] and for the history of the subject we refer to [LV97, ZL70, vL87]. The definitions of algorithmic randomness mentioned above relativize to any oracle $X \in 2^\omega$, in the same way that Turing computations relativize, thus forming the base of a theory of *relative randomness*. In particular, given an oracle X let us denote the class of random sequences relative to X by MLR^X (and $\mathsf{MLR} = \mathsf{MLR}^\emptyset$)[2] and the prefix-free complexity relative to X by K^X. The obvious way to compare the strength of two oracles A, B with respect to relative randomness (as opposed to, for example, relative computation) is to say that A is weaker than B in the case that every sequence which is derandomized by A, is also derandomized by B. Here an oracle *derandomizes* a sequence if the latter has some special properties *relative* to the oracle. Also, A is weaker than B as to the ability to compress strings, if modulo a constant every string gets a shorter prefix-free description relative to B than it does relative to A. These comparison relations between oracles were defined formally by Nies [Nie05].

Definition 1.1 (Nies [Nie05])**.** We say that $A \leq_{LR} B$ if every Martin-Löf random set relative to B is also Martin-Löf random relative to A. We say that $A \leq_{LK} B$ if $K^B(\sigma) \leq^+ K^A(\sigma)$ for all $\sigma \in 2^{<\omega}$.

The relation $\leq_{LR}$ was studied in [BLS08a, BLS08b, Sim07]. The reals B such that $\mathsf{MLR} \subseteq \mathsf{MLR}^B$ (i.e. $B \leq_{LR} \emptyset$) are sometimes called *low for random*.

[1] It is a well known fact that there is a universal prefix-free machine M, i.e. one that describes strings in an optimal way with respect to any other prefix-free machine N and up to a constant: for every finite string σ, if there is a string τ such that $N(\tau) = \sigma$ then there is some τ' with $|\tau'| \leq^+ |\tau|$ such that $M(\tau') = \sigma$.

[2] The notation MLR is taken from [Nie08].

The analog of Schnorr's theorem for relative randomness was recently given by Kjos-Hanssen/Miller/Solomon.

Theorem 1.2 (Kjos-Hanssen/Miller/Solomon [KHMSxx]). *The relations $\leq_{LR}$ and $\leq_{LK}$ are equal.*

This result demonstrates that this relation is a natural and robust way to study relative randomness. The following, then, is a basic question about relative randomness.

Question. Given an oracle $B \in 2^\omega$, how many oracles can compress at most as well as B?

According to the discussion above, this is equivalent to asking how many $A \in 2^\omega$ are there such that $K^B(\sigma) \leq^+ K^A(\sigma)$ for all $\sigma \in 2^{<\omega}$, or even what is the cardinality of the class

$$\mathcal{C}^B = \{A \mid \mathsf{MLR}^B \subseteq \mathsf{MLR}^A\}. \tag{1.1}$$

We notice that $\mathcal{C}^B$ is Borel, hence it is either countable or it contains a perfect set. Let us give a brief history of the attempts that have taken place in order to answer this question. We recall that a set B is *low for Martin-Löf random* if $\mathsf{MLR} \subseteq \mathsf{MLR}^B$, i.e. every Martin-Löf random set is also Martin-Löf random relative to B. This notion was introduced in [KT99], where a non-computable c.e. set with this property was constructed. In the list of open questions on randomness [ASK00] Question 4.4 asked about the cardinality of $\mathcal{C}^\emptyset$ and whether this is a subclass of Δ^0_2. Nies [Nie05] gave a positive answer (see [BLS08a] for a direct proof), thus determining the cardinality of $\mathcal{C}^B$ for $B = \emptyset$. On the other hand, in [BLS08a] it was shown that $\mathcal{C}^B$ is uncountable for $B = \emptyset'$ and more generally for B in $\overline{GL_2}$, i.e. such that $(B \oplus \emptyset')' <_T B''$. A notion similar to lowness for Martin-Löf randomness but weaker, was introduced in [NST05]. Recall the halting probability Ω of a universal prefix-free machine. A set B is low for Ω if Ω is random relative to B. Miller [Mil] showed that $\mathcal{C}^B$ is countable whenever B is low for Ω. This result prompted him to conjecture that $\mathcal{C}^B$ is countable exactly when B is low for Ω, but this remains unknown. Notice that since every 2-random is low for Ω (see [NST05]) the class $\mathcal{C}^B$ is countable for almost all B (all but a set of measure 0). As far as local degree structures are concerned, in [BLS08b] it was shown that there is a superlow c.e. set B such that $\mathcal{C}^B$ contains a perfect Π^0_1 class. Here we show the following definitive result for the case when the oracle is Δ^0_2.

