THE INFORMATION CONTENT OF TYPICAL REALS

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Abstract. The degrees of unsolvability provide a way to study the continuum in algorithmic terms. Measure and category, on the other hand, provide notions of size for subsets of the continuum, giving rise to corresponding notions of 'typicality' for real numbers. We give an overview of the order-theoretic properties of the degrees of typical reals, presenting old and recent results, and pointing to a number of open problems for future research on this topic.

1. Introduction

The study of the structure of the degrees of unsolvability dates back to [KP54]. The same is true of the application of Baire category methods in this study, while the application of probabilistic techniques in relative computation dates back to [dLMSS55]. In this article we give an overview of the state of the art in degree theory in terms of category and measure. Recent interest in this topic has been motivated by questions in algorithmic randomness, but there is an essential distinction to be drawn between much of the research that takes place in algorithmic randomness and our interests here: while in algorithmic randomness one is concerned with understanding the properties of random reals quite generally, here we are interested specifically in the properties of the *degrees* of random reals.

Formalising a notion of typicality essentially amounts to defining a notion of size, and then defining the typical objects to be those which belong to all large sets from some restricted (normally countable) class. It is then interesting to note that, beyond cardinality, there are two basic notions of size for sets of reals. One can think in terms of measure, the large sets being those of measure 1, or in terms of category, the large sets being those which are comeager. In [Kun84] Kunen provided, in fact, a way of formalising the question as to whether these are the only 'reasonable' notions of size for sets of reals.

Of course one cannot expect a real not to belong to any set of measure 0, or not to belong to any meager set, and so to formalise a level of typicality one restricts attention to sets of reals which are definable in some specific sense – one might consider all arithmetically definable sets of reals which are of measure 0, for example, giving a corresponding class of 'typical reals' which do not belong to any such set. Working both in terms of category and measure, we are then interested in establishing the order-theoretic properties of the 'typical' degrees, i.e. those containing typical reals. Despite the large advances in degree theory over the last 60 years, our knowledge on this issue is rather limited. On the other hand, some progress has been made recently, which also points to concrete directions and methodologies for future research. We present old and new results on this topic, and

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ask a number of questions whose solutions will help obtain a better understanding of the properties of typical degrees.

In order to increase readability we have presented a number of results in tables, rather than a list of theorem displays. For the same reason, a number of citations of results are suppressed. For the remainder of this introductory section we discuss some of the background concepts which we shall require from computability theory. For a thorough background in classical computability theory we refer to [Odi89], [DH10] or [Nie09].

1.1. **The algorithmic view of the continuum.** While it is certainly sometimes of interest, in the computability theoretic context, to consider the reals imbued with their full structure as the complete ordered field, most of the time it is common practice to identify reals with their infinite binary expansions. In this context a real is seen principally as coding certain information – as an infinite binary database. One might consider, for example, the problem of deciding which Diophantine equations with integer coefficients have at least one (integer) solution. Each Diophantine equation can be given a natural number code (in some algorithmic fashion which may or may not involve redundancies of various kinds) and, given any such coding, some reals then specify exactly which equations have solutions. Given access to the bits of the real, we may be able to derive the correct answer just by checking if a particular bit of the real is a 0 or a 1. So reals are identified with subsets of the natural numbers, which in turn can be viewed as solutions to problems, and in this context we are often not overly concerned with the properties of the real as an element of the standard algebraic structure, but rather in the nature of the information that it encodes and perhaps the features of the encoding itself.

The Turing reducibility is a preorder that compares reals according to their information content and was introduced (implicitly) by Turing in [Tur39] and was explicitly studied by Kleene and Post in [KP54]. The formal definition makes use of *oracle* Turing machines, which are Turing machines with an extra oracle tape upon which the (potentially non-computable) binary expansion of a real may be written. We say that A is Turing reducible to B (in symbols, $A \leq_T B$) if there is an oracle Turing machine which calculates the bits of A when the binary expansion of B is written on the oracle tape. One can think of this as meaning quite simply that one can move in an algorithmic fashion from the binary expansion of B to the binary expansion of A. So the reducibility formalises the notion that the information in A is encoded in (and hence, can be algorithmically retrieved from) B.

