THE TYPICAL TURING DEGREE

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Abstract. The Turing degree of a real measures the computational difficulty of producing its binary expansion. Since Turing degrees are tailsets, it follows from Kolmogorov's 0-1 law that for any property which may or may not be satisfied by any given Turing degree, the satisfying class will either be of Lebesgue measure 0 or 1, so long as it is measurable. So either the *typical* degree satisfies the property, or else the typical degree satisfies its negation. Further, there is then some level of randomness sufficient to ensure typicality in this regard. We describe and prove a large number of results in a new programme of research which aims to establish the (order theoretically) definable properties of the typical Turing degree, and the level of randomness required in order to guarantee typicality.

A similar analysis can be made in terms of Baire category, where a standard form of genericity now plays the role that randomness plays in the context of measure. This case has been fairly extensively examined in the previous literature. We analyse how our new results for the measure theoretic case contrast with existing results for Baire category, and also provide some new results for the category theoretic analysis.

Contents

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1. Introduction

The inspiration for the line of research which led to this paper begins essentially with Kolmogorov's 0-1 law, which states that any (Lebesgue) measurable tailset is either of measure 0 or 1. The importance of this law for computability theory then stems from the fact that Turing degrees[1] are clearly tailsets—adding on or taking away any finite initial segment does not change the difficulty of producing a given infinite sequence. Upon considering properties which may or may not be satisfied by any given Turing degree, we can immediately conclude that, so long as the satisfying class is measurable[2], it must either be of measure 0 or 1. Thus either the *typical degree* satisfies the property, or else the typical degree satisfies its negation, and this suggests an obvious line of research. Initially we might concentrate on definable properties, where by a definable set of Turing degrees we mean a set which is definable as a subset of the structure in the (first order) language of partial orders. For each such property we can look to establish whether the typical degree satisfies the property, or whether it satisfies the negation. In fact we can do a little better than this. If a set is of measure 1, then there is some level of algorithmic randomness[3] which suffices to ensure membership of the set. Thus, once we have established that the typical degree satisfies a certain property, we may also look to establish the level of randomness required in order to ensure typicality as far as the given property is concerned.

Lebesgue measure though, is not the only way in which we can gauge typicality. One may also think in terms of Baire category. For each definable property, we may ask whether or not the satisfying class is comeager and, just as in the case for measure, it is possible to talk in terms of a hierarchy which allows us to specify levels of typicality. The role that was played by randomness in the context of measure, is now played by a very standard form of genericity. For any given comeager set, we can look to establish the level of genericity which is required to ensure typicality in this regard.

1.1. A heuristic principle.

During our research, we have isolated the following heuristic principle: *if a property holds for all highly random/generic degrees then it is likely to hold for all non-zero degrees that are bounded by a highly random/generic degree*. Here by 'highly random/generic' we mean at least 2-random/generic.[4] Thus, establishing levels of typicality which suffice to ensure satisfaction of a given property, also gives a way of producing lower cones and sets of degrees which are downward closed (at least amongst the non-zero degrees), such that all of the degrees in the set satisfy the given property. For example, by a simple analysis of a theorem

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[1] The Turing degrees were introduced by Kleene and Post in [KP54] and are a measure of the incomputability of an infinite sequence. For an introduction we refer the reader to [Odi89] and [Coo04].

[2] By the measure of a set of Turing degrees is meant the measure of its union.

[3] The basic notions from algorithmic randomness will be described in Section 2. For an introduction we refer the reader to [Nie09] and [DH10].

[4] The relevant forms of randomness, genericity and the corresponding hierarchies will be defined in section 2.

of Martin [Mar67], Kautz [Kau91] showed that every 2-random degree is hyperimmune.[5] In fact, this is just a special case of (1.1).

(1.1) Every non-zero degree that is bounded by a 2-random degree is hyperimmune.

We may deduce (1.1) from certain facts that involve notions from algorithmic randomness. Fixing a universal prefix-free machine, we let Ω denote the halting probability. A set A is called *low for* Ω, if Ω is 1-random relative to A. By [Nie09, Theorem 8.1.18] every non-zero low for Ω degree is hyperimmune. Since every 2-random real is low for Ω (a consequence of van Lambalgen's theorem, see [Nie09, Theorem 3.4.6]) we have (1.1).

In this paper we will give several other examples that support this heuristic principle. Moreover, in Section 5 we give an explanation of the fact that it holds for the measure theoretic case, by showing how to translate standard arguments which prove that a property holds for all highly random degrees, into arguments that prove that the same property holds for all non-zero degrees that are bounded by a highly random degree. The heuristic principle often fails for notions of randomness that are weaker than 2-randomness and we provide a number of counterexamples throughout this paper. It is well known that the hyperimmunity example above fails for weak 2-randomness. However Martin's proof in [Mar67] actually shows that every Demuth random degree is hyperimmune. We shall give examples concerning minimality, the cupping property and the join property, which also demonstrate the principle for highly generic degrees.

1.2. The history of measure and category arguments in the Turing degrees.

Measure and Baire category arguments in degree theory are as old as the subject itself. For example, Kleene and Post [KP54] used arguments that resemble the Baire category theorem construction in order to build Turing degrees with certain basic properties. Moreover de Leeuw, Moore, Shannon and Shapiro [dLMSS55] used a so-called 'majority vote argument' in order to show that if a subset of ω can be enumerated relative to every set in a class of positive measure then it has an unrelativised computable enumeration. A highly influential yet unpublished manuscript by Martin [Mar67] showed that more advanced degree-theoretic results are possible using these classical methods. By that time degree theory was evolving into a highly sophisticated subject and the point of this paper was largely that category and measure can be used in order to obtain advanced results, which go well beyond the basic methods of [KP54]. Of the two results in [Mar67] the first was that the Turing upward closure of a meager set of degrees that is downward closed amongst the non-zero degrees, but which does not contain $\mathbf{0}$, is meager (see [Odi89, Section V.3] for a concise proof of this). Given that the minimal degrees form a meager class, an immediate corollary of this was the fact that there are non-zero degrees that do not bound minimal degrees. The second result was that the measure of the hyperimmune degrees is 1. Martin's paper was the main inspiration for much of the work that followed in this topic, including [Yat76], [Par77] and [Joc80].

Martin's early work seemed to provide some hope that measure and category arguments could provide a simple alternative to conventional degree-theoretic constructions which are often very complex. This school of thought received a serious

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[5] A degree is hyperimmune if it contains a function $f : \omega \to \omega$ which is not dominated by any computable function, i.e. such that for any computable function $g : \omega \to \omega$ there exist infinitely many n with $f(n) > g(n)$. If a degree is not hyperimmune then we say it is hyperimmune-free.

blow, however, with [Par77]. Paris answered positively a question of Martin which asked if the analogue of his category result in [Mar67] holds for measure: are the degrees that do not bound minimal degrees of measure 1? Paris' proof was considerably more involved than the measure construction in [Mar67] and seemed to require sophisticated new ideas. The proposal of category methods as a simple alternative to 'traditional' degree theory had a similar fate. Yates [Yat76] started working on a new approach to degree theory that was based on category arguments and was even writing a book on this topic. Unfortunately the merits of his approach were not appreciated at the time (largely due to the heavy notation that he used) and he gave up research on the subject altogether.

Yates' work in [Yat76] deserves a few more words, however, especially since it anticipated much of the work in [Joc80]. Inspired by [Mar67], Yates started a systematic study of degrees in the light of category methods. A key feature in this work was an explicit interest in the level of effectivity possible in the various category constructions and the translation of this level of effectivity into category concepts (like '$\mathbf{0}'$-comeager' etc.). Using his own notation and terminology, he studied the level of genericity that is sufficient in order to guarantee that a set belongs to certain degree-theoretic comeager classes, thus essentially defining various classes of genericity already in 1974. He analysed Martin's proof that the Turing upper closure of a meager class which is downward closed amongst the non-zero degrees but which does not contain $\mathbf{0}$ is meager, for example (see [Yat76, Section 5]), and concluded that no 2-generic degree bounds a minimal degree. Moreover, he conjectured (see [Yat76, Section 6]) that there is a 1-generic that bounds a minimal degree. These concerns occurred later in a more appealing form in Jockusch [Joc80], where simpler terminology was used and the hierarchy of n-genericity was explicitly defined and studied.

With Jockusch [Joc80], the heavy notation of Yates was dropped and a clear and systematic calibration of effective comeager classes (mainly the hierarchy of n-generic sets) and their Turing degrees was carried out. A number of interesting results were presented along with a long list of questions that set a new direction for future research. The latter was followed up by Kumabe [Kum90, Kum91, Kum93a, Kum93b, Kum00] (as well as other authors, e.g. [CD90]) who answered a considerable number of these questions.

The developments in the measure approach to degree theory were similar but considerably slower, at least in the beginning. Kurtz's thesis [Kur81] is probably the first systematic study of the Turing degrees of the members of effectively large classes of reals, in the sense of measure. Moreover the general methodology and the types of questions that Kurtz considers are entirely analogous to the ones proposed in [Joc80] for the category approach (e.g. studying the degrees of the n-random reals as opposed to the n-generic reals, minimality, computable enumerability and so on). Kučera [Kuč85] focused on the degrees of 1-random reals. Kautz [Kau91] continued in the direction of [Kur81] but it was not until the last ten years (and in particular with the writing of [DH10, Chapter 8]) that the study of the degrees of n-random reals became well known and this topic became a focused research area.

The programme of research undertaken in the present paper can be seen as something new, in the sense that this is the first attempt at a systematic analysis of the *order theoretically* definable properties satisfied by the typical Turing degree, where typicality is gauged in terms of measure (although some previous results do exist, such as those of Sacks and Paris concerning minimality and the bounding of minimal degrees).

2. Technical background, notation and terminology

We let 2^ω denote the set of infinite binary sequences and denote the standard Lebesgue measure on 2^ω by μ. We let $2^{<\omega}$ denote the set of finite binary strings. We use the variables $c, d, e, i, j, k, \ell, m, n, p, q, s, t$ to range over ω; f, g to range over functions $\omega \to \omega$; $\alpha, \beta, \sigma, \tau, \eta, \rho$ to range over $2^{<\omega}$; A, B, C, D, X, Y, Z to range over 2^ω; we use J, S, T, U, V, W to range over subsets of $2^{<\omega}$ and we use F, G, P and Q to range over subsets of 2^ω. We shall also use the variable P to range over the various definable degree theoretic properties. In the standard way we identify subsets of ω and their characteristic functions.

2.1. **Turing functionals, Cantor space, strings and functions.**

For $\sigma \in 2^{<\omega}$ and $A \in 2^\omega$ we write $\sigma * A$ to denote the concatenation of σ and A, and we say that $P \subseteq 2^\omega$ is a tailset if, for all $\sigma \in 2^{<\omega}$ and all $A \in 2^\omega$, $\sigma * A \in P$ if and only if $A \in P$. A set $V \subseteq 2^{<\omega}$ is said to be *downward closed* if, whenever $\tau \in V$, all initial segments of τ are in this set, and is said to be *upward closed* if, whenever $\tau \in V$, all extensions of τ are in this set. We write $[\![V]\!]$ to denote the set of infinite strings which extend some element of V, and we write $\mu(V)$ to denote $\mu([\![V]\!])$. If $V = \{\tau\}$ then we write $[\tau]$ to denote $[\![V]\!]$.

We use the variables Φ, Ψ, Θ and Ξ to range over the Turing functionals, and let Ψ_i be the ith Turing functional in some fixed effective listing of all Turing functionals. Then $\Psi_i^\sigma(n)$ denotes the output of Ψ_i given oracle input σ on argument n. We make the assumption that $\Psi_i^\sigma(n)\uparrow$ unless the computation converges in $< |\sigma|$ steps and $\Psi_i^\sigma(n')\downarrow$ for all $n' < n$ (these assumption are also made for any *given* Turing functional Φ, but we do not worry about adhering to these conventions when constructing Turing functionals). Letting $\langle i, j\rangle$ be a computable bijection $\omega \times \omega \to \omega$, we write $\omega^{[e]}$ to denote the set of all numbers of the form $\langle e, j\rangle$ for some $j \in \omega$.

To help with readability, we shall generally make some effort to maintain a certain structure in our use of variables. In situations in which we consider the actions of a functional, we shall normally use the variables X and τ for sequences and strings in the domain, and the variables Y and σ for sequences and strings in the image. When another functional then acts on the image space, we shall generally use the variables Z and η for sequences and strings in the second image space. The variables X, Y and Z will generally be used in situations in which we are simultaneously dealing with all sets of natural numbers. When a specific set is given for a construction, or has to be built by a construction, then we will use the variables A, B, C and D.

2.2. **Randomness and Martin-Löf tests.**

If each V_i is a set of finite binary strings and the sequence $\{V_i\}_{i\in\omega}$ is uniformly computably enumerable (c.e.), i.e. the set of all pairs (i, τ) such that $\tau \in V_i$ is c.e., then we say that this sequence is a Martin-Löf test if $\mu(V_i) < 2^{-i}$ for all i. Then we say that X is Martin-Löf random if there doesn't exist any Martin-Löf test such that $X \in \bigcap_i [\![V_i]\!]$. It is not difficult to show that there exists a *universal* Martin-Löf test, i.e. a Martin-Löf test $\{V_i\}_{i\in\omega}$ such that X is Martin-Löf random if and only if $X \notin \bigcap_i [\![V_i]\!]$.

These notions easily relativize. We say that $\{V_i\}_{i\in\omega}$ is a Martin-Löf test relative to X if it satisfies the definition of a Martin-Löf test, except that now the sequence need only be uniformly c.e. relative to X. Now Y is Martin-Löf random relative to X if there does not exist any Martin-Löf test relative to X such that $Y \in \bigcap_i [\![V_i]\!]$. Once again, it can be shown that there exists a universal test relative to any oracle, and that, in fact, this universal test can be uniformly enumerated for all oracles. We let $\{U_i\}_{i\in\omega}$ be a uniformly c.e. sequence of operators such that, for any X, $\{U_i^X\}_{i\in\omega}$ is a universal test relative to X. We assume that, for each i and τ, U_i^τ is finite, and is empty unless $|\tau| > i$. We assume furthermore, that the function $\tau \mapsto U_i^\tau$ is computable.

If a subset of Cantor space P is of measure 1, then it is clear that there is some oracle X such that all sets which are Martin-Löf random relative to X belong to P. For $n \geq 1$ we say that X is n-random if it is Martin-Löf random relative to $\mathbf{0}^{(n-1)}$ (and that a degree is n-random if it contains an n-random set). Martin-Löf randomness is in many respects the standard notion of algorithmic randomness. Other randomness notions may be obtained by varying the level of computability in the above definition. For example, a set is weakly 2-random if it is not a member of any Π^0_2 null class. In order to define Demuth randomness, we need to consider the wtt-reducibility. We say $X \leq_{wtt} Y$ if there exists i such that $\Psi_i^X = Y$ and there exists a computable function f such that the use on argument n is bounded by $f(n)$. Let W_i be the ith c.e. set of finite binary strings according to some fixed effective listing of all such sets. We say that X is Demuth random if there is no f which is wtt-reducible to $\emptyset'$, such that $\mu(W_{f(i)}) < 2^{-i}$ and $X \in [\![W_{f(i)}]\!]$ for infinitely many i. Demuth randomness and weak 2-randomness are incomparable notions, both stronger than 1-randomness and weaker than 2-randomness.

2.3. **Turing degrees and hierarchies.**

For basic concepts and techniques in the Turing degrees, like 'minimal degree' and 'hyperimmune-free degree' we refer the reader to [Odi89]. A very good treatment of 'fixed-point-free' (or 'diagonally non-computable') degrees can be found in [Nie09, Section 4.1].

We review the definition of jump classes and the genericity hierarchy which are quite central in degree theory. The generalized jump hierarchy is defined as follows. For $n \geq 1$ a Turing degree is generalized low$_n$ (GL$_n$), if $\mathbf{a}^{(n)} = (\mathbf{a} \vee \mathbf{0}')^{(n-1)}$, and we say that $\mathbf{a}$ is generalized high$_n$ (GH$_n$) if $\mathbf{a}^{(n)} = (\mathbf{a} \vee \mathbf{0}')^{(n)}$. A degree is generalized low if it is GL$_1$ and is generalized high if it is GH$_1$. A degree is low$_n$ if it is GL$_n$ and below $\mathbf{0}'$. A degree is high$_n$ if it is GH$_n$ and below $\mathbf{0}'$. By low is meant low$_1$ and by high is meant high$_1$.[6]

We say that Y is 1-generic relative to X if, for every $W \subseteq 2^{<\omega}$ which is c.e. relative to X:

$$(\exists \sigma \subset Y)[\sigma \in W \ \vee \ (\forall \sigma' \supset \sigma)(\sigma' \notin W)].$$

It is clear that if a set P is comeager then there is some oracle X such that every set which 1- generic relative to X belongs to P. For $n \geq 1$, we say that Y is n-generic if it is 1-generic relative to $\mathbf{0}^{(n-1)}$, and that a degree is n-generic if it contains an n-generic set.

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[6] We note that the definitions of the classes low$_n$ and high$_n$ are not completely standard. In particular, for some authors, $\mathbf{a}$ being low$_n$ simply means that $\mathbf{a}^n = \mathbf{0}^n$, without any requirement that $\mathbf{a} < \mathbf{0}'$ (and similarly for high$_n$).

3. 0-1 LAWS IN CATEGORY AND MEASURE

In the analysis we have considered so far, we have left a gap which we now close. *If* a tailset is measurable then it is either of measure 0 or 1, and there is then some level of randomness that suffices to ensure typicality. If we restrict to considering definable sets of Turing degrees, however (and where by definable we mean definable in the first order language of partial orders), this raises the question, do all such sets have to be measurable? Similarly we may ask, do all such sets have to be either meager or comeager? In this section we make the following two observations, which were hashed out in an email correspondence with Richard Shore and Yu Liang:

(3.1) Whether or not all definable sets of degrees are measurable is independent of ZFC.

(3.2) Whether or not all definable sets of degrees are either meager or comeager is independent of ZFC.