Theorem 1.3. *If B is Δ^0_2 and not low for Martin-Löf random then $\mathcal{C}^B$ contains a perfect Π^0_1 class.*

The proof of Theorem 1.3 makes use of a new method for constructing reals with certain properties below a Δ^0_2 set B which is not low for Martin-Löf random. This can be viewed as a permitting technique, with some similarities but a lot of differences to the simple, promptly simple permitting (see [Soa87]), and the permitting of Δ^0_2 fixed point free reals (see [Kuč86]).

Corollary. *If B is Δ^0_2, then B is low for Martin-Löf random iff $\mathcal{C}^B$ is countable. Furthermore, if $\mathcal{C}^B$ is uncountable then it contains a perfect Π^0_1 set of reals.*

Proof. One direction of the first claim follows from the result in [Nie05] that if B is low for Martin-Löf random then $\mathcal{C}^B$ is a subclass of Δ^0_2, hence countable. The other direction and the second claim follows from Theorem 1.3. □

It is known from [NST05] that there are low for Ω reals B (even in Σ^0_2) which are not low for random. Then the above mentioned result of Miller [Mil] shows that Δ^0_2 is the largest arithmetical class with the property of the above corollary.

2. Preliminaries

In the following, we use c.e. sets of strings to generate subclasses of the Cantor space. For example, a binary string σ is often identified with the clopen set $[\sigma] = \{X \mid \sigma \subset X\}$ and more generally, a set of strings M is often identified with the open set

$$S(M) = \{X \in 2^\omega \mid \exists n(X \upharpoonright n \in M)\}$$

of the Cantor space. Also, boolean operations, inclusion and measure on sets of string refer to the sets of reals that they represent. Thus if $M, N \subseteq 2^{<\omega}$ then we define $\mu(M) := \mu(S(M))$ (where μ is the Lebesgue measure), $M \subseteq N$ iff $S(M) \subseteq S(N)$, $M \cap N := S(M) \cap S(N)$, $M \cup N := S(M) \cup S(N)$ and $M - N := S(M) - S(N)$.

An *oracle Σ^0_1 class* V is an oracle Turing machine which, given an oracle A outputs a set of finite binary strings V^A, representing an open subset of the space 2^ω . The oracle class V can be seen as a c.e. set of axioms $\langle \tau, \sigma \rangle$ (where $\tau, \sigma \in 2^{<\omega}$) so that

$$\begin{aligned} V^A &= \{\sigma \mid \exists \tau(\tau \subset A \ \wedge \ \langle \tau, \sigma \rangle \in V)\} \\ V^\rho &= \{\sigma \mid \exists \tau(\tau \subseteq \rho \ \wedge \ \langle \tau, \sigma \rangle \in V)\} \end{aligned}$$

for $A \in 2^\omega, \rho \in 2^{<\omega}$. We denote the finite approximation of a parameter at stage s of the universal enumeration of c.e. sets by the suffix $[s]$. An oracle Martin-Löf test (U_e) is a uniform sequence of oracle Σ^0_1 classes U_e such that $\mu(U^X_e) < 2^{-(e+1)}$ and $U^X_e \supseteq U^X_{e+1}$ for all $X \in 2^\omega$, $e \in \mathbb{N}$. A real A is called B-random if for every oracle Martin-Löf test (U_e) we have $A \notin \cap_e U^B_e$. A universal oracle Martin-Löf test is an oracle Martin-Löf test (U_e) such that for every $A, B \in 2^\omega$, A is B-random iff $A \notin \cap_e U^B_e$.