When $A \leq_T B$ and $B \leq_T A$, we regard A and B as having the same information content. Hence the preorder $\leq_T$ induces an equivalence relation on the continuum which identifies reals with the same information content. These equivalence classes are called Turing degrees, and we consider the natural ordering on these degrees inherited by the Turing reducibility. Considering reals as solutions to problems (or rather problem sets indexed by natural numbers), the degrees of unsolvability can then be viewed as ordering problems or computational tasks according to their level of difficulty. We often denote by $\mathbf{a}, \mathbf{b}, \mathbf{c}$ the degrees of the problems A, B, C respectively. Hence if $\mathbf{a} \leq \mathbf{b}$ then a solution to (any problem in) $\mathbf{b}$ is enough to provide a solution to (any problem in) $\mathbf{a}$.

1.2. **Properties of degrees.** The Turing degree structure provides a way to formally address and study the classification of problems according to their difficulty. A large part of this study has focused on establishing basic properties of the partial order, beginning already in [KP54]. It was observed there, for example, that the degree structure forms an upper semi-lattice with a least element and the countable predecessor property. Early research then focused on exhibiting degrees with certain structural properties, for instance minimal pairs and minimal degrees. We shall focus here on natural properties which are first order definable in the language for the structure, i.e. the language of partial orders.

Constructing a degree with certain properties reduces to the construction of a representative real in the degree, with analogous properties. In this context, the connection to Baire category, in particular, becomes immediately apparent. Normally the construction of a real whose degree satisfies a given set of properties, proceeds by first breaking down those properties into a countable list of sub-requirements. If the real A to be constructed must be non-computable, for example, then one might break this down into a list of requirements $\mathcal{P}_i : A \neq \phi_i$, where ϕ_i is the ith partial computable function. Very often (at least in many simple cases) this list of requirements can be satisfied by proceeding according to a *finite extension argument*: one defines A as the infinite sequence extending a sequence of finite strings $\{\tau_s\}_{s\in\omega}$, which are defined in stages. At stage $s+1$, given τ_s, we define τ_{s+1} to be a finite extension of τ_s, chosen in such a way that τ_{s+1} being an initial segment of A ensures satisfaction of $\mathcal{P}_s$. When the proof is of this form we are invariably able to conclude that, in fact, the set of reals satisfying the required properties is not only non-empty but is comeager. Given this immediate connection, it is perhaps not surprising that most of the early results describing the first order properties of degrees of typical reals relate to category rather than measure. Generally it is also fair to say that the proofs required for the measure theoretic results tend to be more complicated, and so one might expect that they would appear later in the subject's development.

1.3. **Overview.** The remainder of this paper is organised in three sections. In Section 2 we give precise definitions of typicality in terms of measure and category, and how one can calibrate typicality by refining these definitions in terms of the classical definability hierarchies. We also discuss for which properties of degrees it makes sense to ask our main question (i.e. whether they are typical) and what kind of answers can be expected. We focus on a list of rather basic properties, introduced in Table 1. These properties have received special attention in the study of degrees over the years. In Section 3 we describe all known results concerning which of these properties are typical. We also consider the question as to which properties are inherited by all (non-zero) predecessors of sufficiently typical degrees.

In Section 4 we discuss the similarities and differences between the two faces of typicality that we have considered. After some superficial remarks, we present the recent results of Shore on the theories of the lower cones of typical degrees. Both of these results are motivated by the fact that, fixing a notion of typicality and given two typical degrees, the first order theories of the corresponding lower cones (in the language of partial orders and with the inherited ordering) are equal. The first result determines the level of typicality that is required for this fact to hold (namely, arithmetical genericity or randomness). The second result shows that the two different kinds of typicality give rise to different corresponding theories.

2. Typical degrees and calibration of typicality

2.1. **Large sets and typical reals.** Given a property which refers to a real (or a degree), measure and category arguments may aim to show that there exist reals that satisfy this property by demonstrating that the property is typical. In other words 'most' reals satisfy this property. This approach is based on:

(a) a formalisation of the notion of 'large';
(b) a restriction of the sets of reals that we consider to a countable class;

(c) the definition 'typical real' as a real which occurs in every large set in this restricted class.

Of course, in terms of category 'large' means comeager, while in terms of measure, 'large' means 'of measure 1'. Restricting attention to those reals which belong to every member of a countable class of large sets, still leaves us with a large class:

(i) the intersection of countably many comeager sets is comeager;
(ii) the intersection of countably many measure 1 sets has measure 1.