We consider first how to prove 3.1, the proof of 3.2 will be similar. On the one hand, it is known that there is a generic extension of L not collapsing cardinals nor violating CH, in which every set of reals which is definable (with no parameter) is measurable [She84]. On the other hand, we wish to make use of the fact, due to Slaman and Woodin [SW86], that any set of Turing degrees above $\mathbf{0}''$ is definable as a subset of the Turing degrees if and only if its union is definable in second order arithmetic. Initially there might seem a basic obstacle to using this fact. We wish to construct a set which is of outer measure 1 and whose complement is also of outer measure 1. The degrees above $\mathbf{0}''$ are of measure 0, and so any subset will be measurable. It is easy to see, however, that the result of Slaman and Woodin extends to any set of degrees which is invariant under double jump—meaning that if $\mathbf{a}$ belongs to the set, then all $\mathbf{b}$ with $\mathbf{b}'' = \mathbf{a}''$ are also members. Now, it is easy enough to construct a tailset which is of outer measure 1 and whose complement is also of outer measure 1, a result due to Rosenthal [Ros75]. One simply defines the set using a transfinite recursion which diagonalises against the open sets of measure < 1. This recursion uses a well-ordering of the reals (which suffices to specify a well-ordering of the open sets). If we assume V=L then we have a well-ordering of the reals which is definable in second order arithmetic, and the set constructed will be definable in second order arithmetic. Finally we just have to modify the construction so as to make the set constructed invariant under double jump. This means that whenever we enumerate a real into the set or its complement, we also enumerate all reals which double jump to the same degree. Since we still add only countably many reals into either the set or its complement at each stage of the transfinite recursion, the argument still goes through as it did previously.

In order to prove 3.2 we proceed in almost exactly the same way. The first direction is once again given by Shelah in [She84]. For the other direction, in order to show that there exist ZFC models with a definable set of degrees which is neither meager nor comeager, we once again assume V=L, but we consider this time a transfinite recursion which defines a set which does not satisfy the property of Baire (see [Kec95], for example, for the description of such a construction).

4. METHODOLOGY

In this section we discuss a framework for constructions which calculate the measure of a given degree-theoretic class. By (3.1) no methodology can be completely general, and as one moves to consider more complicated properties it is to be expected that more sophisticated techniques will be required. The methodology we shall present here, however, does seem to be very widely applicable. All previously known arguments of this type, and all of the new theorems we present here, fit neatly into the framework. An informal presentation of the framework is given in Section 4.1.

Given a degree-theoretic property P which holds for almost all reals, we consider (oracle-free) constructions which work for all sets simultaneously and which specify a G_δ null set such that every real is either in this set or satisfies P. By examining the oracle required to produce arbitrarily small open coverings of this G_δ set, we establish a level of randomness which is sufficient for a real to satisfy P. In all known examples it turns out that 2-randomness suffices and, moreover, that every non-zero degree that is bounded by a 2-random also satisfies P. A widely applicable methodology for results of the latter type is given in Section 5. In Section 4.2 we give a number of rather basic facts about measure in relation to Turing computations that will be used routinely in most of the proofs in this paper.

Our framework rests on various ideas from [Mar67], [Par77] and [Kur81], but introduces new features (like the use of measure density theorems) which simplify and refine the classic arguments as well as establishing new results in a uniform fashion.

4.1. All sufficiently random degrees.

The strategy for showing that all sufficiently random sets X satisfy a certain degree-theoretic property is as follows:

(a) Translate the property into a countable sequence of requirements $\{R_e\}_{e\in\omega}$ referring to an unspecified set X.
(b) Devise an 'atomic' strategy which takes a number e and a string τ as inputs and satisfies R_e for a certain proportion of extensions X of τ, where this proportion depends on e and not on τ.
(c) Assemble a construction from the atomic strategies in a *standard way*.

Since steps (a) and (b) are specific to the degree-theoretic property that is studied, we are left to give the details of the procedure that produces the construction, given the requirements and the corresponding atomic strategies. Step (c) involves a construction that proceeds in stages and places 'e-markers' (for $e \in \omega$) on various strings in the full binary tree. Each e-marker is associated with a version of the atomic strategy for R_e from step (b), which looks to satisfy R_e on a certain proportion of the extensions of the string τ on which it is placed. Once an e-marker is placed on τ, we shall say that the marker 'sits on' τ until such a point as it is removed. So a marker may be 'placed on' τ at a specific point of the construction, and then at this and all subsequent points of the construction, until such a point as it is removed, the marker is said to 'sit on' τ. The basic rules according to which markers are placed on strings and removed from them are as follows:

(i) At most one marker sits on any string at any given stage.
(ii) If $\tau \subset \tau'$ and at some stage an e-marker sits on τ' and a d-marker sits on τ, then $d \leq e$.

(iii) If a marker is removed from τ at some stage then any marker that sits on any extension of τ is also removed.

Note that (ii) and (iii) indicate an injury argument that is taking place along each path X. A marker is called *permanent* if it is placed on some string and is never subsequently removed. The basic rules above allow the possibility that, for some $e \in \omega$, many (perhaps permanent) e-markers are placed along a single path. This corresponds to multiple attempts to satisfy R_e along the path.

The construction will strive to address each requirement R_e along the 'vast majority' of the paths X of the binary tree. In particular, it will work with an arbitrary parameter $k \in \omega$ and will produce the required objects (like various reductions that are mentioned in the requirements) along with a set of strings W such that $\mu(W) < 2^{-k}$. *Every real that does not have a prefix in W will satisfy all $R_e, e \in \omega$.* Considering all of the constructions as k ranges over ω, we conclude that P is satisfied by every real except for those in a certain null G_δ set. Since this set may be seen as a Martin-Löf test relative to some oracle, we can also establish a level of randomness that is sufficient to guarantee satisfaction of the property. This is directly related to the oracle that is needed for the enumeration of W. In all of our examples an oracle for $\emptyset'$ suffices to enumerate W, and thus 2-randomness is sufficient to ensure satisfaction of P. In most of our examples we will be able to show that any standard weaker notion of randomness (in particular, weak 2-randomness) fails to be sufficient.

The outcome of the construction with respect to a particular real X will be reflected by the permanent markers that are placed on initial segments of X. In particular, one of the following outcomes will occur:

(1) For every $e \in \omega$ there is a permanent e-marker placed on some initial segment of X.
(2) There exists some $e \in \omega$ such that, for each $d \leq e$, a permanent d-marker is placed on an initial segment of X, and such that infinitely many permanent e-markers are placed on initial segments of X.
(3) There are only finitely many permanent markers placed on initial segments of X.

Note that by rule (ii), if outcome (2) occurs with respect to X then for $j > e$ there will be no permanent j-marker placed on any initial segment of X, and for each $d < e$ there will only be finitely many (permanent or non-permanent) d-markers placed on initial segments of X.

The only successful outcome for X is (1). Failure of the construction with respect to X therefore comes in two forms. Outcome (3) denotes a *finitary* failure. In this case the construction gives up placing markers on initial segments of X, due to the request of an individual marker that sits on an initial segment τ of X. Such a marker may forbid the placement of markers on certain extensions of τ (including a prefix of X), while waiting for some Σ^0_1 event.[7] At any stage during the construction, requests to forbid the placement of markers will only be made for a small measure of sets, and so we will be able define a set of strings V of small measure, such that every real for which outcome (3) occurs has an initial segment in V.

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[7] As an example, this event might be the convergence of a computation which, should it be found, would then allow the marker to effect a successful diagonalisation above all those strings where it has previously paused the construction (in effect) by forbidding the placement of markers.

Outcome (2) denotes an *infinitary* failure, in the sense that the construction insists on trying to satisfy a certain requirement R_e with respect to X by placing infinitely many e-markers on initial segments of it, but the requirement remains unsatisfied with respect to X. The possibility of outcome (2) is a direct consequence of (b), which says that the atomic strategy only needs to satisfy the requirement on a fixed (possibly small) proportion of the reals in its neighbourhood (leaving the requirement unsatisfied on many other reals). Reals for which outcome (2) occurs, are those which happen to always be in the unsatisfied part of the neighbourhood that corresponds to each e-marker. The Lebesgue density theorem tells us, however, that the reals for which outcome (2) occurs cannot form a class of positive measure. In particular it tells us that, for almost all reals in this class, the limit density must be 1. The existence of an element of the class for which the limit density is 1 contradicts the fact that (b) insists the requirement be satisfied for a fixed proportion of strings extending that on which the marker is placed. This class therefore has measure 0, and we can consider a set of strings S of arbitrarily small measure which contains a prefix of every real in the class. Then we can simply let W be the union of V and S.

Such constructions will typically be computable, thus constructing Turing reductions dynamically. Hence the reals for which outcome (2) occurs will typically form a Σ^0_3 class and V and S will usually require an oracle for $\emptyset'$ for their enumeration. This is the reason that 2-randomness is required in all of the results that involve this type of construction.

4.2. **Measure theoretic tricks concerning Turing reductions.** Given a Turing functional Ψ, if we are only interested in computations that Ψ performs relative to a 'sufficiently random' (typically a 2-random) oracle, then we can expect certain features from Ψ. This section discusses features which are particularly useful for the arguments employed in this paper. Section 4.2.1 shows that we may assume all infinite binary sequences in the range of Ψ are incomputable. In Section 4.2.2 we describe a basic fact concerning the measure of the splittings which can be expected to exist for such a functional Ψ (a tool that is essential in certain coding arguments, including the one in Section 9). Finally, in Section 4.2.3 we give a Ψ-analogue of the Lebesgue density theorem which will be an essential tool for extending results to nonzero degrees below a 2-random degree.

4.2.1. *Turing procedures on random input.* We start with the following useful fact, which says that each Turing functional Φ can be replaced with one which restricts the domain to sequences X which Φ-map to sets relative to which X is not random.

Lemma 4.1 (Functionals and relative randomness)**.** *For each Turing functional Φ there is a Turing functional Ψ which satisfies the following for all X:*

(a) *If Ψ^X is total then Φ^X is total, $\Psi^X = \Phi^X$ and X is not Ψ^X-random.*[8]
(b) *If Φ^X is total and X is not Φ^X-random then Ψ^X is total.*

Moreover, an index for Ψ can be obtained effectively from an index for Φ.

Proof. We describe how to enumerate axioms for Ψ, given the functional Φ. Let $\{U_i\}_{i\in\omega}$ be a universal oracle test as described in Section 2. At stage s, for each pair of strings τ, $\sigma = \rho * j$ of length $< s$, if i is the least number such that τ does

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[8] By Y-random is meant Martin-Löf random relative to Y.

not extend any string in U_i^ρ then do the following. If $\Phi^\tau \supseteq \sigma$ and τ extends a string in U_i^σ then enumerate the axiom $\langle \tau, \sigma \rangle$ for Ψ (thus defining $\Psi^\tau \supseteq \sigma$).

Clearly Ψ is obtained effectively from Φ. If Ψ^X is total for some oracle X and $\Psi^X = Y$, then Φ^X is also total and equal to Y. We also claim that in this case $X \in U_i^Y$ for each $i \in \omega$. Towards a contradiction suppose that i is the least number such that $X \notin U_i^Y$. If $i > 0$ then let $\tau \subset X$ and $\sigma = \rho * j$ be such that τ does not extend any string in U_{i-1}^ρ, but does extend a string in U_{i-1}^σ, and such that we enumerate the axiom $\langle \tau, \sigma \rangle$. Let s be the stage at which this axiom is enumerated. If $i = 0$ then let $s = 0$. Then, subsequent to stage s we do not enumerate any new axioms of the form $\langle \tau', \sigma' \rangle$ such that $\tau' \subset X$. This gives us the required contradiction and concludes the verification of property (a). For (b) suppose that $\Phi^X = Y$ and that $X \in U_i^Y$ for all $i \in \omega$. Then, since it cannot be the case for any finite string σ that $X \in U_i^\sigma$ for all i (according to the conventions established in Section 2), it follows that Ψ^X is total. □

In most measure arguments in this paper we will use Turing functionals which do not map to computable reals. This will simplify the constructions.

Definition 4.2 (Special Turing functionals)**.** *A Turing functional Ψ is called special if all infinite strings in its range are incomputable.*

The following lemma (when combined with the fact that any non-empty Π_1^0 class containing only 1-randoms contains a member of every 1-random degree) will be used throughout this paper in order to justify the use of special functionals in various arguments which involve given reductions.

Lemma 4.3 (Obtaining special functionals)**.** *Given a Turing functional Φ and a non-empty Π_1^0 class P which contains only 1-random sequences we can effectively obtain a special Turing functional Ψ which satisfies the following conditions for every 2-random set X in P:*

- (i) *If Ψ^X is total then Φ^X is total and $\Psi^X = \Phi^X$.*
- (ii) *If Φ^X is total and incomputable then Ψ^X is total.*

Proof. Let V be a c.e. set of finite strings such that a real is in P if and only if it does not have a prefix in V. Given V and Φ we produce Ψ as in the proof of Lemma 4.1 with the additional clause that whenever a string τ appears in V at some stage of the construction, we stop enumerating axioms for Ψ of the form $\langle \tau', \sigma' \rangle$ such that τ' extends τ.

Let X be a 2-random member of P. Clearly Ψ^X satisfies (a) and (b) of Lemma 4.1. This shows (i) above. For (ii), we need a notion from [Kuč93]: a set is called a *basis for 1-randomness* if there is a set that computes it and is 1-random relative to it. By [HNS07], bases for 1-randomness are Δ_2^0. On the other hand no 2-random set computes an incomputable Δ_2^0 set. Hence 2-random sets do not bound incomputable bases for 1-randomness and (ii) follows from (b) of Lemma 4.1.

Finally we show that Ψ is special. If Ψ^X is total then X must be a member of P. Therefore it is 1-random. By (a) of Lemma 4.1, totality of Ψ^X means that X is not Ψ^X-random. This shows that Ψ^X is incomputable. □

The use of special functionals in what follows is not necessary but it often simplifies the proofs considerably. The simplification comes from the fact that the use of special functionals will often reduce the number of outcomes that a strategy has.

The following fact is applicable in arguments where we show that some property holds for all non-zero degrees below a sufficiently random degree.

Lemma 4.4 (Special functionals for downward density)**.** *Given Turing functionals* Θ, Φ *and a non-empty* Π^0_1 *class* P *which contains only 1-random reals we can effectively produce a special Turing functional* Ψ *which satisfies the following conditions for every 2-random set* X *in* P*:*

(a) *For all* Y*, if* Ψ^Y *is total then it is equal to* Φ^Y*.*
(b) *If* $\Theta^X = Y$ *and* Φ^Y *is total and incomputable then* Ψ^Y *is total.*

Proof. We describe how to enumerate the axioms for Ψ. Let V be an upward closed computable set of strings which contains initial segments of precisely those reals which are not in P. At stage s, for each triple $\tau, \sigma,\ \eta = \rho * j$ such that all strings in the triple are of length $< s$ and such that $\tau \not\in V$, if i is the least number such that τ does not extend a string in U^{ρ}_i then do the following. If $\Theta^\tau = \sigma$, $\Phi^\sigma \supseteq \eta$ and τ extends a string in U^{η}_i then enumerate the axiom $\langle \sigma, \eta \rangle$ for Ψ.

Clearly (a) holds. If Ψ^Y is total then there is some $X \in P$ such that $\Theta^X = Y$, $\Phi^Y = \Psi^Y$ and X is not random relative to Ψ^Y. Hence Ψ^Y is incomputable, and thus Ψ is special. For (b) suppose that $\Theta^X = Y$ for some 2-random X which is in P such that Φ^Y is total and incomputable. Then X is not random relative to Φ^Y because 2-random reals do not compute incomputable bases for 1-randomness. Therefore the construction will define $\Psi^Y = \Phi^Y$. □

4.2.2. *Measure splittings for Turing functionals.* Recall that a Ψ*-splitting* is a pair of strings τ, τ' such that Ψ^τ and $\Psi^{\tau'}$ are incompatible. When we deal with functionals that operate on a random oracle, a measure theoretic version of this notion is useful.

(4.1) Given a set of reals X and a string τ, the τ-measure of X is the measure of the reals in X with prefix τ, multiplied by $2^{|\tau|}$.

Given a Turing functional Ψ, a string τ and a real number ϵ we say that a pair (U, V) of finite sets of strings is a Ψ*-splitting above* τ if:

- the strings in $U \cup V$ all have the same length and extend τ;
- if $\tau_0 \in U$ and $\tau_1 \in V$ then τ_0 and τ_1 are Ψ-splitting.

Moreover, we say that (U, V) has measure ϵ if $\mu(U) = \mu(V) = \epsilon/2$. A rational number is *dyadic* if it has a finite binary expansion. We define:

$$\pi(\Psi, \sigma) = \mu(\{X \mid \Psi^X \supseteq \sigma\}). \tag{4.2}$$

If U is a prefix-free set of strings and Ψ is a functional then we let $\pi(\Psi, U)$ be the sum of all $\pi(\Psi, \sigma)$ for $\sigma \in U$.

Proposition 4.5. *If* Ψ *is a special Turing functional then for each* $c \in \omega$ *and each* σ *there exists* $\ell \in \omega$ *such that* $\pi(\Psi, \sigma')/\pi(\Psi, \sigma) \leq 2^{-c}$ *for all* $\sigma' \supset \sigma$ *of length* ℓ*.*

Proof. For a contradiction, suppose that there exists some $c \in \omega$ such that for each $\ell \in \omega$ we have $\pi(\Psi, \sigma')/\pi(\Psi, \sigma) > 2^{-c}$ for some string $\sigma' \supset \sigma$ of length ℓ. Then by König's lemma there exists an infinite binary sequence Y extending σ such that $\pi(\Psi, Y\restriction_n)/\pi(\Psi, \sigma) > 2^{-c}$ for all $n \in \omega$. For each n there exists a clopen set V_n such that $\mu(V_n)/\pi(\Psi, \sigma) > 2^{-c-1}$, such that all strings in V_n Ψ-map to extensions of $Y\restriction_n$ and such that $V_{n+1} \subseteq V_n$. By compactness it follows that Y is in the range of Ψ. Moreover Y is computed by a set of reals of positive measure, hence it is computable. This contradicts the fact that Ψ is special. □

A basic fact from classical computability theory is that if some oracle X computes an incomputable set via a Turing reduction Ψ then Ψ-splittings are dense along X. In other words, for every initial segment τ of X there exists a Ψ-splitting such that all strings in the splitting extend τ. The measure theoretic version of this fact is as follows.

Lemma 4.6 (Measure splittings for functionals)**.** *Suppose that Ψ is a special Turing functional, ϵ is a dyadic rational and τ is a string. If there does not exist a Ψ-splitting above τ of measure ϵ then there exists a c.e. set V of strings extending τ such that $\mu(V) \leq 2\epsilon$ and every set extending τ on which Ψ is total has a prefix in V. Moreover, given τ, Ψ and ϵ, an oracle for $\emptyset'$ can find whether or not there exists such a splitting and, if there does not then an index for V.*

Proof. Let ℓ be the least number such that $\pi(\Psi, \sigma) \leq \epsilon/2$ for all strings σ of length ℓ. If the measure of all $X \supset \tau$ such that $|\Psi^X| \geq \ell$ is greater than 2ϵ then there exists a Ψ-splitting above τ of measure ϵ. Otherwise we can let V be the c.e. set of strings $\tau' \supset \tau$ such that $|\Psi^{\tau'}| \geq \ell$. Finally note that the above procedure only involves Σ^0_1 questions, and so can be carried out using an oracle for $\emptyset'$. ☐

The following version of Lemma 4.6 is applicable in arguments where we show that some property holds for all non-zero degrees below a sufficiently random degree.