In [KH07] (see [BLS08a] for a different proof) it was shown that $\mathsf{MLR}^B \subseteq \mathsf{MLR}^A$ iff for some member U of a universal oracle Martin-Löf test, there is a $\Sigma^0_1(B)$ class V^B with $U^A \subseteq V^B$ and $\mu(V^B) < 1$. In the following we fix U to be the second member of a universal oracle Martin-Löf test, so that $\mu(U^X) \leq 2^{-2}$ for all $X \in 2^\omega$. We can choose U and the oracle test (U_i) which is used below, such that U^τ, U^τ_i are clopen sets which are uniformly computable in i, τ (see [BLS08a]). By an LR reduction we mean an oracle Σ^0_1 class V such that $\mu(V^X) < 1$ for all $X \in 2^\omega$, and X is reducible to Y via this reduction if $U^X \subseteq V^Y$. The following lemma is implicit in [BLS08a].

Lemma 2.1. *Let U be a member of an oracle Martin-Löf test, $B \in 2^\omega$ and $m \in \mathbb{N}$. Then there exists $n \in \mathbb{N}$ such that for all $s, t > n$ we have $\mu(U^{B\restriction s} - U^{B\restriction t}) < 2^{-m}$.*

To show Lemma 2.1 we just have to notice that the negation of it would imply that $\mu(U^B) = \infty$, which is absurd. For background on relative randomness via Martin-Löf tests and even simple versions of some of the methods that are used in this paper we refer to [BLS08a, BLS08b]. In the following, trees are thought of as growing upwards.

3. Proof of Theorem 1.3

Given a Δ^0_2 set B which is not low for Martin-Löf random (and an effective approximation of it) we need to construct a perfect Π^0_1 class $[T]$ (where T is a tree representing the class and $[T]$ the infinite paths through T) such that $X \in \mathcal{C}^B$ (i.e. $X \leq_{LR} B$) for all $X \in [T]$. We will do this via a single LR reduction, i.e. we will construct an oracle Σ^0_1 class V with $\mu(V^Z) < 1$ for all $Z \in 2^\omega$, such that $U^X \subseteq V^B$ for all $X \in [T]$.

A standard way to construct a Π^0_1 class, taken from [JS72], is the following. We will define an effective sequence of 1–1 maps $T[s] : 2^{<\omega} \to 2^{<\omega}$ which preserve the ordering and compatibility relations. These can be viewed as uniformly computable perfect trees, and we can consider the set of infinite paths through them:

$$[T[s]] = \{X \mid \forall n \exists \sigma \ (|\sigma| = n \ \wedge \ T_\sigma[s] \supseteq X \restriction n)\}$$

which is a Π^0_1 class. We will also ensure that $[T[s+1]] \subseteq [T[s]]$ for each $s \in \mathbb{N}$ and that $T_\sigma = \lim_s T_\sigma[s]$ exists for each $\sigma \in 2^{<\omega}$. This ensures that

$$[T] = \cap_s [T[s]]$$

is a perfect Π^0_1 class, where T is the limit map $\sigma \to T_\sigma$. Essentially we construct an effective monotone sequence of perfect computable trees $T[s]$ converging to a perfect tree T such that $[T]$ is a Π^0_1 class. In order to achieve $U^X \subseteq V^B$ for all $X \in [T]$ we have to make the tree T very *thin*, in some sense. Indeed, since T is perfect there are continuum many paths though it and so $\cup_{X \in [T]} U^X$ is very likely to have large measure; but we need to achieve $\mu(V^B) < 1$. This conflict is, in a way, similar to the conflict that we meet when we wish to construct a perfect Π^0_1 class which only contains paths with 'low' information. For example consider a direct construction of a perfect Π^0_1 class which only contains generalized low paths (see the methodology in [BLS08b], although this was originally proved indirectly in [Cen99]) or even one which only contains jump-traceable paths, which was constructed in [Nie06]. More related is the case of Theorem 1.3 for $B = \emptyset'$ which was proved in [BLS08a, BLS08b].