Those reals which are typical, we call *generic* for the category case, and *random* if working with measure – both of these notions clearly depending on the countable class specified in (b). The 'default' for the generic case is to consider the countable collection of sets which are definable in first order arithmetic, giving 'arithmetical genericity' (or 'arithmetical randomness' accordingly). It is quite common when discussing genericty to suppress the prefix 'arithmetical', so that by 'generic' is often meant arithmetical genericity. Alternative choices are possible, resulting in stronger or weaker genericity and randomness notions. For example, we may consider a randomness notion that is defined with respect to the hyperarithmetical sets – for more information on such strong notions of randomness we refer to [Nie09, Chapter 9].

In the computability context though, it is normally appropriate to consider a finer hierarchy and much weaker randomness and genericity notions. To this end, we define a set of reals C to be *effectively* null, if there is an algorithm which given any rational $\epsilon > 0$ as input, enumerates an open covering of C of measure $< \epsilon$ – to enumerate an open set means to enumerate $W \subset 2^{<\omega}$ which specifies the set, in the sense that its elements are precisely those infinite strings extending members of W. This definition relativises in the obvious way, so that C is effectively null relative to the real A, if the same condition holds when the algorithm has access to an oracle for A. Then we define a real to be A-random if it does not belong to any set which is effectively null relative to A. A simple but crucial observation is that *any* null set is effectively null relative to *some* A, so that once we have observed that a set is of measure 1, there will always be some level of randomness which suffices to ensure membership of the set. Finally, the hierarchy we normally work with is defined as follows: 1-random means $\emptyset$-random, 2-random means $\emptyset'$-random and, more generally, n-random means $\emptyset^{(n-1)}$-random, so that arithmetical randomness is equivalent to being n-random for all $n \in \omega$.

The generic case works similarly. The definition we give here is not the original, but is equivalent to it, and is more commonly used. Given $W \subseteq 2^{<\omega}$, we say that A *forces* W if there exists $\sigma \subset A$ (denoting that σ is an initial segment of A) such that either:

(1) $\sigma \in W$, or;
(2) σ has no extension in W.

So A forces W if it has some initial segment which is already enough to decide whether A has an initial segment in W. The hierarchy is then defined as follows: a real is 1-generic if it forces every W which is c.e. (computably enumerable), and more generally, is n-generic if it forces every W which is c.e. relative to $\emptyset^{(n-1)}$. Analogously to the random case, if a set of reals C is meager then there exists some A such that being 1-generic relative to A ensures failure to belong to C. It is also easily seen that arithmetical genericity is equivalent to being n-generic for all $n \in \omega$.

Finally, the following notions allow us to consider an even finer hierarchy. We say a real is weakly n-random if it does not belong to any Π^0_n null set. We say that a real is weakly

n-generic if it forces every W which is c.e. relative to $\emptyset^{(n-1)}$ and dense (i.e. any finite string has an extension in W).

Thus the (non-collapsing) hierarchies of genericity and randomness provide a calibration of typicality on the continuum, and have become a subject in their own right. Of course, we are interested in answering the following question:

(2.1) Which definable properties are typical of a degree of unsolvability?

Before we embark to the pursuit of this quest, let us examine the answers that are possible. It is reasonable to expect that some structural properties hold for all generic reals while they fail to hold for all random reals? Indeed, as we noted above, while genericity and randomness might be said to formalise the same intuitive notion of typicality, on a technical level they are very different notions. In fact, as we remark in Section 4, the two classes and their Turing degrees form disjoint classes.

2.2. **Properties of degrees and definability.** Is it reasonable to expect that for every property $\mathcal{P}$ which is definable in some sense, either $\mathcal{P}$ is met by all typical reals or else that $\neg\mathcal{P}$ is met by all typical reals? We have already discussed the fact that there are many levels of typicality that one may consider, for both randomness and genericity. Let $\mathcal{T}$ be a certain level of genericity or randomness. In the case that $\mathcal{P}$ is met by all reals in $\mathcal{T}$ or $\neg\mathcal{P}$ is met or by all reals in $\mathcal{T}$ we say that $\mathcal{P}$ is decided by $\mathcal{T}$. Hence we are essentially asking whether every definable property $\mathcal{P}$ is decided by some typicality level $\mathcal{T}$. At this point let us consider the cases of randomness and genericity separately. For the case of randomness, Kolmogorov's 0-1 law states that any (Lebesgue) measurable tailset[1] is either of measure 0 or 1. We may identify a set of Turing degrees with the set of reals contained in those degrees (i.e. with the union). In this sense, every set of Turing degrees is a tailset. Hence any measurable set of Turing degrees must either be of measure 0 or 1. So if we restrict question (2.1) to arithmetically definable properties of degrees $\mathcal{P}$ (i.e. sets of degrees definable in the structure, and for which the union is arithmetically definable as a set of reals), then the satisfying class is a set of measure 0 or 1, so either all arithmetically random degrees $\mathbf{x}$ satisfy $\neg\mathcal{P}$ or all arithmetically random degrees satisfy $\mathcal{P}$ (respectively).