Lemma 4.7 (Measure splittings for downward density)**.** *Suppose that Θ, Ψ are special Turing functionals, ϵ is a rational number and σ is a string. If there does not exist a Ψ-splitting (U, V) above σ such that $\pi(\Theta, U)$ and $\pi(\Theta, V)$ are at least $\epsilon/2$ then there exists a c.e. set V of strings such that $\mu(V) \leq 2\epsilon$ and every set which Θ-maps to an extension of σ on which Ψ is total has a prefix in V. Moreover given σ, Θ, Ψ and ϵ, an oracle for $\emptyset'$ can find whether or not there exists such a splitting and, if there does not then an index for V.*

Proof. Let ℓ be the least number such that $\pi(\Psi \circ \Theta, \eta) \leq \epsilon/2$ for all η of length ℓ. If the measure of all reals which Θ-map to extensions of any $\rho \supset \sigma$ such that $|\Psi^\rho| \geq \ell$ is $> 2\epsilon$ then there exists a Ψ-splitting (U, V) above σ such that $\pi(\Theta, U)$ and $\pi(\Theta, V)$ are at least $\epsilon/2$. Otherwise we can let V be the c.e. set of strings τ such that Θ^τ extends σ which Ψ-maps to a string of length $\geq \ell$. Finally note that we only ask Σ^0_1 questions, so the above can be done computably in $\emptyset'$. ☐

4.2.3. *Measure density for Turing reductions.* The observations in this section are mainly to be applied in the methodology that is described in Section 5.

Lemma 4.8 (Ψ-totality)**.** *Let Ψ be a Turing functional, $c \in \omega$ and let E be a set of tuples (σ, ℓ) such that the strings occurring in the tuples form a prefix-free set and for each $(\sigma, \ell) \in E$:*

$$\mu(\{X \mid \sigma \subseteq \Psi^X \ \wedge \ |\Psi^X| \geq \ell\}) < 2^{-c} \cdot \pi(\Psi, \sigma). \tag{4.3}$$

Then the class of reals X such that a prefix of Ψ^X occurs in some tuple $(\sigma, \ell) \in E$ and $|\Psi^X| \geq \ell$, has measure $< 2^{-c}$.

Proof. For each $(\sigma, \ell) \in E$ consider the set M_σ of reals X such that $\Psi^X \supseteq \sigma$. The sets M_σ are pairwise disjoint. Moreover, the proportion of the reals X in M_σ with $|\Psi^X| \geq \ell$ is $< 2^{-c}$. Therefore the class of reals X such that a prefix of Ψ^X occurs in some tuple $(\sigma, \ell) \in E$ and $|\Psi^X| \geq \ell$, has measure $< 2^{-c}$. ☐

Finally we give an analogue of the Lebesgue density theorem which refers to a Turing functional Θ and a set of strings V. It says that if F consists of the reals X for which Θ^X is total and does not have a prefix in V, then for almost all $X \in F$ the proportion of the reals that Θ-map to $\Theta^X \upharpoonright_n$ which are in F tends to 1 as $n \to \infty$.

Lemma 4.9 (Θ-density)**.** *Suppose Θ is a Turing functional, V is a set of finite strings and let F_V be the set of reals X such that Θ^X is total and does not extend any strings in V. Then:*

$$\lim_n \frac{\mu\{X_1 \in F_V \mid \Theta^{X_1} \supseteq \Theta^{X_0} \upharpoonright_n\}}{\pi(\Theta^{X_0} \upharpoonright_n)} = 1 \quad \text{for almost all } X_0 \in F_V, \tag{4.4}$$

where $\pi(\sigma) = \pi(\Theta, \sigma)$ and 'almost all' means 'all but a set of measure zero'.

Proof. Without loss of generality we may assume that V is prefix-free. For each $\epsilon \in (0,1)$ define:

$$G_\epsilon = \{X_0 \in F_V \mid \liminf_n \frac{\mu\{X_1 \in F_V \mid \Theta^{X_1} \supseteq \Theta^{X_0} \upharpoonright_n\}}{\pi(\Theta^{X_0} \upharpoonright_n)} < 1 - \epsilon\}. \tag{4.5}$$

It suffices to show that for each $\epsilon \in (0,1)$ there exists a sequence $Q_0 \supseteq Q_1 \supseteq \cdots$ of open sets such that $G_\epsilon \subseteq Q_i$ and $\mu(Q_{i+1}) \leq \mu(Q_i) \cdot (1 - \epsilon)$ for all $i \in \omega$. Indeed, in that case we have $\lim_i \mu(Q_i) = 0$ and so the reals X_0 in F_V that fail (4.4) form a null set. For each i we will define a set of string/number tuples E_i and define:

$$Q_i = \{X \mid \Theta^X \text{ has a prefix in a tuple of } E_i\}.$$

Let E be the set of tuples (σ, ℓ) such that $\ell > |\sigma|$, $\pi(\sigma) > 0$ and the proportion of the reals X with $\Theta^X \supseteq \sigma$, such that either $\Theta^X \upharpoonright \ell$ is undefined or has a prefix in V, is $\geq \epsilon$. We order the strings first by length and then lexicographically. Also, we order E lexicographically, i.e. $(\sigma, m) < (\sigma', n)$ when either $\sigma < \sigma'$, or $\sigma = \sigma'$ and $m < n$.

At step $i = 0$ we define a sequence of tuples by recursion: let (σ_j, ℓ_j) be the least tuple in E such that σ_j is incompatible with σ_k for $k < j$. Let E_0 be the collection of all these tuples. At step $i + 1$ do the following for each string σ which does not have a prefix in V and such that $|\sigma| = \ell$ and $\sigma' \subseteq \sigma$ for some $(\sigma', \ell) \in E_i$. Define a sequence of tuples by recursion, letting (σ'_j, ℓ'_j) be the least tuple in E such that σ'_j extends σ and is incompatible with σ'_k for $k < j$. Let E_{i+1} be the set of all tuples which occur in any sequence defined at step $i + 1$ (i.e. take the union of all the sequences produced for the various σ such that σ does not have a prefix in V, $|\sigma| = \ell$ and $\sigma' \subseteq \sigma$ for some $(\sigma', \ell) \in E_i$).

It follows by induction on i that the set of all strings which are in any tuple in E_i is prefix-free, and that $Q_i \supseteq Q_{i+1}$. By the definition of Q_0 and the minimality of the strings that are enumerated into E_0 we have $G_\epsilon \subseteq Q_0$. For the same reason, at each step $i + 1$ we have $Q_i - Q_{i+1} \subseteq 2^\omega - G_\epsilon$. Hence $G_\epsilon \subseteq Q_i$ for all $i \in \omega$. It remains to show that $\mu(Q_{i+1}) \leq \mu(Q_i) \cdot (1 - \epsilon)$ for all $i \in \omega$. In order to see this note that, at stage $i + 1$, we consider in effect a partition of Q_i into sets Q_σ where σ occurs in a tuple in E_i and $Q_\sigma = \{X \mid \Theta^X \supseteq \sigma\}$. According to the definition of E, we only enumerate into Q_{i+1} at most $1 - \epsilon$ of the measure in each Q_σ. □

4.3. Example: bounding a 1-generic degree.

In this section we demonstrate how to apply the methodology that was discussed in Section 4 by giving a simple proof of a result from [Kur81] and [Kau91] that says that every 2-random degree bounds a 1-generic degree. This result is also discussed in [DH10, Section 8.21].

This is the only level of genericity and randomness where the two notions interact in a non-trivial manner. In fact, it follows from the results in this paper that *every 2-generic degree forms a minimal pair with every 2-random degree.* This fact was originally proved directly in [NST05].

Theorem 4.10 (Kurtz [Kur81] and Kautz [Kau91])**.** *Every 2-random degree bounds a 1-generic degree.*

Proof. Let $\{W_e\}_{e\in\omega}$ be an effective enumeration of all c.e. sets of finite binary strings. It suffices to define a computable procedure which takes $k\in\omega$ as input and returns the index of a $\emptyset'$-c.e. set of strings W with $\mu(W)<2^{-k}$ and a functional Φ such that Φ^X is total and the following condition is met for all $e\in\omega$ and each X which does not have a prefix in W:

$$R_e:\ \exists n\ \left[\Phi^X\upharpoonright_n\in W_e\ \vee\ \forall\sigma\in W_e,\ \Phi^X\upharpoonright_n\not\subseteq\sigma\right].$$

Construction. At stage $s+1\in 2\omega^{[e]}+1$, if $e>k+1$ do the following.

(1) For each e-marker that has not acted and sits on a string τ, if $\Phi^\tau[s]=\sigma$ and there is a proper extension ρ of σ in $W_e[s]$ then enumerate the axiom $\langle\tau*0^e,\rho\rangle$ for Φ, and declare that the marker has *acted*.

(2) Let ℓ be large. For each string τ of length ℓ check to see whether there is some $\tau'\subset\tau$ such that either (a) an e-marker sits on τ' and has not acted, (b) an e- marker sits on τ' that has acted and $\tau'*0^e\subseteq\tau$, or (c) for some $i<e$ an i-marker sits on τ' that has not acted and $\tau'*0^i\subseteq\tau$. If none of these conditions hold then place an e-marker on τ and remove any j-marker that sits on any initial segment of τ for $j>e$.

At stage $s+1\in 2\omega$ let ℓ be large and for each τ of length ℓ enumerate the axiom $\langle\tau,\Phi^\tau[s]*0\rangle$ for Φ unless there is some $e\in\omega$ and a string τ' with an e-marker sitting on it which has not acted, such that $\tau'*0^e\subseteq\tau$.

Verification. We start by noting that the axioms enumerated for Φ are consistent. Indeed, the only point at which an inconsistency could possibly occur is during step (1) of an odd stage $s+1$. During this step, when we enumerate an axiom $\langle\tau*0^e,\rho\rangle$, ρ extends $\Phi^\tau[s]$, and we have not enumerated any axioms with respect to proper extensions of τ which are compatible with $\tau*0^e$.

We consider versions of the outcomes (1)–(3), as described in Section 4, which are modified to consider only $e>k+1$ in the obvious way. For each $e>k+1$ let V_e be the set of strings on which we place a permanent e-marker that never acts. When such a marker is placed on τ the construction will cease placing e-markers on extensions of τ, and V_e is therefore prefix-free. If we let $V=\bigcup_{e>k+1}\{\tau*0^e\mid\tau\in V_e\}$ then $\mu(V)\leq 2^{-k-1}$ and V is c.e. in $\emptyset'$. This deals with the reals for which outcome (3) occurs.

Let Q_e be the set of X such that we place infinitely many e-markers on initial segments of X, but finitely many d-markers for each $d<e$. If $X\in Q_e$ then all but finitely many of the e-markers placed on initial segments of X will be permanent and will act at some stage. We claim that the measure of Q_e is 0. If it was positive, then by the Lebesgue density theorem there would be some $X\in Q_e$ such that the relative measure of Q_e above $X\upharpoonright_n$ tends to 1 as $n\to\infty$. This contradicts the fact that every time a permanent e-marker placed on $X\upharpoonright_n$ acts, a fixed proportion (namely $1/2^e$) of the reals extending $X\upharpoonright_n$ will not receive an e-marker again, and

so will not be in Q_e. Since $\cup_e Q_e$ is Σ^0_3 and has measure 0, we can compute the index of a $\emptyset'$-c.e. set of strings S such that $\mu(S) < 2^{-k-1}$ and every real in $\cup_e Q_e$ has a prefix in S. If we set $W = V \cup S$ then $\mu(W) < 2^{-k}$ and, for every real that does not have a prefix in W, outcome (1) occurs.

Now suppose that outcome (1) occurs for X. This means that, for each $e > k+1$ there is some longest $\tau \subset X$ on which a permanent e marker is placed. There are two possibilities to consider. The first possibility is that $\tau * 0^e \subset X$ and the permanent marker placed on τ acts. Then R_e is satisfied with respect to X, and we Φ-map $\tau * 0^e$ to a proper extension of Φ^τ. The second possibility is that the permanent marker on τ does not act. Then there are no proper extensions of Φ^τ in W_e. At the stage $s+1$ after placing the marker on τ we enumerate an axiom $\langle \tau', \Phi^\tau[s] * 0\rangle$ for some $\tau' \subset X$. Thus, in either case R_e is satisfied with respect to X, and we may also conclude that Φ^X is total. □

Theorem 4.10 says that 2-randomness is sufficient to guarantee bounding a 1-generic. Throughout this paper we will be concerned in establishing optimal results, i.e. the 'weakest' level of randomness or genericity that is sufficient to guarantee some property. In this case, it is not difficult to deal with weak 2-randomness. The following fact is also mentioned in [DH10, comments after Theorem 8.21.4].

Proposition 4.11. *There is a weakly 2-random degree which does not bound any 1-generic degrees.*

Proof. This is a consequence of the following facts: (i) hyperimmune-free 1-random degrees exist and are weakly 2-random, (ii) the hyperimmune-free degrees are downward closed and (iii) 1-generic degrees are not hyperimmune-free. □

We do not know, however, whether every Demuth random bounds a 1-generic.

5. ALL NON-ZERO DEGREES BOUNDED BY A SUFFICIENTLY RANDOM DEGREE

Many degree-theoretic properties P that hold for all sufficiently random degrees also hold for any non-zero degree that is bounded by a sufficiently random degree. In this section we show how the type of construction discussed in Section 4.1, which proves that a property P holds for all sufficiently random degrees, can be modified to show that P holds for all non-zero degrees which are bounded by a sufficiently random degree. Typically, 'sufficient randomness' turns out to be 2-randomness.

5.1. **Methodology.** As in Section 4.1 we break P into a countable list $\{R_e\}_{e\in\omega}$ of simpler requirements. Given a special functional Θ we look to show that P is satisfied by all sets computed by a 2-random via Θ. We have an atomic strategy which takes a number e and a string σ as inputs and satisfies R_e for a certain proportion of the reals that Θ-map to extensions of σ, where this proportion depends on e and not on σ. Given $k \in \omega$ we describe how to assemble a construction (from the atomic strategies) which produces a set of strings W with $\mu(W) < 2^{-k}$ and ensures that all requirements are met for all reals that do not have a prefix in W.

So, to clarify, the construction is similar to the one discussed in Section 4.1, only this time the e-markers are to be placed on initial segments of the images Θ^X rather than the arguments X (whose initial segments may possibly be members of W). As a result of this modification, an e-marker that is placed on some string σ will strive to achieve the satisfaction of R_e for a fixed proportion of the reals that Θ-map to σ, rather than a proportion of the reals extending σ.

The outcomes of the construction refer to reals Y in the image space for Θ, and are the same (1), (2), (3) as listed in Section 4.1. A density argument (based on Lemma 4.9) suffices to show that the reals that Θ-map to reals Y with infinitary outcome (2) form a null Σ^0_3 class. A simple measure counting argument will show that the reals X for which Θ^X is total and has outcome (3), are contained in an open set of measure at most 2^{-k-1}. This way a set of strings W of measure $< 2^{-k}$ can be produced such that for every real X without a prefix in W, if Θ^X is total then it has outcome (1) and therefore satisfies P.

We give some details concerning the standard features of such a construction and its verification. Let us recall what took place in the proof of Theorem 4.10, since this serves as a useful example. When an e-marker was placed on a string τ, what we did in effect was to reserve a proportion 2^{-e} of the total measure above τ. For the strings extending $\tau * 0^e$ we stopped enumerating axioms for Φ, and we waited for a chance to satisfy the genericity requirement directly for these strings. This proportion 2^{-e} then played two vital roles:

(a) We were able to consider the prefix-free set of strings on which permanent e-markers are placed but do not act, and were able to conclude that the measure permanently reserved by these markers is at most 2^{-e}.

(b) We were able to conclude that, when an e-marker placed on τ acts, it permanently satisfies the corresponding requirement for a proportion 2^{-e} of the total measure above τ, so that the Lebesgue density theorem can be applied to show that the set of reals for which outcome (2) occurs is of measure 0.

Now we look to achieve something very similar. We want conditions very similar to (a) and (b) to hold, but now, rather than considering proportions of the measure above the string on which a marker is placed, we must consider proportions of the measure that Θ-maps there. The first important point to note is that we do not actually require the proportions involved in (a) and (b) to be the same. If we have that some modified version of condition (a) applies, where the proportion involved is 2^{-e}, then we shall be happy if condition (b) applies for a smaller proportion—so long as this proportion depends only on e and not on σ we shall be able to apply Lemma 4.9 as desired.

We proceed as follows. Let us write $\pi(\sigma)$ instead of $\pi(\Theta, \sigma)$, and let $\sigma \mapsto q_\sigma$ be a computable map from strings to numbers such that:

$$\sum_\sigma 2^{-q_\sigma} < 2^{-k-3}, \qquad \text{where } \sigma \text{ ranges over all strings.} \tag{5.1}$$

When an e-marker is placed on σ, it is given a corresponding parameter m_σ, which is chosen to be *large*. It then places *submarkers* on all extensions of σ of length m_σ. The atomic strategy for the satisfaction of R_e that we assume given, will be played individually by these submarkers. Each e-marker works with an approximation $\pi^*(\sigma)$ to $\pi(\sigma)$ which is initially the current value $\pi(\sigma)$ at the stage when the marker is placed, and is updated when necessary, so as to maintain the condition that (5.2) holds at stages s where the value of $\pi^*(\sigma)$ is used by the construction:

$$\pi(\sigma)[s] < 2\pi^*(\sigma)[s]. \tag{5.2}$$

Each update causes an injury of the e-marker and causes it to remove its previous submarkers (and all other markers and submarkers placed on proper extensions of σ) and redefine m_σ. Clearly each marker can only be injured finitely many times in this way. This injury is the reason that the atomic strategy is implemented by the submarkers, rather than by the marker itself.

An e-marker that sits on a string σ is initially *inactive*. An inactive marker may only be activated by the construction at a stage s_0 if it has found a *suitable* set of strings $P_\sigma(\sigma')$ above each string σ' on which it has placed a submarker. We then let P_σ be the union of all the various $P_\sigma(\sigma')$, as σ' ranges over the strings on which it has placed submarkers. Here *suitable* means that the strings in $P_\sigma(\sigma')$ are all those extending σ' of some length $\ell_\sigma > m_\sigma$ and furthermore, for $s = s_0$:

$$\pi(P_\sigma)[s] \geq 2^{-k-2} \cdot \pi^*(\sigma)[s] \ \text{ and } \ \forall \rho \in P_\sigma(\sigma')[\pi(\rho)[s] < 2^{-q_{\sigma'}}]. \tag{5.3}$$

Once a marker becomes active it remains so until injured or removed.