Let us denote concatenation of strings by $*$. We let $T[0]$ be the identity map. If we could control B (an assumption which roughly corresponds to the case where $B = \emptyset'$), at stage s_0 we would choose some $\sigma \in 2^{<\omega}$ and enumerate $U^{T_{\sigma*i}[s_0]} - U^{T_\sigma[s_0]}$ into V^B (by enumerating certain strings into V^B) for $i = 0, 1$ with big use c_σ. If at some later stage s we have $\mu(U^{T_\rho[s]} - U^{T_\sigma[s]}) \geq 2^{-2|\sigma|-2}$ for some $\rho \supset \sigma$ with $|\rho| < s$ we would redefine $T_{\sigma*\eta}[s+1] = T_{\rho*\eta}[s]$ for all $\eta \in 2^{<\omega}$, enumerate c_σ into B (evicting $U^{T_{\sigma*i}[s_0]} - U^{T_\sigma[s_0]}$ from V^B) and enumerate $U^{T_{\sigma*i}[s+1]} - U^{T_\sigma[s+1]}$ into V^B for $i = 0, 1$ with new big use c_σ; and so on. Since $\mu(U^\beta) < 2^{-2}$ there can be at most $2^{2|\sigma|}$ changes in the approximation of T_σ (given a final approximation of T_{σ^-}, where σ^- denotes the predecessor of σ) and eventually T_σ will be defined such that $\mu(U^{T_\rho} - U^{T_\sigma}) < 2^{-2|\sigma|-2}$ for all $\rho \supset \sigma$.

These procedures can work simultaneously for all $\sigma \in 2^{<\omega}$ with a typical finite injury effect: when T_σ is redefined, T_ρ is redefined for all $\rho \supseteq \sigma$ and some number c_σ enters B in order to evict the intervals it contributed to V^B under the previous definition. This process makes the tree thinner and thinner, but eventually all nodes reach a limit, thus defining a perfect tree T. Figure 1 shows the full binary tree, and inside it one can see a thinner

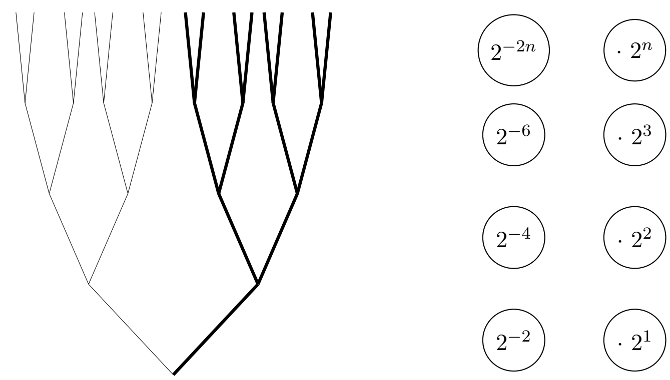

FIGURE 1. Building a perfect tree which is *thin*, in the sense that it is assigned a bounded amount of measure with respect to the oracle Σ^0_1 class U. The figure shows the thin tree as a substree of the full binary tree.

subtree, which is T. The oracle Σ^0_1 class U can be viewed as a computable assignment of measure through the paths of the full binary tree. The first column next to the tree of Figure 1 shows an upper bound on the measure assingned to the various segments of the paths through T (the bound is uniform for each level of the tree). The second column shows the number of segments (which is the same as the number of paths) of each level of T. If

$$C_n = \{T_\sigma \mid \sigma \in 2^{<\omega} \ \wedge\ |\sigma| \leq n\}$$

and $A_n = \cup_{\tau \in C_n} U^\tau$ we have $C_n \subseteq C_{n+1}$, $A_n \subseteq A_{n+1}$ and $V^B = \cup_n A_n$. By induction $\mu(A_n) < \sum_{i=0}^n 2^i \cdot 2^{-(2i+2)} = 2^{-1}$ and so $\mu(V^B) \leq 2^{-1}$. For a detailed presentation of this argument we refer to [BLS08a] (where it is presented as an oracle argument) or [BLS08b] (where it is presented dynamically, as described here).

Now the difficulty is that instead of being able to control B, we merely have the information that B is not low for random. In the argument above, V^B consisted of the union of U^β for $\beta \in T$ because we were able to evict irrelevant strings which entered V^B by the strategy of some T_σ at a time when T_τ, for some $\tau \subseteq \sigma$ had not taken its final value. In the general case we will not be able to do this, so $V^B = (\cup_{\beta \in T} U^\beta) \cup Junk$ where $Junk$ contains the reals which became irrelevant and were not evicted from V^B. We will refine the above ideas and use the fact that B is not low for random (instead of explicitly enumerating into it), in order to achieve $\mu(V^B) < 1$. Let (U_j) be a universal oracle Martin-Löf test and fix a Δ^0_2 approximation $(B[s])$ to B.