More generally we could restrict question (2.1) to Borel properties $\mathcal{P}$, and the same considerations would hold. Ultimately, we would like to consider all properties $\mathcal{P}$ which are definable in the structure of the Turing degrees, in the first order language of partial orders. As was demonstrated in [BDL12, Section 3], however, whether all definable sets of degrees are measurable is independent from ZFC. Hence in this more general form, question (2.1) may not always admit a clear answer. In our discussions on randomness, most of the properties $\mathcal{P}$ that we consider are arithmetically definable. In the cases where $\mathcal{P}$ is not evidently arithmetically definable (see the cupping property in Table 1) we show that (arithmetical) randomness suffices in order to decide this property.

In the case of genericity we can consider the topological 0-1 law which says that tailsets satisfying the property of Baire[2] are either meager or comeager. Since all Borel sets of reals have the property of Baire, if we restrict question (2.1) to Borel properties $\mathcal{P}$ we can expect a definite answer. In other words, in this case there is a level of genericity that decides $\mathcal{P}$. In the case where $\mathcal{P}$ is restricted to the arithmetically definable properties, we can expect that n-genericity for some $n \in \omega$ decides $\mathcal{P}$. We may also want to consider the more general case where $\mathcal{P}$ is definable in the structure of the degrees (again, in the first-order language

[1] A set of reals C is a tailset if for every real A and every $\sigma \in 2^{<\omega}$, $\sigma * A \in C$ iff $A \in C$ (where $*$ denotes concatenation).

[2] A set of reals has the property of Baire if its symmetric difference from some open set is meager.

of partial orders). Unfortunately, it was observed in [BDL12, Section 3] that in this case we cannot expect a definite answer to question (2.1). Indeed, it is independent of ZFC whether all definable sets of degrees are either meager or comeager. On the other hand, it is well known that under the axiom of determinacy AD every set of reals has the property of Baire. Hence in $ZF + AD$ every property $\mathcal{P}$ is decided by some level of genericity.

In conclusion, our project consists of considering various structural properties $\mathcal{P}$ of the Turing degrees and establishing a level of randomness or genericity that decides $\mathcal{P}$. If $\mathcal{P}$ is Borel then we can expect this task to have a clear solution. Otherwise it is possible that none of the standard levels of randomness or genericity decides $\mathcal{P}$. The properties that we consider are in a certain sense simple, although not always (obviously) Borel. Moreover, according to the known results, they all have a typicality level that decides them, and this is often much lower than the level that is guaranteed by their complexity. This latter observation is evident from an inspection of Tables 1 and 2.

2.3. **Very basic questions remain open.** Our search for the properties of the typical degree starts by considering a small collection of properties which can be considered as 'natural' in the sense that they are encountered in most considerations in classical degree theory. Let us start by noting that there are very simple properties, like density, for which it is unknown whether they are typical. This may come as a surprise to the reader, given that the study of the degrees of unsolvability dates back to [KP54].

Question 1. *Are the random degrees dense?*

Formally, is it true that given any two sufficiently random degrees $\mathbf{a} < \mathbf{b}$, the interval $(\mathbf{a}, \mathbf{b})$ is nonempty? A variation of this question can be stated in terms of a property of a single degree: is it true that for any sufficiently random degree $\mathbf{b}$ and any $\mathbf{a} < \mathbf{b}$ the interval $(\mathbf{a}, \mathbf{b})$ is nonempty? In other words (see the relevant entry in Table 1) we ask whether every random degree fails to be a minimal cover. This property can be expressed with 5 alternating quantifiers in arithmetic, so we can expect that either every 5-random real satisfies it or every 5-random real fails to satisfy it. In particular, arithmetical randomness decides this property. Ultimately, this question can be expressed without reference to random degrees:

Question 2. *What is the measure of minimal covers?*

In Section 3 we are going to see that in terms of genericity, typical degrees are minimal covers. On the other hand it is a well known fact that no typical degree is minimal, both in terms of measure and category. Other basic properties that we consider are displayed in Table 1, along with their formal definitions. All of these properties have straightforward first order definitions in arithmetic, except for the cupping property and the property of having a (strong) minimal cover. However, as we discuss in the following, these latter properties are decided by arithmetical randomness (indeed, 2-randomness). In Section 3 we give a full account of the known status of the properties in Table 1 (i.e. whether they are typical). In general, more is known for genericity than randomness.