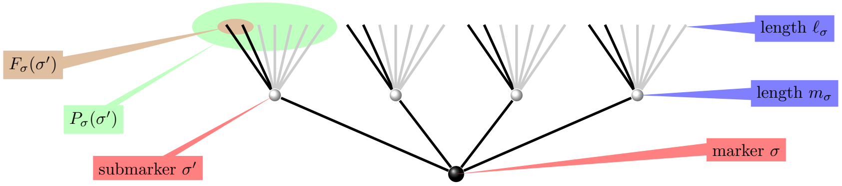

FIGURE 1. A marker and its submarkers

Let us consider first what it means if a permanent marker never becomes active. Proposition 4.5 ensures that for all sufficiently large potential values of ℓ_σ the second inequality of (5.3) will eventually always hold. Since the set of strings on which we place permanent markers which do not become active will be a prefix-free set, Lemma 4.8 then tells us that we can cover the set of all X such that Θ^X is total and extends a string in this prefix-free set, with an open set of measure $< 2^{-k-2}$.

So now suppose that the marker becomes active at some stage s_0. The second condition of (5.3) allows us to consider a subset $F_\sigma(\sigma')$ of each $P_\sigma(\sigma')$ such that for $s = s_0$:

$$0 \leq \pi(F_\sigma(\sigma'))[s] - 2^{-e} \cdot \pi(P_\sigma(\sigma'))[s] \ < \ 2^{-q_{\sigma'}}. \tag{5.4}$$

In other words, the measure mapping to $F_\sigma(\sigma')$ is a good approximation to a 2^{-e} slice of the measure mapping to $P_\sigma(\sigma')$. This immediately gives us, for $s = s_0$:

$$\pi(F_\sigma(\sigma'))[s] < 2^{-e} \cdot \pi(\sigma')[s] + 2^{-q_{\sigma'}}. \tag{5.5}$$

So (5.5) gives us a modified version of condition (a) which holds at stage s_0, since the submarker on σ' will try to satisfy its requirement directly on the reals that Θ-map to extensions of the strings in $F_\sigma(\sigma')$ by reserving this measure. In fact, it does just a little bit better than this, since the requirement only requires any conditions to be satisfied in the case that Θ^X is total. Take the union of all the $F_\sigma(\sigma')$ as σ' ranges over the strings on which submarkers are placed by the marker on σ, and then replace each string in $F_\sigma(\sigma')$ with the shortest initial segment of it which is long enough to be incompatible with all strings in $P_\sigma(\sigma') - F_\sigma(\sigma')$. Call this set D_σ. If the marker placed on σ is permanent, then for any X such that Θ^X extends a string in D_σ, we shall not have to place further e-markers on initial segments of Θ^X. It is therefore the strings which Θ-map to extensions of strings in this set D_σ with which we have to work to get our modified version of condition (b). By the first inequality of (5.3) and the first inequality of (5.4), we get that for $s = s_0$:

$$2^{-k-2-e} \cdot \pi^*(\sigma)[s] \leq \pi(D_\sigma)[s]. \tag{5.6}$$

It follows from 5.2 in other words, that the measure of the reals which Θ-map to extensions of strings in D_σ is more than a certain fixed proportion of $\pi(\sigma)$. For $s = s_0$ we have our modified version of condition (b):

$$2^{-k-3-e} \cdot \pi(\sigma)[s] \leq \pi(D_\sigma)[s]. \tag{5.7}$$

Now what we have to do is to maintain (5.5) and (5.7) at stages $s > s_0$. Actually, maintaining (5.7) does not initially seem very problematic. While $\pi(D_\sigma)[s]$ may increase as s increases, (5.2) guarantees that $\pi(\sigma)[s]$ will not increase by any problematic amount—or rather that if it does, then this will constitute one of only finitely many injuries to the marker on σ. Maintaining (5.5), however, requires us to do a little bit of work. It may be the case that as s increases, $\pi(F_\sigma(\sigma'))$ increases for some σ' on which a submarker has been placed, so that (5.5) no longer holds. In this case, we wish to remove some strings from $F_\sigma(\sigma')$. We can immediately do this if the second condition of (5.3) still holds for all $\rho \in F_\sigma(\sigma')$. In this case we can remove strings from $F_\sigma(\sigma')$ so that:

$$2^{-e} \cdot \pi(\sigma')[s] \leq \pi(F_\sigma(\sigma'))[s] < 2^{-e} \cdot \pi(\sigma')[s] + 2^{-q_{\sigma'}}. \tag{5.8}$$

This action may remove strings from D_σ but it does not threaten satisfaction of (5.7), since we still have that $\pi(F_\sigma(\sigma'))[s] \geq 2^{-e} \cdot \pi(\sigma')[s] \geq 2^{-e} \cdot \pi(\sigma')[s_0]$. We still have to deal, however, with the case that the second condition of (5.3) no longer holds for all $\rho \in F_\sigma(\sigma')$. In this case, we simply choose ℓ to be large, and replace each string $\rho \in F_\sigma(\sigma')$ with all extensions of ρ of length ℓ, to form a new $F_\sigma(\sigma')$. This does not threaten satisfaction of (5.7) because it does not change D_σ. Moreover, Proposition 4.5 ensures that we will only have to redefine $F_\sigma(\sigma')$ in this way finitely many times.

These considerations allow for an argument along the lines of Section 4.1. The basic features of the methodology, such as the measure counting which deals with outcome (3) and the density argument which deals with outcome (2), remain essentially the same. In constructions of this form, the submarkers are primarily responsible for ensuring that the requirements are met. It is the submarkers that can act. The markers themselves can only change between being inactive and active.

5.2. **Example: downward density for 1-generic degrees.**

In this section we prove Theorem 5.1 which says that every non-zero degree that is bounded by a 2-random degree **a** bounds a 1-generic degree. This is a strengthening of a result from [Kur81] (also discussed in [DH10, Section 8.21]), which asserted that the 1-generic degrees are downward dense in almost all degrees (i.e. the class of degrees **a** with the above property has measure 1).

Theorem 5.1. *Every non-zero degree that is bounded by a 2-random degree bounds a 1-generic degree.*

Proof. Let $\{W_e\}_{e\in\omega}$ be an effective enumeration of all c.e. sets of strings and suppose that B is 2-random and computes an incomputable set A via the Turing reduction Θ. By Lemma 4.3 we may assume that Θ is special. It suffices to define a computable procedure which takes as input $k \in \omega$ and returns the index of a $\emptyset'$-c.e. set of strings W with $\mu(W) < 2^{-k}$ and a functional Φ such that, if $\Theta^X = Y$ and X does not have a prefix in W, then Φ^Y is total and for all e:

$$R_e: \ \exists n \ \left[\Phi^Y \upharpoonright_n \in W_e \ \vee \ \forall \eta \in W_e, \ \Phi^Y \upharpoonright_n \not\subseteq \eta\right].$$

We follow the methodology and notation of Section 5.

Construction. At Stage 0 place a k+4-marker on the empty string.

At stage $s+1 \in 2\omega^{[e]}$, if $e > k+3$ then for each e-marker that sits on a string σ, proceed according to the first case below that applies.

(1) If (5.2) does not hold, let $\pi^*(\sigma) = \pi(\sigma)[s]$, declare that the e-marker on σ is *injured* and is inactive. Remove any markers and submarkers that sit on proper extensions of σ. Let m_σ be large and place a submarker on each extension of σ of length m_σ.

(2) Otherwise, if the marker is inactive and (5.3) holds for some set of strings $P_\sigma(\sigma')$ for each submarker σ', declare that the marker is *active*, and define $F_\sigma(\sigma')$ for each submarker σ' to be a subset of $P_\sigma(\sigma')$ such that (5.4) holds. Moreover for each submarker σ' and for each extension ρ of σ' in $P_\sigma(\sigma') - F_\sigma(\sigma')$, define Φ^ρ to be $\cup_{\rho' \subset \rho} \Phi^{\rho'}$ concatenated with 0.

(3) Otherwise, for each submarker σ' which has not acted, such that there is an extension η of $\Phi^{\sigma'}[s]$ in $W_e[s]$, define Φ^ρ to be the least such η for all $\rho \in F_\sigma(\sigma')$. In this case, remove all markers and submarkers that sit on proper extensions of σ' and declare that the submarker has acted. For each submarker σ' which has not acted, such that there is no extension η of $\Phi^{\sigma'}[s]$ in $W_e[s]$ and such that (5.5) no longer holds, there are two possibilities to consider. If the second condition of (5.3) still holds for all $\rho \in F_\sigma(\sigma')$, then remove strings from $F_\sigma(\sigma')$ so that (5.8) holds. If ρ is removed from $F_\sigma(\sigma')$ then define Φ^ρ to be $\cup_{\rho' \subset \rho} \Phi^{\rho'}$ concatenated with 0. If the second condition of (5.3) does not hold then choose ℓ to be large, and replace each string $\rho \in F_\sigma(\sigma')$ with all extensions of ρ of length ℓ, to form a new $F_\sigma(\sigma')$.

At stage $s+1 \in 2\omega+1$ let ℓ be large and do the following for each string ρ of length ℓ, provided that if σ is the longest prefix of it on which a marker is placed, then this marker is active. Let σ' be the string of length m_σ which is an initial segment of ρ, and let e be the index of the marker placed on σ. If the submarker on σ' has not acted then put an $(e+1)$-marker on ρ, unless ρ extends a string in $F_\sigma(\sigma')$. If the submarker on σ' has acted, then put an $e+1$ or e marker on ρ depending on whether it has a prefix in $F_\sigma(\sigma')$ or not (respectively).

Verification. First we show that the axioms enumerated for Φ are consistent. The only steps of the construction at which we enumerate axioms for Φ are in clauses (2) and (3) of the even stages. Consider first the case that (2) applies at stage s. Then, prior to this stage, we have not enumerated any axioms for Φ with respect to strings extending the submarkers (since whenever the marker is injured because clause (1) applies we redefine m_σ to be large). The axioms enumerated at this point are therefore unproblematic. Consider next the case that (3) applies at stage s. For each $\rho \in F_\sigma(\sigma')$ for which we enumerate an axiom, this string is mapped to an extension of $\Phi^{\sigma'}[s]$, and we have not previously enumerated axioms with respect to proper extensions of σ' which are compatible with ρ.

Let T_0 be the set of strings σ on which we place a permanent marker that is always inactive after its last injury. No markers are placed above inactive markers, and upon every injury through clause (1) a marker removes all markers placed on proper extensions. The set T_0 is therefore prefix-free. Moreover, for each $\sigma \in T_0$ we have that (4.3) holds for $c = k + 2$ and for all sufficiently large ℓ. By Lemma 4.8 we can find an index of a $\emptyset'$-c.e. set of strings V_0 such that $\mu(V_0) < 2^{-k-2}$ and, if Θ^X is total and has a prefix in T_0, then X has a prefix in V_0.

For each $e > k + 3$ let T_e be the set of strings on which we place permanent submarkers which do not act, which are placed by permanent e-markers which are eventually always active. If a permanent e-marker is placed on σ, which places a permanent submarker on σ' which does not act, then the construction will not place e-markers on extensions of σ'. Therefore each T_e is a prefix-free set. Let J_e be the union of all $F_\sigma(\sigma')$ such that $\sigma' \in T_e$ and the submarker on σ' is placed by a marker on σ. Since we maintain (5.5) it follows that:

$$\pi(J_e) < \sum_{\sigma' \in T_e} 2^{-q_{\sigma'}} + \sum_{\sigma' \in T_e} 2^{-e} \cdot \pi(\sigma').$$

Summing over all e it follows that we can find an index for a $\emptyset'$-c.e. set of strings V_1, such that $\mu(V_1) < 2^{-k-2}$ and any X such that Θ^X extends a string in some J_e has an extension in V_1.

So far we have dealt with the reals for which outcome (3) occurs. Next we wish to show:

(5.9) The class of reals X such that $\Theta^X = Y$ and for some e there are infinitely many permanent e-markers that are placed on initial segments of Y, has measure zero.

For a contradiction, assume that $e > k + 3$ and that the class of reals X such that $\Theta^X = Y$ and there are infinitely many permanent e-markers that are placed on initial segments of Y, is of positive measure. Let D_e be the union of all the final values D_σ such that a permanent e-marker is placed on σ. Now consider the set of X such that Θ^X is total and does not extend any string in D_e. This is a superset of the set of reals X such that $\Theta^X = Y$ and there are infinitely many permanent e-markers that are placed on initial segments of Y. Applying Lemma 4.9 to Θ and D_e we conclude that there exists X such that for any $\epsilon > 0$, there exists a permanent e-marker placed on $\sigma \subset \Theta^X$ which is eventually active, for which the proportion of reals Θ-mapped to extensions of σ which do not map to extensions of any string in D_σ, is $< \epsilon$. This contradicts (5.7). Since the class of (5.9) is a null Σ^0_3 class, there is a $\emptyset'$-c.e. set of strings S such that $\mu(S) < 2^{-k-1}$ and every real in the class has a prefix in S. Moreover, an index for S can be computed from an index for the given Σ^0_3 class. We let $W = V_0 \cup V_1 \cup S$.

Finally then, suppose that $\Theta^X = Y$ is total, and that for every $e > k + 3$ there is a permanent e marker placed on some initial segment of Y. Let σ be the longest initial segment of Y on which a permanent e-marker is placed. This e-marker will become active. Let σ' be the initial segment of Y on which the marker on σ places a permanent submarker. If Y extends a string in $F_\sigma(\sigma')$ then the submarker acts, and in doing so properly extends Φ^Y and ensures that R_e is satisfied with respect to Y. Otherwise Y does not extend a string in $F_\sigma(\sigma')$. In this case R_e is automatically satisfied with respect to Y because there do not exist any extensions of $\Phi^{\sigma'}$ in W_e. The length of Φ^Y is increased the last time that σ is declared active. $\square$

6. Bounding a minimal degree

First of all let us consider some background. Cooper showed that all high degrees below $\mathbf{0}'$ bound minimal degrees, and this was extended by Jockusch [Joc77] who used the recursion theorem in order to show that, in fact, all degrees which are GH_1 bound minimal degrees. This was shown to be sharp by Lerman [Ler86], who constructed a high_2 degree which does not bound any minimal degrees. Next let us consider what happens when we consider Baire category.

6.1. **Category.**

As discussed in the introduction, the degrees which do not bound minimals form a comeager class [Mar67], and the level of genericity that guarantees this property turns out to be 2-genericity [Yat76, Joc80]. On the other hand Chong and Downey [CD90] and (independently) Kumabe [Kum90] constructed a 1-generic degree which bounds a minimal degree. As a point of interest, one can also show that there are non-zero hyperimmune-free degrees bounded by 1-generics [Lew07, DY06], (as well as hyperimmune-free degrees that are not bounded by any 1-generic degree).

6.2. **Measure.**

A sufficiently random degree does not bound minimal degrees. This follows from a paper by Paris [Par77], where it is shown that the degrees with minimal predecessors form a class of measure 0. A substantial refinement of this result was given by Kurtz [Kur81] (also see [DH10, Section 7.21.4]), who showed that for almost all degrees $\mathbf{a}$ (i.e. all but a set of measure 0) if $\mathbf{0} < \mathbf{b} \leq \mathbf{a}$ then $\mathbf{b}$ bounds a 1-generic degree. In other words, for almost all degrees $\mathbf{a}$ the class of 1-generic degrees is downward dense below $\mathbf{a}$. Since 1-generic degrees are not minimal (by [Joc80]) this implies Paris' result. Both of these arguments, however, were achieved by way of contradiction and do not allow a clear view of the level of randomness that is required. In [DH10, Section 7.21.4, Footnote 15], for example, the authors note that the precise level of randomness which guarantees Kurtz's result was not known. In Section 4.3 we answered this question by proving that every non-zero degree bounded by a 2-random computes a 1-generic.

Corollary 6.1. *If a degree is 2-random then it does not have minimal predecessors.*

Proof. This is a consequence of Theorem 5.1, since 1-generic degrees cannot be minimal. □

In the remainder of this section we show that these results are optimal. In other words, 2-randomness cannot be replaced with any of the standard weaker forms of randomness. It is not hard to show that *there is a Demuth random degree which bounds a minimal degree*. By [Nie09, Theorem 3.6.25] there is a Demuth random real which is Δ^0_2. All 1-random degrees, and so all Demuth random degrees, are fixed point free. Kučera's technique of fixed point free permitting shows that all fixed point free Δ^0_2 degrees bound non-zero c.e. degrees. By [Yat70] every non-zero c.e. degree bounds a minimal degree.

In order to show that there is a weakly 2-random degree which bounds a minimal degree we will use the following characterization of weak 2-randomness.

(6.1) A 1-random real is weakly 2-random iff it forms a minimal pair with $\mathbf{0}'$.

This characterization was proved in [DNWY06] and was essentially based on a theorem by Hirschfeldt and Miller on Σ^0_3 null classes (see [DH10, Theorem 6.2.11] or [Nie09, Theorem 5.3.16] for more details). As mentioned previously, in [Joc77] it was shown that every generalized high degree bounds a minimal degree. Hence to exhibit a weakly 2-random degree which bounds a minimal degree it suffices to exhibit a generalized high weakly 2-random degree. Given (6.1) it suffices to show that every Π^0_1 class of positive measure has a member of generalized high degree which forms a minimal pair with $\mathbf{0}'$. For more basis theorems of this type (involving Π^0_1 classes and degrees which form a minimal pair with $\mathbf{0}'$) we refer the reader to [BDN11, Sections 2,3]. Note that this statement, which will be proved as Theorem 6.2, is not true for all Π^0_1 classes with no computable paths. Indeed, it is well known that there is such a class for which all members are generalized low ([Cen99]).

The proof of Theorem 6.2 uses a basic strategy for dealing with the minimal pair requirements in Π^0_1 classes (as in [BDN11, Section 2.1]) combined with the method of Kučera [Kuč85] for coding information into the jump of a random set. A detailed presentation of the latter can be found in [BDN11, Section 1.2]. Coding into random sets (or their jumps) is based on the following fact from Kučera [Kuč85]. Let $\{P_e\}_{e\in\omega}$ be an effective enumeration of all Π^0_1 classes. We say τ is P_e-extendible if it has an infinite extension in P_e.

(6.2) There exists a Π^0_1 class P of positive measure and a computable function g of two arguments such that, for all P-extendible strings τ and all $e \in \mathbb{N}$, if $P \cap P_e \cap [\tau] \neq \emptyset$ there exist at least two $P \cap P_e$-extendible strings of length $g(|\tau|, e)$ with common prefix τ.