For each $\sigma \in 2^{<\omega}$ we often identify T_σ with the *strategy* to define the value of T on σ. Strategy T_σ has a quota parameter $p_\sigma \in \mathbb{N}$. It will make use of U_{p_σ} and will construct a Σ^0_1 class E_σ which 'attempts' to cover $U^B_{p_\sigma}$. It will also enumerate an oracle Σ^0_1 class V_σ and its goals will be the following.

Goals of strategy T_σ.

- $T_\sigma[s]$ reaches a limit as $s \to \infty$.
- $U^{T_{\sigma * i}} - U^{T_\sigma} \subseteq V^B_\sigma$ for $i = 0, 1$.
- $\mu(V^X_\sigma) < \mu(U^X_{p_\sigma})$ for all $X \in 2^\omega$.

Eventually we set $V^B = V_{-1} \cup (\cup_{\sigma \in 2^{<\omega}} V^B_\sigma)$, where $V_{-1} = \lim_s U^{T_\emptyset[s]}$ which is a Σ^0_1 class (as $T_\emptyset$ is approximated monotonically) and $\mu(V_{-1}) < 2^{-2}$. Note that the third clause implies that $\mu(V^B_\sigma) < 2^{-p_\sigma}$ which, by appropriate choice of the quotas p_σ, will be used to show that $\mu(V^B) < 1$. When all T_σ strategies are put together the finite injury effect will cause some p_σ to change finitely many times.

3.1. **Strategy T_σ in isolation.** In this section we restrict our attention to T_ρ for $\rho \supseteq \sigma$. We define a strategy which approximates T_σ, satisfying the goals outlined above, without any assumptions about the approximation of T_ρ, $\rho \supset \sigma$ other than monotonicity and the preservation of ordering and compatibility relations in $T[s]$ restricted to arguments $\supseteq \sigma$. In particular, we do not assume the convergence of T_ρ for any $\rho \subseteq \sigma$. Order the strings as usual, first by length and then lexicographically. We assume that p_σ is a given constant. We will construct an auxiliary oracle Σ^0_1 class F_σ such that $F^\tau_\sigma \subseteq U^\tau_{p_\sigma}$ and $\mu(F^\tau_\sigma) = \mu(V^\tau_\sigma)$ for all $\tau \in 2^{<\omega}$. In this way, every bit of measure in V_σ will be tied up with a bit of equal measure in U_{p_σ}. For each s we let $\eta_\sigma[s]$ be the least $\eta \subset B[s]$ such that $\mu(U^\eta_{p_\sigma} - F^\eta_\sigma[s]) > 0$ (equivalently, $\mu(U^\eta_{p_\sigma}) - \mu(V^\eta_\sigma) > 0$). Also let $C_\sigma[s] = U^{\eta_\sigma[s]}_{p_\sigma} - U^{(\eta_\sigma[s])^-}_{p_\sigma}$.

The main idea is that we wish to define T_σ in a way such that $U^{T_\rho} - U^{T_\sigma}$ is very small for all $\rho \supset \sigma$. As discussed above, this is possible but we also wish to ensure that $Z_i = U^{T_{\sigma*i}} - U^{T_\sigma} \subseteq V_\sigma^B$ for $i = 0, 1$ while keeping $\mu(V_\sigma^B)$ small. We demand $U^{T_\rho} - U^{T_\sigma}$ be very small (an amount corresponding to the measure of some interval $C_\sigma[s]$ which seems to be in the universal class U_{p_σ} with use $\eta_\sigma[s]$) and enumerate Z_i into $V_\sigma^B[s]$ with the same use $\eta_\sigma[s]$. If our demand was too strong and we need to redefine T_σ, the amount enumerated into $V_\sigma^B[s]$ is useless and we wish to remove it. So we put $C_\sigma[s]$ into E_σ, thus threatening to cover $U_{p_\sigma}^B$. Either B will change so that the useless amount is removed from V_σ^B, or E_σ will cover a part of $U_{p_\sigma}^B$. Eventually we can argue that either V^B does not contain much useless measure, or E_σ covers a universal class relative to B. In the latter case $\mu(E_\sigma) = 1$ since B is not low for random and this will imply that for the path β carved by the redefinitions of T_σ, U^β has too much measure, which is a contradiction as $\mu(U^\beta) < 2^{-2}$. The big picture can be described as follows. The fact that B is not low for random means that any Σ_1^0 cover E_σ of the universal class relative to B must have measure 1. While we try to find a final value for T_σ, we enumerate a cover E_σ in such a way that each time we move T_σ, some measure is added in E_σ and an analogous amount of measure is added in U^{T_σ} (for the new value of T_σ, which extends the previous one). The construction operates in such a way that if T_σ moves indefinitely, E_σ covers the universal class relative to B. This leads to a contradiction as the measure of it must be 1, and roughly the same amount of measure (say, a half of the previous amount) must occur in U^{T_σ}. We now give the formal details of strategy T_σ. In the following module and the construction, when a parameter is not explicitly redefined it retains its previous value.