3. Properties of the typical degrees and their predecessors

3.1. **Some properties of the typical degrees.** We summarise the status of the properties of Table 1 with respect to the typical degrees in the validity columns of Table 2. Especially in the case of randomness, there are some notable gaps in our knowledge, including the property of being a minimal cover that was discussed in Section 2.3. These gaps are indicated with a question mark in the corresponding entries of the validity columns of Table 2.

Join	$\forall \mathbf{x} < \mathbf{a}\ \exists \mathbf{y} < \mathbf{a}\ (\mathbf{x} \neq \mathbf{0} \to (\mathbf{x} \vee \mathbf{y} = \mathbf{a}))$
Meet	$\forall \mathbf{x} < \mathbf{a}\ \exists \mathbf{y} < \mathbf{a}\ ((\mathbf{y} \neq \mathbf{0})\ \&\ \mathbf{x} \wedge \mathbf{y} = \mathbf{0})$
Cupping	$\forall \mathbf{y} > \mathbf{a}\ \exists \mathbf{x} < \mathbf{y}\ (\mathbf{a} \vee \mathbf{x} = \mathbf{y})$
Complementation	$\forall \mathbf{x} < \mathbf{a}\ \exists \mathbf{y} < \mathbf{a}\ (\mathbf{x} \neq \mathbf{0} \to (\mathbf{x} \wedge \mathbf{y} = \mathbf{0}\ \&\ \mathbf{x} \vee \mathbf{y} = \mathbf{a}))$
Top of a diamond	$\exists \mathbf{x}\ \exists \mathbf{y}\ (\mathbf{x} \neq \mathbf{0}\ \&\ \mathbf{x} \vee \mathbf{y} = \mathbf{a}\ \&\ \mathbf{x} \wedge \mathbf{y} = \mathbf{0})$
Being a minimal degree	$\mathbf{a} > \mathbf{0}\ \&\ (\mathbf{0}, \mathbf{a}) = \emptyset$
Bounding a minimal degree	$\exists \mathbf{x} \leq \mathbf{a}\ [\mathbf{x} > \mathbf{0}\ \&\ (\mathbf{0}, \mathbf{x}) = \emptyset]$
Being a minimal cover	$\exists \mathbf{x} < \mathbf{a}\ [(\mathbf{x}, \mathbf{a}) = \emptyset]$
Having a minimal cover	$\exists \mathbf{y} > \mathbf{a}\ [(\mathbf{a}, \mathbf{y}) = \emptyset]$
Being a strong minimal cover	$\exists \mathbf{x} < \mathbf{a}\ [(\mathbf{0}, \mathbf{a}) = (\mathbf{0}, \mathbf{x}]]$
Having a strong minimal cover	$\exists \mathbf{y} > \mathbf{a}\ [(\mathbf{0}, \mathbf{a}] = (\mathbf{0}, \mathbf{y})]$

TABLE 1. Properties of a degree **a**.

There are many more open problems here, however, than just those indicated by question marks. In fact, there are open questions associated with almost every row of the table. This is because, as discussed previously, our project does not end upon deciding whether a property is typical. We are also interested in determining the exact levels of the typicality hierarchies which suffice to decide each property. This information is displayed in the columns 'Level' and 'Fails' of Table 2. Here '2' means 2-genericity or 2-randomness, and similarly for '1'. Also 'w2' denotes weak 2-randomness or weak 2-genericity, and similarly for 'w1'. The level of the typicality hierarchy that is indicated under column 'Level' is the lowest level of the hierarchy which is known to decide the corresponding property. Similarly, the level of the typicality hierarchy that is indicated under column 'Fails' is the highest level where reals have been found that give opposite answers to the validity of the corresponding property – or simply the highest level at which it is known that the property can fail, in the case that we do not know whether typical degrees satisfy the property. An optimal result has been achieved in the cases where the two levels are consecutive. All of the other cases can be seen as open problems.