Note that (6.2) also holds for every Π^0_1 subclass of P in place of P. Moreover, according to [Kuč85] the class P can be assumed to contain only 1-random reals and may be chosen to have measure that is arbitrarily close to 1. As a consequence, for each string τ, if $P \cap [\tau]$ is nonempty then it has positive measure.

Roughly speaking, constructing a random set A whose jump A' has a certain computational power, involves an oracle construction that looks like forcing with Π^0_1 classes, but typically involves injury amongst the Π^0_1 conditions. In particular, a sequence $\{Q_s\}_{s\in\omega}$ of Π^0_1 classes of 1-random reals is defined in stages, along with a monotone sequence $\{\tau_s\}_{s\in\omega}$ of strings (so that ultimately we can define $A = \cup_s \tau_s$) but we do not always have $Q_s \supseteq Q_{s+1}$. The coding of a certain event (which is Σ^0_1 relative to the oracle used to run the construction) into A' is associated with a certain class Q_s. Then the $Q_{s'}$ for $s' > s$ are defined as subclasses of Q_s and the $\tau_{s'}$ for $s' > s$ are extendible in Q_s. If and when the aforementioned Σ^0_1 event occurs, however, the construction defines an initial segment of A in such a way as to ensure $A \notin Q_s$. This action codes the event into A' and may cause injury to lower priority requirements (whose satisfaction relied on a Π^0_1 condition that may no longer be valid). This intuitive description may be helpful in visualising the proof of Theorem 6.2.

Theorem 6.2. *Given a Π^0_1 class P of positive measure there is $A \in P$ which is generalized high and forms a minimal pair with $\emptyset'$. Moreover $A \leq_T \emptyset''$.*

Proof. The construction is a forcing argument with Π^0_1 classes of positive measure, in which we allow finite injury amongst the Π^0_1 conditions (and the requirements that these represent). The construction will proceed in stages, computably in $\emptyset''$, defining a Π^0_1 class Q_s and a string τ_s at stage $s \in \omega$. We will have $\tau_s \subset \tau_{s+1}$ for each s and will eventually define $A = \cup_s \tau_s$. However, we may have $Q_s \not\supseteq Q_{s+1}$, which indicates an injury that is caused by the coding of $(A \oplus \emptyset')'$ into A'. The minimal pair requirements may be expressed as follows:

$$R_e : \text{If } \Psi_e^{\emptyset'} \text{ is total and incomputable then } \Psi_e^{\emptyset'} \neq \Psi_e^{A}.$$

Stages in $2\omega^{[e]}$ will be devoted to the satisfaction of R_e. We may need to act (finitely) many times for each R_e due to the injuries to requirements that may occur. Stages in $2\omega + 1$ will be devoted to coding $(A \oplus \emptyset')'$ into A'. In particular, stages in $2\omega^{[e]} + 1$ are devoted to satisfying the requirement N_e that we code into A' whether or not e belongs to $(A \oplus \emptyset')'$. By [Kuč85] we may assume that the given class P is the same as the class of (6.2), with the additional properties mentioned in the paragraph below it. Let $\tau_0 = \emptyset$, $Q_0 = P$ and consider the function g of (6.2). For the purposes of this proof we assume that if $n \in \omega^{[e]}$ then either $n + 1 \in \omega^{[e+1]}$ or $n + 1 \in \omega^{[0]}$. The following construction of (Q_s) and τ_s will also involve the definition of auxiliary classes Q_e^* and functions f_e.

Construction. At stage $s + 1 \in 2\omega^{[e]}$ let j_s be an index for Q_s. Let ρ_0 and ρ_1 be, respectively, the leftmost and rightmost extensions of τ_s which are extendible in Q_s and are of length $g(|\tau_s|, j_s)$. Check to see whether there exists n such that $\Psi_e^{\emptyset'}(n) \downarrow = m$ and:

$$Q_s \cap [\rho_1] \cap \{X \mid \Psi_e^X(n) \downarrow \neq m \ \vee \ \Psi_e^X(n) \uparrow\} \neq \emptyset. \tag{6.3}$$

If there is such n then consider the least one, set Q_{s+1} equal to the Π_1^0 class of (6.3) and define $\tau_{s+1} = \rho_1$. Otherwise, let $Q_{s+1} := Q_s \cap [\rho_0]$ and define $\tau_{s+1} = \rho_0$.

At stage $s + 1 \in 2\omega^{[e]} + 1$ let j_s be an index for Q_s.

We consider first the case that Q_e^* and f_e are undefined. In this case proceed as follows. Let $Q_e^* = Q_s$ and define f_e by recursion: $f_e(0) = |\tau_s|$ and $f_e(k + 1) = g(f_e(k), j_s)$. Also, let Q_{s+1} consist of all elements of Q_s except those that extend any string ρ which satisfies the following: there exists $k \in \omega$ and τ of length $f_e(k)$, such that ρ is the leftmost extension of τ of length $f_e(k + 1)$ which is extendible in Q_s. By the choice of g it follows that Q_{s+1} is a non-empty Π_1^0 class. Also let τ_{s+1} be the leftmost one-bit extension of τ_s which is extendible in Q_{s+1}.

If Q_e^*, f_e are defined at stage $s + 1$, let t be the stage at which they were last defined (i.e. the greatest stage $\leq s$ such that these values were undefined at the beginning of the stage and were made defined according to the instructions for that stage). If N_e *acted* after stage t or $\Psi_e^{\tau_s \oplus \emptyset'}[s] \uparrow$, then let $Q_{s+1} = Q_s$ and let τ_{s+1} be the leftmost one-bit extension of τ_s which is extendible in Q_s. On the other hand, if $\Psi_e^{\tau_s \oplus \emptyset'}[s] \downarrow$, then let ρ be the least Q_s-extendible extension of τ_s of length in $f_e(\omega)$ and define τ_{s+1} to be the leftmost extension of ρ of length $f_e(|\rho|)$ which is extendible in Q_e^*. In the latter case define $Q_{s+1} = Q_e^*$, declare that N_e has *acted* at this stage and make Q_j^*, f_j undefined for all $j > e$. Note that when determining the value of $\Psi_e^{\tau_s \oplus \emptyset'}[s]$, the construction uses the true initial segment of $\emptyset'$ of length s, and not the result of enumerating $\emptyset'$ for s steps.

Verification. Let $A = \cup_s \tau_s$ and note that $A \in P$. First, we show by induction on e that each N_e acts finitely often (with Q_e^* and f_e eventually being permanently defined). Suppose that this holds for all N_j, $j < e$. At the first stage s_0 in $2\omega^{[e]} + 1$ after the last action of some N_j, $j < e$ the construction will define Q_e^* and f_e. By the choice of s_0 it follows that these values will never subsequently be made undefined. Therefore after stage s_0 requirement N_e can act at most once. This concludes the induction step.

We show next that A satisfies all $R_e, e \in \omega$. Pick $e \in \omega$ and consider the least stage $s+1$ in $2\omega^{[e]}$ which is greater than all the stages at which some N_j acts for $j < e$. Then $A \in Q_{s+1}$ because we have $Q_j^* \subseteq Q_{s+1}$ for all j such that N_j acts in later stages. If Q_{s+1} is defined according to (6.3) then clearly $\Psi_e^{\emptyset'}(n) \neq \Psi_e^A(n)$. If, on the other hand, we define $Q_{s+1} := Q_s \cap [\rho_0]$, this means that either $\Psi_e^{\emptyset'}$ is partial or Ψ_e^X is total for all $X \in Q_s \cap [\rho_1]$ and agrees with $\Psi_e^{\emptyset'}$. The latter condition implies that $\Psi_e^{\emptyset'}$ is computable. In either case A satisfies R_e.

It remains to show that $(A \oplus \emptyset')' \leq_T A'$. First of all note that the construction is not only computable in $\emptyset''$ (so that $A \leq_T \emptyset''$) but also $A \oplus \emptyset'$. Indeed, the only place where we used more than $\emptyset'$ in order to define τ_{s+1} and Q_{s+1} was in stages $2\omega^e$. In these stages, in order to decide which clause we follow it suffices to calculate ρ_0 and ρ_1 (using $\emptyset'$) and check which of these strings the set A extends. If it extends ρ_1 then we defined Q_{s+1} according to (6.3); otherwise we followed the second clause.

The algorithm which calculates $(A \oplus \emptyset')'$ from A' is as follows. Given $e \in \omega$ suppose that we have used the oracle for A' to calculate $(A \oplus \emptyset')' \upharpoonright_e$ and the least stage s_e after which no $N_j, j < e$ *acts*. Let $t_e > s_e$ be the least in $2\omega^e + 1$. Then by stage t_e the parameters Q_e^*, f_e have reached their eventual values. Moreover, using $A, \emptyset'$, we may play back the construction up to this stage and calculate the final values of Q_e^* and f_e. Then $e \in (A \oplus \emptyset')'$ if and only if N_e *acts*, and this happens if and only if there exists $k \in \omega$ such that $A \upharpoonright_{f_e(k+1)}$ is the leftmost extension of $A \upharpoonright_{f_e(k)}$ of length $f_e(k+1)$ which is extendible in Q_e^*. Once we have determined whether N_e acts subsequent to stage t_e, this suffices to specify s_{e+1}. ☐

We can now obtain the desired result.

Corollary 6.3. *There is a weakly 2-random degree which bounds a minimal degree.*

Proof. This is a consequence of (6.1), combining the fact from Jockusch [Joc77] that every GH$_1$ degree bounds a minimal degree, and the application of Theorem 6.2 to a nonempty Π_1^0 class which consists entirely of Martin-Löf random paths. ☐

Note that by Theorem 6.2, the degree of Corollary 6.3 may be chosen below $\mathbf{0}''$.

Theorem 6.2 may be seen as a dramatic strengthening of the result proved in [LMN07], that there exists a weakly 2-random set which is not generalized low. It also gives a rather simple positive answer to [Nie09, Problem 3.6.9] which asks whether all weakly 2-random sets are array computable, since array computable sets A are generalized low$_2$. This problem was first solved in [BDN11, Section 5] where a much stronger result was shown using a different but more complicated argument. It was shown there that for every function f there exists a function g which is computable in a weakly 2-random set and which is not dominated by f.

7. Minimal covers

First of all we consider some background. The most well known theorem here is the result of Jockusch that there exists a cone of minimal covers [Joc73]. This follows from the fact that the corresponding Gale-Stewart game is determined. By considering a pointed tree such that every path through the tree is a play of the game according to the winning strategy, we conclude that either there is a cone of minimal covers, or else a cone of degrees which are not minimal covers. Clearly the latter is impossible. Next let us consider what happens when we consider Baire category.

7.1. **Category.**

The degrees that are minimal covers form a comeager set, so a sufficiently generic degree is a minimal cover of some other degree. In fact, Kumabe [Kum93a] showed that for each $n > 1$, every n-generic is a minimal cover of an n-generic. The question left open here, is as to whether or not this result is sharp:

Question 1. *Is every 1-generic degree a minimal cover?*

At the time of writing it seems likely that Durrant and Lewis are able to answer this question in the negative.

7.2. **Measure.**

Not very much is known as regards the measure theoretic case here. The basic question remains:

Question 2. *What is the measure of the degrees which are minimal covers?*

By [Kur81, Kau91] (also see [DH10, Section 8.21.3]) every 2-random degree is c.e. relative to some degree strictly below it. Hence we may deduce that every 2-random degree bounds a minimal cover. This follows by relativizing the proof from [Yat70] that every non-zero c.e. degree bounds a minimal degree. Thus, if we are to believe the heuristic principle, that properties satisfied by all highly random degrees are likely to hold for all non-zero degrees below a highly random, then we would expect the answer to Question 2 to be 1.

8. Strong minimal covers and the cupping property

A degree $\mathbf{a}$ is a strong minimal cover of another degree $\mathbf{b} < \mathbf{a}$ if for all degrees $\mathbf{x} < \mathbf{a}$ we have $\mathbf{x} \leq \mathbf{b}$. Notice that a strong minimal cover is not the join of two lesser degrees. All the known examples of degrees that fail to have a strong minimal cover satisfy the *cupping property*. A degree $\mathbf{a}$ is said to have this property if for all $\mathbf{c} > \mathbf{a}$ there exists $\mathbf{b} < \mathbf{c}$ such that $\mathbf{a} \vee \mathbf{b} = \mathbf{c}$. Clearly, a degree which has a strong minimal cover fails to satisfy the cupping property. However it is not known if the converse holds.

8.1. **Category.**

It is important to distinguish between the degrees that *are* a strong minimal cover and the degrees which *have* a strong minimal cover. The strong minimal covers form a meager class: if $A \oplus B$ is 1-generic then the Turing degrees of A and B are strictly less than the degree of $A \oplus B$. Hence strong minimal covers are not 1-generic. On the other hand, the degrees which satisfy the cupping property form a comeager class, and so the degrees which have a strong minimal cover also form a meager class. In fact, Jockusch [Joc80, Section 6] showed that every 2-generic degree has the cupping property and thus fails to have a strong minimal cover. This can easily be extended to the weakly 2- generics, by showing that all weakly 2-generics are a.n.r.[9], since it was shown in [DJS96] that all a.n.r. degrees satisfy the cupping property. In order to show that every weakly 2-generic set A is a.n.r., consider the function g_A which specifies the number of consecutive 0s in the obvious way, so that if

$$A = 11001111000011\cdots$$

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[9] Recall that A is array non-recursive (a.n.r.) if, for every $f \leq_{wtt} \emptyset'$ there exists $g \leq_T A$ which is not dominated by f.

then $g_A(0) = 2$ and $g_A(1) = 4$, for example. Given $f \leq_T \emptyset'$ (we do not require $f \leq_{wtt} \emptyset'$), let $h \leq_T \emptyset'$ be a function which on input σ outputs $\tau \supset \sigma$ with $g_B(|\sigma|) > f(|\sigma|)$ for all $B \supset \tau$. For every l, let $V_l = \{h(\sigma) : \ |\sigma| > l\}$. Each V_l is dense, so A must have an initial segment in each V_l. Thus g_A is not dominated by f.

On the other hand, Kumabe [Kum00] constructed a 1-generic degree with a strong minimal cover.

8.2. **Measure.** The strong minimal covers form a null class. Indeed, if $A \oplus B$ is 1-random then the Turing degrees of A and B are strictly less than the degree of $A \oplus B$. Hence strong minimal covers are not 1-random. We shall show in Section 9 that, in fact, every non-zero degree bounded by a 2-random satisfies the join property, and this suffices to show that no degree bounded by a 2-random is a strong minimal cover. On the other hand, the measure of the degrees which *have* a strong minimal cover is 1. Barmpalias and Lewis showed in [BL12] that every 2-random degree has a strong minimal cover, and so fails to satisfy the cupping property. In the same paper we pointed out that this result fails if 2-randomness is replaced with weak 2-randomness.

Theorem 8.1. *Every degree that is bounded by a 2-random degree has a strong minimal cover. Hence no such degree has the cupping property.*

Proof. We assume that the reader is familiar with the proof described in [BL12] and describe only the modifications required to give the stronger result. Recall that $T \subseteq 2^{<\omega}$ is *perfect* if it is non-empty and, for all $\tau \in T$, there exist incompatible strings τ_0, τ_1 which extend τ and belong to T. Our main task in the proof of [BL12] is to show that there exists $f \leq_T \emptyset'$ such that, for any $j, n \in \omega$, $f(j, n) = e$ which satisfies:

- $\mu(W_e^{\emptyset'}) < 2^{-n}$;
- if $X \notin [\![W_e^{\emptyset'}]\!]$ and X computes T which is perfect via Ψ_j, then it computes a perfect pointed $T' \subseteq T$.

Here $W_e^{\emptyset'}$ is the eth set of strings which is c.e. relative to $\emptyset'$. In order to specify $W_e^{\emptyset'}$ we consider a computable construction which enumerates axioms for two functionals Φ and Ξ. The idea is that, if $X \notin [\![W_e^{\emptyset'}]\!]$ and X computes T which is perfect via Ψ_j, then Ξ^X will be some perfect $T' \subseteq T$ and, for all Y which are paths through T', $\Phi^Y = X$. During the course of constructing Φ and Ξ, we consider various sets S of finite strings τ for which Ψ_j^τ is of at least a certain length. Then we enumerate axioms for Φ and Ξ in such a way that, for a high proportion of the strings in S, Ξ^τ is an appropriate subtree $T' \subseteq \Psi_j^\tau$ and, for all $\sigma \in \Xi^\tau$, Φ^σ is an initial segment of τ of appropriate length. During this process it may be that $\tau, \tau' \in S$ and τ is incompatible with τ' but $\Psi_j^\tau = \Psi_j^{\tau'}$. In this case we might define Ξ^τ and $\Xi^{\tau'}$ differently. The small modification required in order to give the stronger result is simply to remove this possibility. Now the idea is that if $X \notin [\![W_e^{\emptyset'}]\!]$ and X computes T which is perfect via Ψ_j, then Ξ^T will be some perfect $T' \subseteq T$ and, for all Y which are paths through T', $\Phi^Y = T$. Now when $\Psi_j^\tau = \Psi_j^{\tau'}$, it is this single value which we must consider as the oracle input for Ξ, rather than the two values τ and τ' as previously. There is no longer the possibility of mapping to two distinct values. This does not cause any problems, because now we are only required to ensure that, if X doesn't have any initial segment in $W_e^{\emptyset'}$ and $\Psi_j^X = T$ is perfect, then for all Y which are paths through Ξ^T, $\Phi^Y = T$, i.e. it is only the value T that Y must compute rather than the various X such that $\Psi_j^X = T$, so there is no need to map to two distinct values anyway. □

The following fact was first obtained in [NST05, Theorem 3.14 and Remark 3.15] via a direct argument.

Corollary 8.2 (Nies, Stephan and Terwijn [NST05]). *Every 2-random degree forms a minimal pair with every 2-generic degree.*

Proof. As mentioned previously, Jockusch showed that all 2-generics satisfy the cupping property. Martin [Mar67] showed that, if $\mathbf{a}$ is n-generic and $\mathbf{0} < \mathbf{b} < \mathbf{a}$ then $\mathbf{b}$ bounds an n-generic. Since the degrees which satisfy the cupping property are upward closed, it follows that all non-zero degrees below a 2-generic satisfy the cupping property and are therefore not bounded by a 2-random. □

9. The join property

A degree $\mathbf{a}$ satisfies the join property if for every non-zero degree $\mathbf{b} < \mathbf{a}$ there exists $\mathbf{c} < \mathbf{a}$ such that $\mathbf{b} \vee \mathbf{c} = \mathbf{a}$. The strongest positive result here [DGLM13] is that all non-GL$_2$ degrees satisfy the join property. The degrees which satisfy the join property, however, are not upward closed, and it remains open as to whether $\mathbf{0}'$ can be defined as the least degree such that all degrees above satisfy the join property.