T_σ **routine at stage** $s+1$.

(1) If for some $\rho \supset \sigma$ of length $s+1$ we have $\mu(U^{T_\rho[s]} - U^{T_\sigma[s]}) \geq \mu(C_\sigma[s])/2$, pick the least such and define $T_{\sigma*\tau}[s+1] = T_{\rho*\tau}[s]$ for all $\tau \in 2^{<\omega}$. Also enumerate $C_\sigma[s]$ into E_σ, enumerate $U_{p_\sigma}^{\eta_\sigma[s]}$ into $F_\sigma^{\eta_\sigma[s]}$ and also some dummy clopen set into $V_\sigma^{\eta_\sigma[s]}$ in order to make $\mu(F_\sigma^{\eta_\sigma[s]}) = \mu(V_\sigma^{\eta_\sigma[s]})$.

(2) Otherwise enumerate $M_i = (U^{T_{\sigma*i}[s]} - U^{T_\sigma[s]}) - V_\sigma^{\eta_\sigma[s]}[s]$ into $V_\sigma^{\eta_\sigma[s]}$ for $i = 0, 1$, and enumerate a clopen subset of $C_\sigma[s] - F_\sigma^{\eta_\sigma[s]}[s]$ of measure $\mu(M_0 \cup M_1)$ into $F_\sigma^{\eta_\sigma[s]}$.

Verification of T_σ **routine**. We verify that the T_σ routine satisfies its goals of strategy T_σ as mentioned above. By induction on the stages it follows that $\mu(V_\sigma^\tau[s]) = \mu(F_\sigma^\tau[s])$ for all s and the second clause of the T_σ routine is well defined. Indeed, supposing this for stages $\leq s$, since $\mu(V_\sigma^\tau[s]) = \mu(F_\sigma^\tau[s])$, if the second clause was applied at $s+1$ we must have $\mu(M_0 \cup M_1) < \mu(C_\sigma[s] - F_\sigma^{\eta_\sigma[s]}[s])$ because otherwise the first clause would apply. On the other hand if $\mu(F_\sigma^\tau[s])$ increases at $s+1$, it is clear that $\mu(V_\sigma^\tau[s])$ will increase by the same amount, and the induction step is complete. It is also clear from the T_σ routine that $F^\tau_\sigma[s] \subseteq U^\tau_{p_\sigma}$ for all $s \in \mathbb{N}$ and $\tau \in 2^{<\omega}$. Hence $\mu(V^X_\sigma) \leq \mu(U^X_{p_\sigma})$ for all $X \in 2^\omega$.

Second, we show that T_σ will reach a final value. Suppose for a contradiction that $\lim_s T_\sigma[s] = X$, where X is an infinite string. Then $C_\sigma[s]$ does not reach a limit, as in that case $\mu(U^X) = \infty$ by the first step of the T_σ routine (each time $T_\sigma[s]$ changes, $\mu(U^{T_\sigma})$ increases by $\mu(C_\sigma[s])$). We claim that $U^B_{p_\sigma} \subseteq E_\sigma$. Indeed, if this was not the case consider the least $\tau \subset B$ such that $U^\tau_{p_\sigma} \not\subseteq E_\sigma$. We must have $U^\tau_{p_\sigma} - F^\tau_\sigma \neq \emptyset$ because the only place in the T_σ routine where all of $U^\tau_{p_\sigma}$ is enumerated into F^τ_σ is the first clause, but in that case $U^\tau_{p_\sigma}$ is enumerated in E_σ. When $B \upharpoonright |\tau|$ settles, the value of $\eta_\sigma[s]$ would settle on τ and so $C_\sigma[s]$ would reach a limit, and this is impossible by the discussion above. Hence $U^B_{p_\sigma} \subseteq E_\sigma$ and since $\mathsf{MLR} \not\subseteq \mathsf{MLR}^B$ (by [KH07]) we have $\mu(E_\sigma) = 1$. But by the T_σ routine, every time $\mu(E_\sigma)$ increases by some amount $r \in \mathbb{Q}$, U^{T_σ} increases by at least $r/2$. So $\mu(U^{T_\sigma}) \geq 1/2$ which contradicts the choice of U.