As an example, let us examine the join property. By [BDL12] all 2-random degrees satisfy the join property. On the other hand, in [Lew12] it was shown that all low 1-random degrees fail to satisfy the join property. We do not know the answer with respect to the weakly 2-random degrees. In terms of genericity, it was shown in [Joc80] that all 2-generic degrees satisfy the join property. This was extended in [BDL12] (via a different argument) to all 1-generic degrees. On the other hand, by [Kur81, Kur83] every hyperimmune degree[3] contains a weakly 1-generic set. Hence the join property fails for some weakly 1-generic degrees.

Another example is the property of being the join of a minimal pair of degrees. We express this fact by saying that the degree is the 'top of a diamond'. It is a rather straightforward observation that every 1-generic degree is the top of a diamond. Such basic facts

[3]A degree is called hyperimmune if it computes a function which is not dominated by any computable function.

Properties	Generic			Random		
	Validity	Level	Fails	Validity	Level	Fails
Join	✓	**1**	**w1**	✓	**2**	**1**
Meet	✓	**2**	**w1**	**?**	**?**	**w1**
Cupping	✓	**w2**	**1**	✗	**2**	**w2**
Complementation	✓	**2**	**w1**	**?**	**?**	**1**
Top of a diamond	✓	**1**	**w1**	✓	**w2**	**w1**
Being a minimal degree	✗	**1**	**w1**	✗	**1**	**w1**
Bounding a minimal degree	✗	**2**	**1**	✗	**2**	**w2**
Being a minimal cover	✓	**2**	**w1**	**?**	**?**	**1**
Having a minimal cover	✓	**0**	**-**	✓	**0**	**-**
Being a strong minimal cover	✗	**1**	**w1**	✗	**1**	**w1**
Having a strong minimal cover	✗	**w2**	**1**	✓	**2**	**w2**

Table 2. Validity of the properties for a typical degree.

about the generic degrees were established in [Joc80] (an analogous observation, following from van Lambalgen's Theorem, the fact that bases for 1-randomness are Δ^0_2 and the fact that weakly 2-random degrees form a minimal pair with $\mathbf{0}'$, is that every weakly 2-random degree is the top of a diamond). On the other hand, using the fact from [Kur81, Kur83] that every hyperimmune degree contains a set which is weakly 1-generic (and a set which is weakly 1-random), it follows that there are degrees of weakly 1-generic sets (and degrees of weakly 1-random sets) which are not the top of a diamond.

Many of the results in Table 2 regarding genericity were obtained in [Kum90, Kum93a, Kum93b, Kum00], solving a number of questions in [Joc80]. For example, it was shown that every 2-generic degree is a minimal cover. We do not know if there exists a 1-generic degree or a weakly 2-generic degree which is not a minimal cover. Kumabe also showed that every 2-generic degree satisfies the complementation property. It is not known if this can be extended to 1-generic degrees, and the complementation property for random reals is an open problem.

A general theme in the study of typical reals from an algorithmic point of view, is that information introduces order and hence makes reals special and less typical. In other words, a degree that has high information content (e.g. it can compute the halting problem) fails to be typical. More precisely, 1-generic reals are incomplete (i.e. fail to compute the halting problem) and weakly 2-random reals are incomplete. There are a number of results that support this intuition, both in terms of genericity and in terms of randomness. Another more sophisticated example from [Ste06] is that weakly 2-random reals cannot compute a complete extension of Peano arithmetic. Despite these facts, it turns out that there is no bound on the information that joins of typical reals can have. Since we have not found this basic fact in the literature, we present it here.

Properties	**Bounded by a generic**		**Bounded by a random**	
	Validity	Level	Validity	Level
Join	✓	**2**	✓	**2**
Meet	**?**	**?**	**?**	**?**
Cupping	✓	**2**	✗	**2**
Complementation	**?**	**?**	**?**	**?**
Top of a diamond	**?**	**?**	✓	**2**
Being a minimal degree	✗	**2**	✗	**2**
Bounding a minimal degree	✗	**2**	✗	**2**
Being a minimal cover	**?**	**?**	**?**	**?**
Being a strong minimal cover	✗	**2**	✗	**2**
Having a strong minimal cover	✗	**2**	✓	**2**

Table 3. Validity of the properties, for a nonzerodegree which is bounded above by a typical degree.

Theorem 3.1. *Let $\mathcal{V}$ be a null or a meager set of degrees, and let* $\mathbf{d}$ *be a degree. Then there exist degrees* $\mathbf{x}, \mathbf{y}$ *which are not in* $\mathcal{V}$ *and* $\mathbf{d} < \mathbf{x} \vee \mathbf{y}$.