9.1. **Category.** The degrees which satisfy the join property form a comeager class. Indeed, Jockusch [Joc80, Section 6] showed that every 2-generic degree satisfies the join property. He also showed that every degree that is bounded by a 2-generic degree satisfies the join property. In this section we show that every 1-generic degree has the join property. The coding that we employ is based on the classic and elegant method that was used in [PR81] for the proof that $\mathbf{0}'$ has the join property.

Theorem 9.1. *Every 1-generic degree satisfies the join property.*

Proof. We suppose we are given A which is 1-generic and also an incomputable set $B <_T A$. We may suppose that B is not c.e., since anyway $B \oplus \bar{B}$ is not c.e. when B is incomputable, and is of the same degree as B. We wish to construct $C <_T A$ such that $A \leq_T B \oplus C$. In order to do this we suppose given an arbitrary set X and we build C_X. For some X we will have that C_X is a partial function, but C_A will be total and will be the required joining partner for B.

Let Ψ be such that $\Psi^A = B$, and assume that this functional satisfies all of the conventions satisfied by any Ψ_j as specified in Section 2. We also assume that, for any ρ and any n, if $\Psi^\rho(n)\downarrow$ then $\Psi^\rho(n) \in \{0,1\}$. Let $\sigma_m = 0^m1$. Let $\psi(X;n)$ be the use of the computation $\Psi^X(n)$ (so that if $\Psi^X(n)\uparrow$ then $\psi(X;n)\uparrow$). We define a function f_X, which may be partial. For any n, if $\psi(X;n)\uparrow$ then let $f_X(n)$ be undefined, and otherwise let ρ be the initial segment of X of length $\psi(X;n)$. If there exists m such that $\rho * \sigma_m \subset X$ then let $f_X(n) = |\rho * \sigma_m|$.

We consider given some fixed effective splitting search procedure which enumerates all unordered pairs $\{\rho_0, \rho_1\}$ such that ρ_0 and ρ_1 are Ψ-splitting but there does not exist any ρ_2 such that either $\rho_2 \subset \rho_0$ and ρ_2 and ρ_1 are Ψ-splitting, or $\rho_2 \subset \rho_1$ and ρ_2 and ρ_0 are Ψ-splitting. So the procedure enumerates all pairs of strings which are Ψ-splitting and such that neither string can be replaced by a proper initial segment to form a new splitting. In order to define C_X, we define a sequence of strings $\{\tau_{X,s}\}_{s\geq 0}$ so that $C_X = \bigcup_s \tau_{X,s}$. As we define the sequence $\{\tau_{X,s}\}_{s\geq 0}$ we also define sequences $\{n_{X,s}\}_{s\geq 1}$ and $\{\rho_{X,s}\}_{s\geq 0}$. The sequence $\{n_{X,s}\}_{s\geq 1}$ just keeps track of which bit of Ψ^X we make use of at each stage of the construction. The sequence $\{\rho_{X,s}\}_{s\geq 0}$ records the initial segment of X used by the end of stage s. This means that for all $Y \supset \rho_{X,s}$ the construction will run in an identical way up to the end of stage s.

The construction is required to be a little more subtle than it might initially seem.

Construction. Stage 0. Define $\tau_{X,0} = \rho_{X,0} = \emptyset$.
Stage $s+1 \in \omega^{[i]}$. Search until a first pair $\{\rho_0, \rho_1\}$ is enumerated by the splitting search procedure such that both of ρ_0 and ρ_1 extend $\rho_{X,s}$ and one of these strings, ρ_0 say, is an initial segment of X.[10]

For use in the verification it is also useful to enumerate a certain set $V_{X,i}$. Let $\{\rho_2, \rho_3\}$ be the first pair enumerated by the splitting search procedure such that both of ρ_2 and ρ_3 extend $\rho_{X,s}$. Let n_0 be the least such that $\Psi^{\rho_2}(n_0)\downarrow\neq \Psi^{\rho_3}(n_0)\downarrow$, let $d \in \{2,3\}$ be such that $\Psi^{\rho_d}(n_0) = 1$ and enumerate ρ_d into $V_{X,i}$.

Now we pay attention again to the pair $\{\rho_0, \rho_1\}$. Let n_1 be the least such that $\Psi^{\rho_0}(n_1)\downarrow\neq \Psi^{\rho_1}(n_1)\downarrow$. The remaining instructions for the stage are divided into steps $t \geq 0$.

Step t. Check to see whether there exists a least n with $n_1 \leq n \leq n_1 + t$ such that either:[11]

(a) $\Psi^X(n) = 1$ and there does not exist any Ψ_i-splitting above $\tau_{X,s} * \sigma_n$ with the strings of length $\leq f_X(n)$, or;

(b) $\Psi^X(n) = 0$ and there does exist a Ψ_i-splitting above $\tau_{X,s} * \sigma_n$ with the strings of length $\leq f_X(n)$.

If there exists no such n, then proceed to step $t+1$, otherwise let n be the least such and define $n_{X,s+1} = n$. If case (a) applies for n, then define $\rho_{X,s+1}$ to be the initial segment of X of length $f_X(n_1+t)$ and define $\tau_{X,s+1} = \tau_{X,s} * \sigma_n * X(s)$. If case (b) applies for n, then let τ and τ' be the first Ψ_i-splitting above $\tau_{X,s} * \sigma_n$ found by some fixed computable search procedure. Let n_2 be the least such that $\Psi_i^{\tau}(n_2)\downarrow\neq \Psi_i^{\tau'}(n_2)\downarrow$ and let $\tau'' \in \{\tau, \tau'\}$ be such that $\Psi_i^{\tau''}(n_2) \neq X(n_2)$. Let $m = f_X(n_1+t)$ and define $\rho_{X,s+1}$ to be the initial segment of X of length m (note that $m \geq n_2$). Define $\tau_{X,s+1} = \tau'' * X(s)$. For future reference, when case (b) occurs we also enumerate $\rho_{X,s+1}$ into the set $S_{X,i}$. This records that we have managed to directly diagonalize for Ψ_i at this stage. Whether case (a) or case (b) applies, proceed to stage $s+2$.

Verification. Since Ψ^A is total and there exist infinitely many n such that $A(n) = 1$, it follows that f_A is total. Also, since Ψ^A is total and incomputable, for every initial segment ρ of A there exists a pair $\{\rho_0, \rho_1\}$ enumerated by the splitting search

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[10] It may be the case that no such pair is enumerated, in which case the construction simply continues this search for ever and $\tau_{X,s+1}$ remains undefined.

[11] Note that when we write "$\Psi^X(n)$" in case (a) and case (b) this denotes its final value; if $\Psi^X(n)\uparrow$ or $f_X(n)\uparrow$ for any n with $n_1 \leq n \leq n_1 + t$ then the construction with respect to X does not terminate at stage $s+1$ and we perform no further instructions.

procedure such that both of these strings extend ρ and one of them is an initial segment of A. In order to show that C_A is total, it therefore suffices to show that when the construction is run for $X = A$ there are only finitely many steps t run at each stage of the construction. So suppose otherwise, and let s be the least such that are an infinite number of steps run at stage $s+1$ of the construction. Let n_1 be as defined in the instructions for that stage. Then, for all $n \geq n_1$, if $n \in B$ then there does exist a Ψ_i-splitting above $\tau_{X,s} * \sigma_n$, and if $n \notin B$ then there does not exist a Ψ_i-splitting above $\tau_{X,s} * \sigma_n$. This means that B is c.e., contrary to assumption.

Having established that $C = C_A$ is total, we wish to show next that $B \oplus C$ can compute the sequence $\{\tau_{A,s}\}_{s \geq 0}$, and that therefore $A \leq_T B \oplus C$. Suppose inductively that $B \oplus C$ has already been able to decide $\tau_{A,s}$. Then there exists a unique n such that $\tau_{A,s} * \sigma_n \subset C$. This value of n is $n_{A,s+1}$. By checking whether $n \in B$ or not, $B \oplus C$ can now decide whether case (a) or case (b) applied for n at the step when $\tau_{A,s+1}$ was defined, and this is sufficient information to be able to determine $\tau_{A,s+1}$.

We are therefore left to prove that $C <_T A$. Fix $i \in \omega$. Let $S = \bigcup_X S_{X,i}$. If there is some initial segment of A in S then it is clear that $A \neq \Psi_i^C$, so suppose otherwise. Next suppose there exists a stage $s+1$ such that:

(1) $s+1 \in \omega^{[i]}$;
(2) For the step t at which stage $s+1$ terminates, case (a) applies for $n_{A,s+1}$.
(3) Putting $n = n_{A,s+1}$, there does not exist any Ψ_i-splitting above $\tau_{A,s} * \sigma_n$.

In this case it is clear that Ψ_i^C is either partial or computable, so $A \neq \Psi_i^C$.

Finally, suppose that neither of these two cases occur. This means that as we run the construction for $X = A$, for every $s+1 \in \omega^{[i]}$ and for $n = n_{A,s+1}$, case (a) applies for n at the step of stage $s+1$ at which we define $\tau_{A,s+1}$, but *actually* there does exist some Ψ_i-splitting above $\tau_{A,s} * \sigma_n$. Now we look to derive a contradiction, by showing that for each $\rho \subset A$ there are strings in S extending ρ.

Let $V = \bigcup_X V_{X,i}$. Since A is 1-generic and V is c.e. and all initial segments of A have extensions in V, it follows that there are infinitely many strings in V which are initial segments of A. Now we have to establish exactly what this means. Suppose $\rho \in V$ and $\rho \subset A$. Then there exists some X such that ρ is enumerated into $V_{X,i}$ during stage $s+1$ of the construction for X. Since $\rho \subset A$ it must be the case that $\rho_{X,s} \subset A$. This means that, up until the end of stage s the constructions for X and A are identical and $\rho_{X,s} = \rho_{A,s}$. Therefore ρ is also enumerated into $V_{A,i}$ at stage $s+1$ of the construction for A and the pairs $\{\rho_0, \rho_1\}$ and $\{\rho_2, \rho_3\}$ as specified in the instructions for that stage are identical. Without loss of generality, suppose that $\rho_0 = \rho_2 \subset A$ and let $n_0 = n_1$ be as defined in the instructions of the construction for A at that stage. Then $\Psi^{\rho_0}(n_0) = 1$. There are now two possibilities to consider.

First, suppose that $n_{A,s+1} = n_0$. Then case (a) applies for n_0 at step 0 when we define $\tau_{A,s+1}$ but actually there does exist a Ψ_i-splitting above $\tau_{A,s} * \sigma_{n_0}$. Let r be greater than the length of the strings in the first such splitting. Then $\rho_1 * \sigma_r$ is a string in S extending $\rho_{A,s}$. This follows because, when we run the construction for any $Y \supset \rho_1 * \sigma_r$, it will be identical to the construction for A up until the end of stage s. Then $\{\rho_0, \rho_1\}$ will be the first pair enumerated by the splitting search procedure such that both of ρ_0 and ρ_1 extend $\rho_{Y,s}$ and one of these strings is an initial segment of Y. Now *here* is the crucial point: at step $t = 0$ in stage $s+1$

of the construction for Y we find that $\Psi^Y(n_0)\downarrow = 0$ and that there does exist a Ψ_i-splitting above $\tau_{Y,s} * \sigma_{n_0}$ with the strings of length less than $f_Y(n_0)$.

Next suppose that $n_{A,s+1} \neq n_0$. Since $\Psi^A(n_0) = 1$ this means that there does exist a Ψ_i-splitting above $\tau_{A,s} * \sigma_{n_0}$. Once again, choosing r sufficiently large it follows that $\rho_1 * \sigma_r$ is a string in S extending $\rho_{A,s}$.

We have shown that every initial segment of A has extensions in S. Since S is a c.e. set, and A is 1-generic but does not have any initial segment in S, this gives the required contradiction. □

9.2. **Measure.** The degrees which satisfy the join property form a class of measure 1. Indeed, we show the following.

Theorem 9.2. *Every 2-random degree satisfies the join property.*

Proof. Suppose that A is 2-random and $\Psi^A = B$ for some incomputable set B and a Turing functional Ψ. We will exhibit a set $C <_T A$ such that $C \oplus B \equiv_T A$. By Lemma 4.3, we may assume that Ψ is special. In order to establish the existence of such a set C it suffices to define a computable procedure which takes a number $k \in \omega$ as input and returns (indices of) a $\emptyset'$-c.e. set of strings W with $\mu(W) < 2^{-k}$ and a Turing functional Φ such that the following is satisfied for all sets X which do not have a prefix in W:

$$\Psi^X \text{ is total} \Rightarrow (\Phi^X \text{ is total} \wedge X \leq_T \Psi^X \oplus \Phi^X \wedge X \not\leq_T \Phi^X). \tag{9.1}$$

Since A is 2-random there will be some $k \in \omega$ such that A does not have a prefix in the set W produced by the computable procedure with input k. If we let $C = \Phi^A$ for the functional Φ that is produced by the procedure with input k, then C has the desired properties.

Let us fix $k \in \omega$. The procedure on input k will also produce the reduction Ξ which establishes $X \leq_T \Psi^X \oplus \Phi^X$ in (9.1). For ease of notation we let the oracle inputs for Ξ appear as arguments and not as superscripts.

Since 1-generic degrees do not bound 1-random degrees, in order to ensure that $A \not\leq_T \Phi^A$ it suffices to ensure that Φ^A is 1-generic. We therefore look to satisfy the following requirements for all X that do not have a prefix in W, where $\{W_e\}_{e\in\omega}$ is an effective enumeration of all upward closed c.e. sets of strings:

$$R_e: \ \Psi^X \text{ is total } \Rightarrow \ \exists n[(\Phi^X \restriction_n \in W_e) \ \vee \ \forall \sigma \in W_e(\Phi^X \restriction_n \not\subseteq \sigma)].$$

The construction fits the general description of Section 4.1. The purpose of an e-marker that is placed on a string τ is to enumerate axioms for Φ and Ξ, and to ensure that R_e is satisfied for a fixed proportion of the extensions X of τ. We describe only roughly how the marker operates now, the precise instructions will deviate just slightly from this rough description.

The marker begins by searching for a Ψ-splitting (V, V') above τ, of τ-measure 2^{-e}. Until such a splitting is found the marker is *inactive*. If and when the splitting is found, the marker becomes *active*. Upon finding the splitting the marker discards some strings from V and V', so that V' is still of τ-measure at least $2^{-(e+2)}$ and so that $\mu(V)/\mu(V \cup V') = 2^{-e}$ (this may involve extending the length of the strings as necessary). Once active, the marker enumerates axioms for Φ and Ξ on the strings in V' and restrains the placement of markers on extensions of the strings in V. Finally, if and when an extension σ of Φ^τ appears in W_e, it defines Φ^ρ to be an extension of σ for all strings $\rho \in V$ and lifts the restraint on the placement of markers on strings extending those in V. In this event we say that the marker has *acted*.

According to Lemma 4.6, if the marker remains inactive then its actions may cause Φ^X to be partial although Ψ^X is not partial, for τ-measure at most $2^{-(e-1)}$. Once the marker becomes active, it may cause Φ^X to be partial for those X extending strings in V, but this is only 2^{-e} of the total proportion of strings in $V \cup V'$. Once active, the marker ensures R_e is satisfied for at least a fixed proportion of the reals extending τ, where this proportion depends solely on e.

Construction of Φ and Ξ. At stage 0 place a $k+4$-marker on the empty string.

At stage $s+1 \in 2\omega^{[e]}+1$, if $e > k+3$ then perform the following instructions, otherwise go to the next stage. Order the strings on which e-markers sit, first by length and then from left to right. For each such τ and its marker in turn, perform the following instructions for the first of cases (a) and (b) which applies (or if neither case applies then do nothing).

(a) If the marker is *inactive* and there is a Ψ-splitting (V, V') of τ-measure 2^{-e} above τ in which the strings are of length $\leq s$ then proceed as follows. Discard some strings from V and V', so that V' is still of τ-measure at least $2^{-(e+2)}$ and so that $\mu(V)/\mu(V \cup V') = 2^{-e}$ (we can assume the strings are long enough to do this). Take each $\rho \in V'$ in turn and enumerate the axioms $\Phi^\rho = \Phi^\tau * 0^{n_\rho}1$ and $\Xi(\Psi^\rho, \Phi^\rho) = \rho$, where n_ρ is chosen to be large at the time of the enumeration (and so increases as we proceed through the various ρ). Declare the marker to be *active*.

(b) If the marker is *active* with splitting (V, V') but has not acted and there is some $\rho \in V'$ and some extension σ of Φ^ρ in $W_e[s]$ then proceed as follows. Choose the least such extension σ and, taking each $\rho' \in V$ in turn, define $\Phi^{\rho'} = \sigma * 0^{n_{\rho'}}1$ and $\Xi(\Psi^{\rho'}, \Phi^{\rho'}) = \rho'$, where $n_{\rho'}$ is chosen to be large at the time of the enumeration. Remove any markers that sit on extensions of the strings in $V \cup V'$ and declare that the marker has *acted*.

At stage $s+1 \in 2\omega+2$ let ℓ be large and proceed as follows for each string τ of length ℓ (starting from the leftmost string and moving right). Let ρ be the longest initial segment of τ on which a marker sits and let e be the index of the marker. If the e-marker is active with splitting (V, V') but has not acted and τ has a prefix in V' then place an $(e+1)$-marker on τ. If the e-marker is active with splitting (V, V') but has not acted and τ does not have a prefix in $V \cup V'$ then place an e-marker on τ. If the e-marker has acted place an e-marker on τ, unless τ has a prefix in V in which case place an $(e+1)$-marker on τ. If a marker was placed on τ, define Φ^τ to be $\cup_{\rho \subset \tau} \Phi^\rho$ concatenated with $0^{n_\tau}1$, where n_τ is chosen to be large at the time of the enumeration.

Verification. It is clear that the axioms enumerated for Φ and Ξ are consistent. The only point at which this condition could possibly be violated is when a marker on τ with splitting (V, V') acts and defines $\Phi^{\rho'} = \sigma * 0^{n_{\rho'}}1$ and $\Xi(\Psi^{\rho'}, \Phi^{\rho'}) = \rho'$ for each $\rho' \in V$. Here σ extends Φ^ρ for some $\rho \in V'$ which is incompatible with each $\rho' \in V$. These axioms remain consistent with those previously enumerated, however, precisely because (V, V') is a Ψ-splitting.