Next we show that $\eta_\sigma[s]$ reaches a limit. If this did not happen, by the fact that $B[s]$ converges to B we have that $U^\tau_{p_\sigma} = F^\tau_\sigma$ for all $\tau \subset B$. Since U_{p_σ} is universal there are infinitely many $\tau \subset B$ such that $U^\tau_{p_\sigma} - U^{\tau^-}_{p_\sigma} \neq \emptyset$. Again by the convergence of $B[s]$ to B we have that for each such τ there is some stage s such that $\eta_\sigma[s] = \tau$. So

$$\forall n\ \exists s\ [\eta_\sigma[s] \subset B\ \wedge\ |\eta_\sigma[s]| > n]. \tag{3.1}$$

Choose a stage s_0 such that $T_\sigma[s] = T_\sigma[s_0]$ for all $s \geq s_0$, and choose $m \in \mathbb{N}$, $\tau_0 \supset \sigma$ such that $\mu(U^{T_{\tau_0}[s_0]} - U^{T_\sigma[s_0]}) > 2^{-m}$. Now by Lemma 2.1 choose some $n \in \mathbb{N}$ such that $\mu(U^\tau - U^{\tau^-}) < 2^{-m}$ for all $\tau \subset B$ of length $> n$. By (3.1) there is some $s > \max\{s_0, |\tau_0|\}$ such that $\eta_\sigma[s] \subset B$, $|\eta_\sigma[s]| > n$ and since $\mu(C_\sigma[s]) < 2^{-m}$ and $|\tau_0| < s$ the value of T_σ would change at $s > s_0$ by clause (2) of the T_σ routine, a contradiction.

Finally we show that for the final values of $T_\sigma, T_{\sigma*i}$ we have $U^{T_{\sigma*i}} - U^{T_\sigma} \subseteq V^B_\sigma$ (the values $T_{\sigma*i}$, $i = 0, 1$ may be infinite limits as the T_σ routine does not assume that $T_{\sigma*i}[s]$ converges after finitely many stages). Let t_0 be the least stage such that $\eta_\sigma[t] = \eta_\sigma[t_0]$ for all $t \geq t_0$. Then $C_\sigma[t] = C_\sigma[t_0]$ for all $t \geq t_0$ and $U^{T_{\sigma*i}} - U^{T_\sigma}$ will keep on being enumerated into $V^{\eta_\sigma[t_0]}_\sigma$ by clause 2 of the T_σ routine. But $\eta_\sigma[t_0] \subset B$, so $U^{T_{\sigma*i}} - U^{T_\sigma} \subseteq V^B_\sigma$.

3.2. **All strategies together.** The T_σ routines can work together with a finite injury effect. We let $p_{\sigma,j} = 2|\sigma| + j + 4$ and $n_\sigma[s]$ is the number of times that T_σ has been injured by stage s. Also, the routines will use $E_{\sigma,j}$, $F_{\sigma,j}$, $V_{\sigma,j}$, where $j = n_\sigma[s]$, at stage $s+1$ (i.e. they change parameters each time they are injured). So we also need to redefine $\eta_\sigma[s]$ to be the least $\eta \subset B[s]$ such that $\mu(U^\eta_{p_{\sigma,j}} - F^\eta_{\sigma,j}[s]) > 0$ (equivalently, $\mu(U^\eta_{p_{\sigma,j}}) - \mu(V^\eta_{\sigma,j}) > 0$) and also let $C_\sigma[s] = U^{\eta_\sigma[s]}_{p_{\sigma,j}} - U^{(\eta_\sigma[s])^-}_{p_{\sigma,j}}$, where $j = n_\sigma[s]$. Eventually we define $V_\sigma = \cup_s V_{\sigma,j_s}$, where $j_s = n_\sigma[s]$ (so that they are Σ^0_1).

General T_σ routine at stage $s+1$. Let $j = n_\sigma[s]$.