Proof. Let D, X, Y denote representatives of the degrees $\mathbf{d}, \mathbf{x}, \mathbf{y}$ respectively. Since there are no maximal degrees, it suffices to show that $\mathbf{d} \leq \mathbf{x} \vee \mathbf{y}$. Let V be the union of the degrees in $\mathcal{V}$. Suppose that $\mathcal{V}$ is null, so that V is a null set of reals. Then there exists a closed set P of positive measure that is disjoint from V. It is a basic fact from measure theory that P contains a real Z which has an indifferent set of digits (s_i) with respect to P, in the sense that any modification in the bits of Z on positions s_i results in a real which is in P. Algorithmic refinements of this fact were studied in [FMN09]. Without loss of generality let us assume that (s_i) is increasing. For each $t \in \omega$ which is not a term of (s_i) define $X(t) = Y(t) = Z(t)$. Moreover for each i let $X(s_i) = D(i)$ and $Y(s_i) = 1 - D(i)$. Then clearly $X \oplus Y$ can compute D.

The case when $\mathcal{V}$ is meager is similar, based on the fact that every comeager set of reals contains a real which is indifferent in P with respect to a sequence of positions. Algorithmic versions of this fact were studied in [Day13]. □

3.2. **Properties of typical degrees are inherited by the non-zero degrees they compute.** Is it reasonable to expect that the properties of typical degrees also hold for the non-zero degrees that they bound? This is equivalent to the expectation that the upward closure of any 'small' class of degrees that does not contain $\mathbf{0}$ is 'small'. This is not true in general, although the upper cone of degrees above any given non-zerodegree is both meager and null. Martin's category theorem (from [Mar67], see [Joc80]) says that if C is a meager downward closed set of degrees then the upward closure of $C - \{\mathbf{0}\}$ is meager. On the other hand if a meager class C is not downward closed then the upward closure of $C - \{\mathbf{0}\}$ can be comeager (see [Joc80]). An application of Martin's category theorem was that the set of degrees that bound minimal degrees is meager. An analogous result in terms of measure was shown in [Par77]. In particular, it was shown that if $\mathcal{C}$ is the null class of minimal degrees, then the upward closure of $\mathcal{C}$ is also null. However there exist downward closed null classes $\mathcal{C}$ of degrees such that $\mathcal{C} - \{\mathbf{0}\}$ has measure 1. For example, let $\mathcal{C}$ consist of the degrees bounded by 1-generics. Then $\mathcal{C}$ is null and by [Kur81] its upward closure has measure 1. Hence the exact analogue of Martin's theorem in terms of measure is not true. We are yet to find a counterexample, however, which is 'naturally' definable (as a subset of the Turing degrees, rather than as a set of reals definable in arithmetic), and we do not know if there is some analogue of Martin's category theorem in terms of measure. Let us express this problem in terms of the following, somewhat vague, question:

Question 3. *Which null classes of degrees have null upward closure?*

Our discussion shows that whether the non-zeropredecessors of a typical degree **a** inherit a property of **a** depends on the type of the property in question. However many of the basic degree theoretic properties that are known to hold for generic degrees are also known to hold for the nonzero degrees that are bounded by generic degrees. A number of results in [Joc80] follow this heuristic principle. For example, the cupping and join properties are satisfied by all non-zero degrees bounded by any 2-generic degree. Curiously enough, the same phenomenon is common in the random degrees. For example, the join property is shared by all non-zero degrees that are bounded by a 2-random degree, and the cupping property fails for all such degrees. This observation also relates to many of the arguments that are used to obtain these results. In [BDL12] we presented a methodology for examining whether a property is typical of a random degree, which extends to a methodology for examining whether the property is typical of a nonzero degree which is bounded by a random degree. Usually, the latter argument tends to be more involved than the former, but both rest on the same ideas. This methodology rests on the work of other authors, for example [Par77] where it was shown that sufficiently random degrees do not bound minimal degrees. However it is refined considerably, which allows to obtain more precise classifications like the fact that 2-random degrees do not bound minimal degrees. Such results are often shown to be tight by providing counter-examples for lower levels of typicality. For example, it was shown that there are weakly 2-random degrees that bound minimal degrees.