It is also clear that for each real X, one of the outcomes (1), (2) or (3) as described in Section 4.1 must occur. Once an e-marker placed on τ becomes active, it ensures that at least a certain proportion of the reals extending τ do not have infinitely many e-markers placed on their initial segments, and so, as previously observed, it follows by the Lebesgue density theorem that the set of reals for which outcome (2) occurs is a Σ^0_3 set of measure 0. We may compute the index of a set of strings S which is c.e. in $\emptyset'$, which is of measure $< 2^{-k-1}$ and such that all reals for which outcome (2) occurs have a prefix in S.

Now suppose that outcome (1) occurs for X. For any $e > k+3$ let τ be the longest initial segment of X on which a permanent e-marker is placed. Let (V, V') be the splitting for the marker placed on τ. Suppose the marker on τ does not act and X extends a string in V'. In this case R_e is satisfied and the lengths of Φ^X and $\Xi(\Psi^X, \Phi^X)$ are properly increased by the marker on τ. Otherwise the marker acts and X extends a string in V, but this allows us to draw the same conclusion.

It remains to show that we can find the index of a set of strings V which is c.e. in $\emptyset'$, such that $\mu(V) \le 2^{-k-1}$, and such that any X for which outcome (3) occurs either has Ψ^X partial, or else has an initial segment in V. We can then put $W = V \cup S$. So consider the set of strings τ that hold a permanent marker which remains inactive. This is a prefix-free set. For each τ in the set, if e is the index of the marker that sits on τ then we can (uniformly) find the index of a set of strings of τ-measure $\le 2^{-e+1}$ which contains an initial segment of any extension of τ on which Ψ is total. Since we only consider $e > k+3$, taking the union over all such τ gives a set of measure $\le 2^{-(k+2)}$.

Next, fix $e > k+3$ and consider all those τ on which a permanent e-marker is placed, which is eventually active but does not act. If (V_0, V_0') is the splitting corresponding to one such τ and (V_1, V_1') is the splitting corresponding to a different one, then any string in $V_0 \cup V_0'$ is incompatible with any string in $V_1 \cup V_1'$. Since the measure of V is always 2^{-e} of the total measure of $V \cup V'$, the measure of the union of all corresponding sets V is at most 2^{-e}. Taking the union over all $e > k+3$, we obtain a set of measure $< 2^{-(k+3)}$ as required. □

To what extent is Theorem 9.2 optimal? It is not too difficult to show that there exist Demuth randoms that do not satisfy the join property. This follows from the result of [Lew12] that all low fixed point free degrees fail to satisfy the join property, and the fact [Nie09, Theorem 3.6.25] that there exist low Demuth random reals. The following question remains open:

Question 3. *Does there exist a weakly 2-random degree which does not satisfy the join property?*

Next we use a very slightly modified version of the machinery developed in Section 5 in order to prove another instance of our heuristic principle. The original machinery could certainly have been specified in such a way that no modification would be required for this application, but this would have made the proof of Theorem 5.1 seem more complicated.

Theorem 9.3. *Every degree that is bounded by a 2-random degree satisfies the join property.*

Proof. Suppose that A is a 2-random set that computes an incomputable set B via Θ. We need to show that B has the join property. If B is of 1-generic degree then the theorem holds by Theorem 9.1, so suppose otherwise. By Lemma 4.3 we may assume that Θ is special. Suppose that B computes an incomputable set C

via a Turing functional Ψ. By Lemma 4.4 we may assume that Ψ is special. In order to show that there is some $D <_T B$ such that $D \oplus C \equiv_T B$, it suffices to define a computable procedure which takes a number k and returns (indices of) a $\emptyset'$-c.e. set of strings W with $\mu(W) < 2^{-k}$, and a Turing functional Φ such that the following holds for all sets X which do not have a prefix in W:

$$\Theta^X = Y \text{ and } \Psi^Y \text{ is total} \Rightarrow \Phi^Y \text{ is total} \wedge Y \leq_T \Psi^Y \oplus \Phi^Y \wedge Y \not\leq_T \Phi^Y. \tag{9.2}$$

In order to see that this suffices, consider the sequence of procedures with input $k \in \omega$. Since A is 2-random, for some $k \in \omega$ the corresponding procedure will produce Φ such that the right hand side of the implication in (9.2) holds with $Y = B$. In other words, $D \oplus C \equiv_T B$ and $D <_T B$ where $D = \Phi^B$. Actually, since we assumed that Θ^A is not of 1-generic degree, it suffices to replace $Y \not\leq_T \Phi^Y$ in (9.2) with the requirement that Φ^Y is 1-generic.

It remains to define and verify this procedure with input Θ, Ψ and $k \in \omega$. The procedure will also produce a Turing functional Ξ for the reduction $Y \leq_T \Psi^Y \oplus \Phi^Y$ in (9.2). We look to satisfy the following requirements for all X which do not have a prefix in W:

$$R_e: \ \Theta^X = Y \text{ and } \Psi^Y \text{ is total} \Rightarrow \begin{cases} \Phi^Y \text{ is total and } \Xi(\Psi^Y, \Phi^Y) = Y \text{ and} \\ \exists n \left[\Phi^Y \restriction_n \in W_e \vee \forall \eta \in W_e,\ \Phi^Y \restriction_n \not\subseteq \eta\right] \end{cases}$$

where $\{W_e\}$ is an effective enumeration of all upward closed c.e. sets of strings. Note that for ease of notation we let the oracles in Ξ appear as arguments and not as superscripts. We define a construction which deviates only slightly from the framework described in Section 5. Just as described there, markers are initially inactive, but now submarkers are also initially inactive and must wait to be made active. In defining the construction we make use of the following inequalities:

$$\pi(T_\sigma)[s] \geq 2^{-k-2} \cdot \pi^*(\sigma)[s]. \tag{9.3}$$

$$\pi(\rho)[s] < 2^{-q_{\sigma'}}. \tag{9.4}$$

$$0 \leq \pi(F_\sigma(\sigma'))[s] - 2^{-e} \cdot \pi(P_\sigma(\sigma'))[s] < 2^{-q_{\sigma'}}. \tag{9.5}$$

Construction of Φ, Ξ. At Stage 0 place a k+4-marker on the empty string.

At stage $s + 1 \in 2\omega^{[e]}$, if $e > k + 3$ then consider each string σ on which an e-marker sits in turn (ordered first by length and then from left to right), and proceed according to the first case below that applies.

1. If (5.2) does not hold, let $\pi^*(\sigma) = \pi(\sigma)[s]$, declare that the e-marker on σ is *injured* and is inactive. Remove any markers and submarkers that sit on proper extensions of σ. Let m_σ be large and place a submarker on each extension of σ of length m_σ.
2. Otherwise, if the marker is inactive and (9.3) holds, where T_σ is the set of all strings extending σ of length m_σ, then declare the marker to be active and define $s_\sigma = s$.
3. If the marker is already active, then proceed as follows for each submarker placed on a string σ' by σ, according to the first case below which applies.
    (a) If the submarker is inactive and there exists a Ψ-splitting (U, V) above σ' such that $\pi(U)[s] \geq \pi(V)[s] \geq 2^{-e}\pi(\sigma')[s_\sigma]$ and such that (9.4) holds for all $\rho \in U \cup V$, then declare the submarker to be active. In this case let $F_\sigma(\sigma')$ be a subset of U such that (9.5) holds, defining

$P_\sigma(\sigma') = F_\sigma(\sigma') \cup V$. Take each $\rho \in V$ in turn and enumerate the axioms $\Phi^\rho = \Phi^{\sigma'} * 0^{n_\rho}1$ and $\Xi(\Psi^\rho, \Phi^\rho) = \rho$, where n_ρ is chosen to be large at the time of the enumeration (and so increases as we proceed through the various ρ).

(b) If the submarker is *active* but has not acted and there is some $\rho \in P_\sigma(\sigma') - F_\sigma(\sigma')$ and some extension η of Φ^ρ in $W_e[s]$ then proceed as follows. Choose the least such extension η and, taking each $\rho' \in F_\sigma(\sigma')$ in turn, define $\Phi^{\rho'} = \eta * 0^{n_{\rho'}}1$ and $\Xi(\Psi^{\rho'}, \Phi^{\rho'}) = \rho'$, where $n_{\rho'}$ is chosen to be large at the time of the enumeration. Remove any markers that sit on extensions of the strings in $P_\sigma(\sigma')$ and declare that the submarker has *acted*.

(c) If the previous cases do not apply and the second inequality of (9.5) no longer holds then there are two possibilities to consider. If (9.4) still holds for all $\rho \in F_\sigma(\sigma')$, then remove strings from $F_\sigma(\sigma')$ so that (9.5) holds. If not then choose ℓ to be large, and replace each string $\rho \in F_\sigma(\sigma')$ with all extensions of ρ of length ℓ, to form a new $F_\sigma(\sigma')$ (whenever we redefine $F_\sigma(\sigma')$ we also consider $P_\sigma(\sigma')$ to be redefined accordingly, $P_\sigma(\sigma') = F_\sigma(\sigma') \cup V$).

At stage $s + 1 \in 2\omega + 1$ let ℓ be large and do the following for each string ρ of length ℓ. Let σ be the longest initial segment of ρ on which a marker sits. Let σ' be the string of length m_σ which is an initial segment of ρ, and let e be the index of the marker placed on σ. If the submarker placed on σ' is not active, then we do not place any marker on ρ, so suppose otherwise. If the submarker on σ' has not acted and ρ has a prefix in $P_\sigma(\sigma') - F_\sigma(\sigma')$ then place an $(e + 1)$-marker on ρ. If the submarker on σ' has not acted and ρ does not have a prefix in $P_\sigma(\sigma')$ then place an e-marker on ρ. If the submarker has acted place an e-marker on ρ, unless ρ has a prefix in $F_\sigma(\sigma')$, in which case place an $(e+1)$-marker on ρ. If a marker was placed on ρ, define Φ^ρ to be $\cup_{\rho' \subset \rho} \Phi^{\rho'}$ concatenated with $0^{n_\rho}1$, where n_ρ is chosen to be large at the time of the enumeration.

Verification. The question of consistency for Φ and Ξ is only trivially different than the case for Theorem 9.2. We are therefore left to specify W such that $\mu(W) < 2^{-k}$ and W has an initial segment of every X such that $\Theta^X = Y$, Ψ^Y is total and either outcome (2) or (3) holds for Y. First of all consider those $\Theta^X = Y$ for which outcome (3) applies. There are three possibilities. First, it may be the case that a permanent marker is placed on $\sigma \subset Y$, which never becomes active. By Lemma 4.8 we can find the index for a $\emptyset'$-c.e. set of strings V_0 such that $\mu(V_0) < 2^{-k-2}$ and V_0 contains an initial segment of every X for which Θ^X is total and has such a marker placed on an initial segment. The second possibility is that the first case does not apply but a permanent submarker is placed on an initial segment of Θ^X which never becomes active. Since the strings on which such submarkers are placed form a prefix-free set and we only work with $e > k+3$, Lemma 4.7 directly provides us with a set V_1 such that $\mu(V_1) \leq 2^{-k-3}$ and which contains an initial segment of every X such that $\Theta^X = Y$ is total, Ψ^Y is total, and such that such a submarker is placed on an initial segment of Y. The last possibility is that Θ^X extends a string in (the final value) $F_\sigma(\sigma')$ for some permanent submarker which does not act and which is placed by an e-marker on σ. Since, for fixed e, the union of all the various $P_\sigma(\sigma')$ corresponding to such submarkers forms a prefix-free set, and since we maintain the second inequality of (9.5) it follows that, summing over all $e > k+3$, we can find the index for an $\emptyset'$-c.e. set of strings V_2 such that $\mu(V_2) < 2^{-k-2}$ and V_2 contains an initial segment of every X for which Θ^X extends a string in one of these $F_\sigma(\sigma')$.

Finally, we must show that the set of X such that Θ^X is total and has outcome (2) is a Σ^0_3 set of measure 0. Now suppose that a permanent marker is placed on σ which becomes active at stage s_σ. We wish to find a prefix-free set of strings V_σ extending σ such that $\pi(V_\sigma)$ is at least a fixed proportion of $\pi(\sigma)$ and no e-markers are placed on strings extending those in V_σ. Then the result will follow by Lemma 4.9. Subsequent to the last injury of the marker on σ we maintain (5.2), and activation of the marker requires that (9.3) holds. If the marker places a permanent submarker on σ' which does not become active, then no markers will be placed on extensions of σ', so we can immediately enumerate all such σ' into V_σ. Now we consider each of the σ' on which the marker places a permanent submarker which becomes active, and we look to enumerate a set of strings $D_\sigma(\sigma')$ into V_σ, such that all these strings extend σ' and $\pi(D_\sigma(\sigma'))$ is at least a fixed proportion of $\pi(\sigma')[s_\sigma]$. We consider approximations to $D_\sigma(\sigma')$ and then take the final value. At each stage define $D_\sigma(\sigma')$ by replacing each string in $F_\sigma(\sigma')$ with the shortest initial segment which is incompatible with all strings that are not in $F_\sigma(\sigma')$ (so this set changes as $F_\sigma(\sigma')$ does). Now at stage s_0 at which the submarker is activated, we have that $\pi(P_\sigma(\sigma') - F_\sigma(\sigma'))[s_0] \geq 2^{-e-1} \cdot \pi(\sigma')[s_\sigma]$, and by (9.5) we therefore have that $\pi(D_\sigma(\sigma'))[s_0] \geq 2^{-2e-1} \cdot \pi(\sigma')[s_\sigma]$. We wish to show by induction that this condition is maintained at subsequent stages. First note that the strings in $P_\sigma(\sigma') - F_\sigma(\sigma')$ do not subsequently change. When we redefine $F_\sigma(\sigma')$ by extending the length of the strings, this does not change D_σ. When we remove strings from $F_\sigma(\sigma')$ at a stage s we maintain satisfaction of the first inequality in (9.5). So since

$$\pi(P_\sigma(\sigma') - F_\sigma(\sigma'))[s] \geq \pi(P_\sigma(\sigma') - F_\sigma(\sigma'))[s_0]$$

we get that $\pi(D_\sigma(\sigma'))[s_0] \geq 2^{-2e-1} \cdot \pi(\sigma')[s_\sigma]$ still holds. □

Corollary 9.4. *Every non-zero degree below a 2-random degree is the supremum of two lesser degrees. Hence 2-random degrees do not bound strong minimal covers.*

Proof. This is a consequence of Theorem 4.10 and Theorem 9.3. □

10. BEING THE TOP OF A DIAMOND

We say that a Turing degree $\mathbf{c}$ is the top of a diamond if there exist $\mathbf{a}, \mathbf{b} < \mathbf{c}$ such that $\mathbf{a} \vee \mathbf{b} = \mathbf{c}$ and $\mathbf{a} \wedge \mathbf{b} = \mathbf{0}$. As will be discussed in the following sections, all sufficiently generic degrees satisfy the complementation property, which is a strictly stronger condition than being the top of a diamond so long as the degree concerned is not $\mathbf{0}$ or minimal. Since we do not know the measure of the degrees which satisfy the complementation property or even the meet property, however, it is interesting to consider the property of being the top of a diamond for the measure-theoretic case.

It is well known that every 2-random degree is the top of a diamond. This is a simple consequence of van Lambalgen's theorem that we mentioned in Section 1.1 and the result in [HNS07] that was discussed in the proof of Lemma 4.3. We show that, in fact, the same property is shared by all nontrivial degrees with a 2-random upper bound.

Theorem 10.1. *Every non-zero degree that is bounded by a 2-random degree is the join of a minimal pair of 1-generic degrees.*

Proof. Assume that $C = \Theta^D$ where D is 2-random and C is incomputable. It follows from Theorem 5.1 and (the proof of) Theorem 9.3 that C is the join of two 1-generic sets. Here we will show that C is the join of two 1-generic sets which form a minimal pair. We will construct the 1-generic sets via two functionals Φ and Ψ. As before we may assume that Θ is special. Given $k \in \omega$ we define a construction which suffices to specify the index for a $\emptyset'$-c.e. set of strings W, such that $\mu(W) < 2^{-k}$, and such that for all X which do not have a prefix in W and such that $\Theta^X = Y$ is total the following requirements are satisfied:

$$\text{For all } e \in 3\omega + 1,\ \ R_e:\ \exists n\ \left[\Phi^Y \restriction_n \in W_{\frac{e-1}{3}}\ \vee\ \forall \sigma \in W_{\frac{e-1}{3}},\ \Phi^Y \restriction_n \not\subseteq \sigma\right];$$

$$\text{For all } e \in 3\omega + 2,\ \ R_e:\ \exists n\ \left[\Psi^Y \restriction_n \in W_{\frac{e-2}{3}}\ \vee\ \forall \sigma \in W_{\frac{e-2}{3}},\ \Psi^Y \restriction_n \not\subseteq \sigma\right].$$

We also need to make Φ^C and Ψ^C a minimal pair. A standard approach to building a minimal pair of sets is to use an approximation via finite strings $\{\alpha_s\}$ and $\{\beta_s\}$ with $A = \lim_s \alpha_s$ and $B = \lim_s \beta_s$. In order to ensure that Ψ_d^A and Ψ_e^B do not both compute the same incomputable set, at some stage s, we look for $\alpha' \supseteq \alpha_s$, $\beta' \supseteq \beta_s$ and $m \in \omega$ such that

$$\Psi_d^{\alpha'}(m) \downarrow \neq \Psi_e^{\beta'}(m) \downarrow . \tag{10.1}$$

If such a pair of extensions are found we set $\alpha_{s+1} = \alpha'$ and $\beta_{s+1} = \beta'$. Failure to find such extensions implies that if $\Psi_d^A = \Psi_e^B$ is total, then it is computable. By using Posner's trick it suffices to meet the following requirements for all $e \in 3\omega$ and for all X which do not have a prefix in W and such that $\Theta^X = Y$ is total:

$$R_e:\ \ \Psi_{\frac{e}{3}}(\Phi^Y) \text{ is not total, or } \Psi_{\frac{e}{3}}(\Phi^Y) \neq \Psi_{\frac{e}{3}}(\Psi^Y), \text{ or } \Psi_{\frac{e}{3}}(\Phi^Y) \text{ is computable.}$$

Note that here, for ease of notation, we sometimes let oracle inputs appear as arguments rather than suffixes. We say that a string ρ is an *e-failure at stage s*, if there exist ρ_1, ρ_2 extending ρ, such that for some n:

$$\Psi_{\frac{e}{3}}(\Phi^{\rho_1})[s] \restriction_n = \Psi_{\frac{e}{3}}(\Psi^{\rho_1})[s] \restriction_n \neq \Psi_{\frac{e}{3}}(\Phi^{\rho_2})[s] \restriction_n = \Psi_{\frac{e}{3}}(\Psi^{\rho_2})[s] \restriction_n .$$

Note that if $e \in 3\omega$ and ρ is not an e-failure at any stage, then requirement R_e is achieved on all extensions of ρ.