(1) If there is some $\tau \supset \sigma$ such that $|\tau| = s+1$ and $\mu(U^{T_\tau[s]} - U^{T_\sigma[s]}) \geq \mu(C_\sigma[s])/2$, define $T_{\sigma*\rho}[s+1] = T_{\tau*\rho}[s]$ for all $\rho \in 2^{<\omega}$. Also enumerate $C_\sigma[s]$ into $E_{\sigma,j}$, enumerate $U^{\eta_\sigma[s]}_{p_{\sigma,j}}$ into $F^{\eta_\sigma[s]}_{\sigma,j}$ and also some dummy clopen set into $V^{\eta_\sigma[s]}_{\sigma,j}$ in order to make $\mu(F^{\eta_\sigma[s]}_{\sigma,j}) = \mu(V^{\eta_\sigma[s]}_{\sigma,j})$.

(2) Otherwise, enumerate $M_i = (U^{T_{\sigma*i}[s]} - U^{T_\sigma[s]}) - V^{\eta_\sigma[s]}_{\sigma,j}[s]$ into $V^{\eta_\sigma[s]}_{\sigma,j}$ for $i = 0, 1$, and enumerate a clopen subset of $C_\sigma[s] - F^{\eta_\sigma[s]}_{\sigma,j}[s]$ of measure $\mu(M_0 \cup M_1)$ into $F^{\eta_\sigma[s]}_{\sigma,j}$.

We say that T_σ *requires attention at stage* $s+1$ if one of the following holds:

(i) There is some $\tau \supset \sigma$ of length $s+1$ such that $\mu(U^{T_\tau[s]} - U^{T_\sigma[s]}) \geq \mu(C_\sigma[s])/2$.

(ii) $T_{\sigma*i}$ has changed value for $i = 0$ or $i = 1$ since the last stage where T_σ received attention.

Construction. At stage $s+1$ let σ be the least string such that T_σ requires attention. Run general routine T_σ and if clause 1 of the routine was applied (i.e. if it required attention through clause (i)) for all $\tau \supset \sigma$ say that T_τ is injured.

Verification. By inductively applying the verification of the T_σ routine of subsection 3.1 we have that T_σ converges for all $\sigma \in 2^{<\omega}$. So T is a perfect tree, and $[T]$ is a Π^0_1 class since $[T] = \cap_s [T[s]]$ and $[T[n+1]] \subseteq [T[n]]$ for all $n \in \mathbb{N}$. It remains to show that $\mu(V^B) < 1$ and that $U^\beta \subseteq V^B$ for all $\beta \in [T]$. By inductively applying the verification of the T_σ routine of subsection 3.1 we have that $\mu(V^\rho_{\sigma,j_s}) \leq \mu(U^\rho_{p_{\sigma,j_s}})$ for all s and $\sigma, \rho \in 2^{<\omega}$, where $j_s = n_\sigma[s]$. If $J_\sigma = \{n_\sigma[s] \mid s \in \mathbb{N}\}$ then

$$\mu(V^\rho_\sigma) \leq \sum_{j \in J_\sigma} \mu(U^\rho_{2|\sigma|+j+4}) \leq \sum_{j \in J_\sigma} 2^{-2|\sigma|-j-4} \leq 2^{-2|\sigma|-3}.$$

Hence $\mu(V^\rho) \leq 2^{-2} + \sum_{\sigma \in 2^{<\omega}} 2^{-2|\sigma|-3} \leq 2^{-1}$. In particular, $\mu(V^B) < 1$. Finally by inductively applying the verification of the T_σ routine we get that $U^{T_{\sigma*i}} - U^{T_\sigma} \subset V^B_{\sigma,n_\sigma}$ for all $\sigma \in 2^{<\omega}$, $i = 0, 1$, where $T_{\sigma*i} = \lim_s T_{\sigma*i}[s]$, $T_\sigma = \lim_s T_{\sigma*i}[s]$ and $n_\sigma = \lim_s n_\sigma[s]$. Clearly we also have $U^{T_\emptyset} \subseteq V_{-1}$. So $U^\beta \subseteq V^B$ for all $\beta \in [T]$, and this completes the proof. We wish to conclude with a question.

Question. Can the Π^0_1 class $[T]$ of Theorem 1.3 be made such that it contains no low for Martin-Löf random paths?

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