In Table 3 we display the properties that are known to hold for all non-zero degrees that are bounded by typical degrees. The reader may observe that most of the properties of Table 2 that hold for typical degrees also hold for the non-zero degrees that are bounded by typical degrees (in fact we do not have natural counter-examples). For example, 2-random degrees all have strong minimal covers, as do all non-zero degrees that are bounded by 2-randoms. The fact that all non-zero degrees bounded by 2-randoms satisfy join, however, implies that there are no strong minimal covers below any 2-random degree. Weak 2-randoms do sometimes bound strong minimal covers, since they sometimes bound minimal degrees. These results are from [BDL12]. In the same paper it was shown that every degree that is bounded by a 2-random degree is the top of a diamond. This result can be seen as a weak version of the complementation property. The latter as well as the meet property are open problems, as we indicate in Tables 2 and 3.

In terms of genericity, we do not know whether complementation is satisfied by all non-zero degrees bounded by 2-generics, or even whether such degrees will always satisfy the meet property (although the latter may not be a difficult problem). We do not know whether every non-zero degree that is bounded by a 2-generic degree is a minimal cover.

4. Genericity and randomness

A discussion comparing randomness and genericity can be found in [DH10, Section 8.20]. As reals, generics and randoms are certainly very different. For example, their Turing degrees form disjoint classes. In fact, no 1-random real is computable in a 1-generic. Furthermore, every generic degree forms a minimal pair with every sufficiently random degree, and this already happens on the level of 2-generics and 2-randoms (see [NST05]). A result which reveals additional relations between the two notions, was proved in [BDL12]. It was shown that every nonzerodegree that is bounded by a 2-random degree is the join of a minimal pair of 1-generic degrees.

An inspection of Table 2 shows that all of the properties that we have considered are decided by level 2 of the genericity hierarchy. Moreover, some of them are even decided earlier, for example the property of having a strong minimal cover which does not hold for any weakly 2-generic degree (see [BDL12, Section 8.1]) but there are some 1-generic degrees which satisfy it (by [Kum00]). For the case of random degrees, there are some unknown cases, but most of the properties that we consider are also decided on the second level of the hierarchy of randomness.

The following question therefore comes into focus. Is it possible that there is a finite level H_n of the hierarchy which is sufficient to decide all sentences φ for the lower cone, in the sense that $\forall \mathbf{x} \in H_n$, $\mathcal{D}(\leq \mathbf{x}) \models \varphi$ or $\forall \mathbf{x} \in H_n$, $\mathcal{D}(\leq \mathbf{x}) \models \neg\varphi$? This question was raised in an early draft of [BDL12, Section 12] and for the case of generic degrees it was independently raised by Jockusch much earlier (personal communication with Richard Shore). It was recently answered in the negative.

Theorem 4.1 ([Sho12])**.** *There are sentences φ_n such that, for $n > 2$, $\mathcal{D}(\leq \mathbf{x}) \models \varphi_n$ for every $(n+1)$-generic or $(n+1)$-random $\mathbf{x}$ but such that $\mathcal{D}(\leq \mathbf{x}) \not\models \varphi_n$ for some n-generics and n-randoms.*

We note that the sentences φ_n are the same for the generic and the random case. Moreover, they are obtained via a process of interpreting arithmetic inside the corresponding degree structures. As a result of this, they are not considered 'natural' and one may consider the task of discovering familiar properties which separate at least the first few levels of the randomness and genericity hierarchies.

Another issue that was raised in [BDL12, Section 12] was whether the theory below an arithmetically generic degree is the same as the theory below an arithmetically random degree. This question makes sense, since given any two arithmetically generic degrees $\mathbf{x}, \mathbf{y}$, the theories of the structures $\mathcal{D}(\leq \mathbf{x})$ and $\mathcal{D}(\leq \mathbf{y})$ are equal. Similarly the theories of the lower cone for any two arithmetically random degrees are equal. An inspection of Table 2 shows that the only properties there where the generic and random degrees differ, are the cupping property and the property of having a strong minimal cover. These, however, are not properties pertaining to the lower cone, so they do not provide an answer to our question. An answer was given in [Sho12], again via the methodology of coding:

Theorem 4.2 ([Sho12])**.** *There is a sentence φ such that $\mathcal{D}(\leq \mathbf{x}) \models \varphi$ for every 3-random degree $\mathbf{x}$ and $\mathcal{D}(\leq \mathbf{x}) \models \neg\varphi$ for every 3-generic degree $\mathbf{x}$.*

We note that, as with the case of Theorem 4.1, the question remains as to whether one can find 'natural' examples of sentences φ which separate the theories of the lower cones below arithmetically random and arithmetically generic degrees. Table 2 points to some candidates, for example the complementation and meet properties, and the property of being a minimal cover.

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