If, for $e \in 3\omega$, we place an e-marker on a string σ, and a submarker on σ' then $F_\sigma(\sigma')$ is the set of strings extending σ' which we may think of as guessing that a pair of extensions can be found as per (10.1). However, we also need to ensure that Y is computable in the join of Ψ^Y and Φ^Y. Assume that at some stage s, we have $\rho \in F_\sigma(\sigma')$ and $\Psi^\rho = \alpha$ and $\Phi^\rho = \beta$. We look for extensions of α and β on which we can achieve our requirement but also on which we can encode ρ. For any two strings $\rho_0, \rho_1 \in P_\sigma(\sigma') - F_\sigma(\sigma')$, we will ensure that ρ_0 and ρ_1 are both Φ-splitting and Ψ-splitting (this is easily achieved since we control these functionals). If ρ_0 and ρ_1 are both e-failures, then let $\alpha' = \Phi^{\rho_0}$ and $\beta' = \Psi^{\rho_1}$. We can ensure that when the submarker on σ' goes to act we have $\alpha \subseteq \alpha'$ and $\beta \subseteq \beta'$. Up until this point, there has been no need to encode anything into the join of α' and β'. Hence, at this point, we could define $\Phi^\rho \supseteq \alpha'$ and $\Psi^\rho \supseteq \beta'$ and then set some extension of the join of α' and β' to compute ρ. Now by the e-failure condition there is an n, and α_0 and α_1 extending α' such that $\Psi_{\frac{e}{3}}(\alpha_0) \restriction_n \neq \Psi_{\frac{e}{3}}(\alpha_1) \restriction_n$. Additionally there is an m, and β_0 and β_1 extending β' such that $\Psi_{\frac{e}{3}}(\beta_0)\upharpoonright_m \neq \Psi_{\frac{e}{3}}(\beta_1)\upharpoonright_m$. Hence we can find $i, j \in \{0, 1\}$ such that $\Psi_{\frac{e}{3}}(\alpha_i)\upharpoonright_{\min(n,m)} \neq \Psi_{\frac{e}{3}}(\beta_j)\upharpoonright_{\min(n,m)}$. Thus we can achieve success on all strings $\rho \in F_\sigma(\sigma')$ by defining Φ and Ψ on these strings to extend $\alpha_i * \rho$ and $\beta_j * \rho$ respectively.

We make use of the following inequalities:

$$\pi(\sigma)[s] < 2\pi^*(\sigma)[s]. \tag{10.2}$$

$$\pi(P_\sigma)[s] \geq 2^{-k-2} \cdot \pi^*(\sigma)[s] \quad \text{and} \quad \forall \rho \in P_\sigma(\sigma')[\pi(\rho)[s] < 2^{-q_{\sigma'}}]. \tag{10.3}$$

$$0 \leq \pi(F_\sigma(\sigma'))[s] - 2^{-e} \cdot \pi(P_\sigma(\sigma'))[s_\sigma] \; < \; 2^{-q_{\sigma'}}. \tag{10.4}$$

Construction of Φ and Ψ. At Stage 0 place a $k+4$-marker on the empty string. At stage $s+1 \in 2\omega^{[e]}$, if $e > k+3$, then for each e-marker that sits on a string σ, proceed according to the first case below that applies:

1. If (10.2) does not hold then redefine $\pi^*(\sigma) = \pi(\sigma)[s]$. If the e-marker on σ is currently *active*, then declare the marker to be *inactive*. For all strings $\rho \in F_\sigma(\sigma')$, define Φ^ρ to be $\cup_{\rho' \subset \rho} \Phi^{\rho'}$ concatenated with ρ, and define Ψ^ρ to be $\cup_{\rho' \subset \rho} \Psi^{\rho'}$ concatenated with ρ. Remove any markers and submarkers that sit on proper extensions of σ. Let m_σ be large and place a submarker on each extension of σ of length m_σ.
2. Otherwise, if the marker is inactive and (10.3) holds for some set of strings $P_\sigma(\sigma')$ for each submarker σ', where the strings in $P_\sigma(\sigma')$ are all those extending σ' of a certain length, declare that the marker is *active* and define $s_\sigma = s$. For each submarker σ', define $F_\sigma(\sigma')$ to be the least initial segment of $P_\sigma(\sigma')$ under the lexicographical ordering such that (10.4) holds.
3. If the marker is active then for each submarker σ' of σ which has not acted perform the following tasks:
   - (a) If (10.4) does not hold there are two possibilities. If the second inequality of (10.3) holds when we only allow the quantifier to range over strings in $F_\sigma(\sigma')$, then remove strings from $F_\sigma(\sigma')$ so that (10.4) does hold. Otherwise let ℓ be large and replace each string in $F_\sigma(\sigma')$ with all extensions of length ℓ.
   - (b) For each extension ρ of σ' in $P_\sigma(\sigma') - F_\sigma(\sigma')$, define Φ^ρ to be $\cup_{\rho' \subset \rho} \Phi^{\rho'}$ concatenated with ρ.
   - (c) For each extension ρ of σ' in $P_\sigma(\sigma') - F_\sigma(\sigma')$, define Ψ^ρ to be $\cup_{\rho' \subset \rho} \Psi^{\rho'}$ concatenated with ρ.
   - (d) If $e \in 3\omega$ and $\rho \in P_\sigma(\sigma')$ is an e-failure at the current stage, but has not been so at any previous stage in $2\omega^{[e]}$ since the marker on σ was last made active, then remove all markers from ρ and any extensions.
   - (e) If $e \in 3\omega$ and there exist two distinct strings $\rho_1, \rho_2 \in P_\sigma(\sigma') - F_\sigma(\sigma')$ such that ρ_1 and ρ_2 are both e-failures then there must exist strings $\rho_1' \supseteq \rho_1$ and $\rho_2' \supseteq \rho_2$ such that $\Psi_{\frac{e}{3}}(\Phi^{\rho_1'})[s]$ and $\Psi_{\frac{e}{3}}(\Psi^{\rho_2'})[s]$ are incomparable. For all $\rho \in F_\sigma(\sigma')$ define:

     $$\Phi^\rho[s+1] = \Phi^{\rho_1'}[s] * \rho \text{ and } \Psi^\rho[s+1] = \Psi^{\rho_2'} * \rho.$$

     Declare that the submarker on σ' has acted and remove all markers and submarkers that sit on proper extensions of σ'.

(f) If $e \in 3\omega + 1$, and σ' can act because there exists a string in $W_{\frac{e-1}{3}}$ extending $\Phi^{\sigma'}$, then for all $\rho \in F_\sigma(\sigma')$ define Φ^ρ as per Theorem 5.1 but define Ψ^ρ to be $\cup_{\rho' \subset \rho} \Psi^{\rho'}$ concatenated with ρ. Similarly for the case $e \in 3\omega + 2$.

At stage $s+1 \in 2\omega + 1$ let ℓ be large. For each string ρ of length ℓ find the longest initial segment σ with a marker. Let e be such that marker on σ is an e-marker. If the marker is inactive, then do not place a marker on ρ. If the marker is active let $\sigma' \subseteq \rho$ be the unique string on which there sits a submarker of the e-marker on σ. If the submarker has acted then if ρ extends a string in $F_\sigma(\sigma')$ place an $(e+1)$-marker on ρ, otherwise place an e-marker. If the marker has not acted then let σ'' be the unique initial segment of ρ in $P_\sigma(\sigma')$. If $\sigma'' \in F_\sigma(\sigma')$ then do not place a marker on ρ. If $\sigma'' \in P_\sigma(\sigma') - F_\sigma(\sigma')$ is an e-failure then place an e-marker on ρ, otherwise place an $(e+1)$-marker.

Verification. The analysis of outcomes (2) and (3) occurs exactly as in the proof of Theorem 5.1 with one small adjustment. Suppose $e \in 3\omega$ and let T be the set of strings on which we place permanent submarkers which do not act, which are placed by permanent e-markers which are eventually always active. Let J be the union of all (the final values) $F_\sigma(\sigma')$ such that $\sigma' \in T$ and the submarker on σ' is placed by a marker on σ. For any $\sigma' \in T$, let $S(\sigma') = \{\rho : \rho \in P_\sigma(\sigma') - F_\sigma(\sigma')$ and ρ is not an e-failure$\}$. No extension of any string in $S(\sigma')$ has an e-marker placed on it. Since the submarker never acts, there is at most one string in $P_\sigma(\sigma')$ which is an e-failure, and so $S(\sigma')$ contains all the initial elements of $P_\sigma(\sigma') - F_\sigma(\sigma')$ with the possible exception of one string that is an e-failure. The fact that we maintain (10.4) therefore means that $\pi(F_\sigma(\sigma')) - 2^{-e} \cdot \pi(S_\sigma(\sigma') \cup F_\sigma(\sigma')) < 2 \cdot 2^{-q_{\sigma'}}$. The union of all $F_\sigma(\sigma') \cup S_\sigma(\sigma')$ as σ' ranges over the elements of T forms a prefix-free set. This suffices to show that $\pi(J)$ is sufficiently small.

We now consider those Y for which outcome (1) occurs. In order to show that all genericity and minimal pair requirements are satisfied with respect to Y, for each $e > k+3$ consider the longest initial segment of Y on which a permanent submarker is placed by a permanent e-marker. Either the submarker does not act and Y extends a string in $P_\sigma(\sigma') - F_\sigma(\sigma')$, which is not an e-failure if $e \in 3\omega$, or else the submarker acts and Y extends a string in $F_\sigma(\sigma')$. In either case the requirement is satisfied. Finally we need to show that Y is computable in the join of Φ^Y and Ψ^Y. Recall that if a marker is placed on σ, then at any stage P_σ is the union of all the various $P_\sigma(\sigma')$ for submarkers σ'. First note that if ρ is in some P_σ, then the construction will enumerate at most one Φ^ρ axiom and at most one Ψ^ρ axiom. Secondly, this is the only way in which the construction enumerates Φ and Ψ axioms.

Lemma 10.2. *At any stage, if ρ_0 and ρ_1 are distinct elements of P_σ on which both Φ and Ψ are defined, then either Φ^{ρ_0} is incompatible with Φ^{ρ_1} or Ψ^{ρ_0} is incompatible with Ψ^{ρ_1}.*

Proof. First assume that this is the first P_σ defined for the e-marker on σ. Let $\alpha = \bigcup_{\sigma' \subset \sigma} \Phi^{\sigma'}$ and $\beta = \bigcup_{\sigma' \subset \sigma} \Psi^{\sigma'}$. In this case we have that no axioms have been enumerated for any ρ' with $\sigma \subseteq \rho' \subset \rho_0$ or $\sigma \subseteq \rho' \subset \rho_1$. If $e \in 3\omega + 1$, then this implies that $\Psi^{\rho_0} = \beta * \rho_0$ and $\Psi^{\rho_1} = \beta * \rho_1$. The case for $e \in 3\omega + 2$ is similar with Φ in place of Ψ. If $e \in 3\omega$, then we have $\Phi^{\rho_i} = \alpha * \rho_i$ unless $\rho_i \in F_\sigma(\sigma')$ at some stage when the submarker on σ' acts. Hence we only need to consider the case when at least one string has this property. Assume ρ_0 has this property. We have that: $\Phi^{\rho_0} = \alpha * \rho_2 * \rho * \rho_0$ and $\Psi^{\rho_0} = \beta * \rho_3 * \rho' * \rho_0$ for some strings ρ_2 and ρ_3 which are e-failures in $P_\sigma(\sigma')$, and some finite strings ρ and ρ'. Note that $\Phi^{\rho_0} \supseteq \alpha * \sigma'$ so if Φ^{ρ_0} is comparable with Φ^{ρ_1}, then this implies that $\rho_1 \in P_\sigma(\sigma')$ and $\rho_1 = \rho_2$. In this case, $\Psi^{\rho_1} \supseteq \beta * \rho_1$. Now because ρ_1 is incomparable with ρ_3 we have that Ψ^{ρ_0} is incomparable with Ψ^{ρ_1}.

The lemma follows from an induction on the number of times the marker is made inactive because (10.2) does not hold. The strings ρ_0 and ρ_1 must extend different elements in some least P_σ, at which point the above argument holds. □

Given the above lemma we can define a Turing functional Γ such that if Φ^Y and Ψ^Y are total, then $\Gamma(\Phi^Y \oplus \Psi^Y) = Y$ as follows. If for any string ρ, the main construction enumerates a Φ-axiom $\langle \rho, \alpha \rangle$ and a Ψ-axiom $\langle \rho, \beta \rangle$ then enumerate a Γ-axiom $\langle \alpha \oplus \beta, \rho \rangle$. □

11. The meet and complementation properties

We say that a degree $\mathbf{c}$ satisfies the meet property if, for all $\mathbf{b} < \mathbf{c}$ there exists a non-zero $\mathbf{a} \leq \mathbf{c}$ with $\mathbf{b} \wedge \mathbf{a} = \mathbf{0}$. We say that a degree $\mathbf{c}$ satisfies the complementation property if, for all non-zero $\mathbf{b} < \mathbf{c}$ there exists a non-zero $\mathbf{a} < \mathbf{c}$ with $\mathbf{b} \wedge \mathbf{a} = \mathbf{0}$ and $\mathbf{b} \vee \mathbf{a} = \mathbf{c}$.

In [GMS04] it was shown that all generalized high degrees have the complementation property. It remains open, however, as to whether this result is sharp. In particular we do not know if all GH_2 degrees satisfy the complementation property. It is also unknown if all GH_2 degrees satisfy the meet property. In fact, we do not even know if all non-GL_2 degrees satisfy the complementation property.

11.1. Category.

Kumabe [Kum93b] showed that every 2-generic satisfies the complementation property, and so also satisfies the meet property. The remaining questions are as to the extent to which this result is sharp:

Question 4. *Do all 1-generics satisfy the complementation property?*

Again, the case for the meet property is also unknown:

Question 5. *Do all 1-generics satisfy the meet property?*

We would expect a negative answer to Question 5.

11.2. Measure.

For the case of measure, nothing is known.

Question 6. *What is the measure of the degrees which satisfy the complementation property? How about the meet property?*

We would expect the answer to both parts of Question 6 to be 0.

12. The typical lower cone

We close by considering some questions which concern what happens to the theory of the lower cone in the limit. For any degree $\mathbf{a}$ let $\mathbf{D}[\leq \mathbf{a}]$ denote the set of degrees below $\mathbf{a}$ with the inherited ordering relation, and let $\mathrm{Th}[\leq \mathbf{a}]$ be the (first order) theory of this structure. If ϕ is any sentence in the first order language of partial orders, then the set of all A such that, for $\mathbf{a} = deg(A)$, $\phi \in \mathrm{Th}[\leq \mathbf{a}]$, is arithmetical and is therefore either meager or comeager and either of measure 0 or measure 1. Thus there exist C_ϕ and D_ϕ such that either all C_ϕ-generic sets A have $\phi \in \mathrm{Th}[\leq \mathbf{a}]$ or else all C_ϕ-generic sets A have the negation of ϕ in $\mathrm{Th}[\leq \mathbf{a}]$, and either all D_ϕ-random sets A have $\phi \in \mathrm{Th}[\leq \mathbf{a}]$ or else all D_ϕ-random sets A have the negation of ϕ in $\mathrm{Th}[\leq \mathbf{a}]$. Taking C Turing above all C_ϕ and D Turing above all D_ϕ, we conclude that for all sufficiently generic degrees $\mathbf{a}$ and $\mathbf{b}$, $\mathrm{Th}[\leq \mathbf{a}] = \mathrm{Th}[\leq \mathbf{b}]$, and for all sufficiently random degrees $\mathbf{a}$ and $\mathbf{b}$, $\mathrm{Th}[\leq \mathbf{a}] = \mathrm{Th}[\leq \mathbf{b}]$. Let us call these theories $\mathrm{Th}[\leq \mathrm{Gen}]$ and $\mathrm{Th}[\leq \mathrm{Ran}]$ respectively. We discussed earlier, that all sufficiently random degrees have a strong minimal cover, while all sufficiently generic degrees satisfy the cupping property. These are not properties which pertain to the lower cone, however, so the following question remains open:

Question 7. *Is there a natural order theoretic property which distinguishes* $\mathrm{Th}[\leq \mathrm{Gen}]$ *and* $\mathrm{Th}[\leq \mathrm{Ran}]$*?*

Recently, Richard Shore used coding methods in order to distinguish $\mathrm{Th}[\leq \mathrm{Gen}]$ and $\mathrm{Th}[\leq \mathrm{Ran}]$.

Theorem 12.1 ([Sho13])**.** *There is a sentence φ in the language of partial orders such that $\mathcal{D}(\leq \mathbf{x}) \models \varphi$ for every 3-random degree $\mathbf{x}$ and $\mathcal{D}(\leq \mathbf{x}) \models \neg\varphi$ for every 3-generic degree $\mathbf{x}$.*

However the sentence φ of the theorem is obtained via coding of the arithmetic into the degrees, so it can hardly be considered as 'natural'. In this sense, Question 7 remains open.

While it is clear that arithmetical randomness and genericity suffices for determining the theory of the structure below a given degree, one might also ask for proof that this is the exact level required:

Question 8. *Does there exist $k \in \omega$, such that for all $\mathbf{a}$ and $\mathbf{b}$ which are k-random/generic,* $\mathrm{Th}[\leq \mathbf{a}] = \mathrm{Th}[\leq \mathbf{b}]$*?*

Richard Shore has recently answered this question in the negative.

Theorem 12.2 ([Sho13])**.** *There are sentences φ_n in the language of partial orders such that, for $n \geq 2$, $\mathcal{D}(\leq \mathbf{x}) \models \varphi_n$ for every $(n+1)$-generic or $(n+1)$-random $\mathbf{x}$ but such that $\mathcal{D}(\leq \mathbf{x}) \models \neg\varphi_n$ for some n-generics and n-randoms.*

Finally, we give some remarks on the complexity of $\mathrm{Th}[\leq \mathbf{a}]$ for a sufficiently generic or random $\mathbf{a}$. Greenberg and Montalbán [GM03] showed that if the 1-generic degrees are downward dense in an ideal $\mathcal{J}$ (that is, every nonzero $\mathbf{a} \in \mathcal{J}$ bounds a 1-generic) then the first order true arithmetic is many-one reducible to the theory of $(\mathcal{J}, \leq)$. Theorem 5.1 says that the 1-generic degrees are downward dense in the degrees below a 2-random degree. Therefore if $\mathbf{a}$ is 2-random then $\mathrm{Th}[\leq \mathbf{a}]$ interpretes true arithmetic. The case for 2-generics is also true and was explicitly stated in [GM03].

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