PA degrees, $\Pi^0_1$ classes and relative randomness

George Barmpalias

Victoria University of Wellington

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No incomplete random degree can be PA.

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Theorem (Barmpalias and Lewis)

Every PA degree is the join of two random degrees.
Corollary

Incomplete PA degrees give examples of pairs of random degrees whose join is not random.
Let $C$ be of PA degree. We wish to find randoms $A, B$ such that $C \equiv_T A \oplus B$.

- We would like to start with a $\Pi^0_1$ class $P$ of randoms and find $A, B$ inside $P$,
- using the fact that $C$ computes a member of every nonempty $\Pi^0_1$ class.
- We would like to find a perfect tree in $P$, which we can use to code $C$ into the join of two of its paths.
Proof idea

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- We would like to find a perfect tree in $P$, which we can use to code $C$ into the join of two of its paths.
If $T$ is a partial function from $2^{<\omega}$ to $2^{<\omega}$ we say that $T$ is a tree if for every $\sigma \in 2^{<\omega}$ and $i \in \{0, 1\}$ such that $T(\sigma \ast i) \downarrow$:

- $T(\sigma) \downarrow \subset T(\sigma \ast i)$;
- $T(\sigma \ast (1 - i)) \downarrow | T(\sigma \ast i)$.

A tree $T$ is perfect if $T(\sigma) \downarrow$ for all $\sigma$. A finite tree $T$ of level $n$ is the restriction of a tree (as a map) to strings of length $n$. 
Trees

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Trees in $P$

- There are perfect trees in $P$.
- By a theorem of Figueira, Miller, Nies for every $X \in P$ there is a sequence of positions in $X$ such that for any alternation of the digits of $X$ on these positions, the resulting real is still in $P$. 
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- We could use this tree to define $A, B$ in $P$ such that $A \oplus B \geq_T C$:
- Get $A$ from $X$ by making the $n$th special position equal to $C(n)$
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The plan...

- Then we would only need to find such a homogenous tree which is computable from $C$.
- The set of homogenous trees of this type is a closed set in the space of perfect trees with the topology generated by the finite trees (as basic open sets).
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However... 

- However the class of trees of this type does not form a $\Pi^0_1$ class!
- In fact, the topological space $T$ of perfect trees (or even trees of this type) is not compact, hence not homeomorphic to the Cantor space.
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A way out . . .

- We work inside a compact subspace of $\mathcal{T}$.
- Let $f : \mathbb{N} \to \mathbb{N}$ be an increasing function.
- Function $f$ defines a partition on any given infinite string $A$.
- Let $(\sigma_A(i))$ be the unique sequence of strings such that $|\sigma_A(i)| = f(i)$ and

$$A = \sigma_A(0) \ast \sigma_A(1) \ast \ldots$$ (1)

- Say that $A, B$ are *piecewise f-different from level n* if $\sigma_A(i) \neq \sigma_B(i)$ for all $i \geq n$. 
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The subspace

• For any such pair $A, B$ define the tree $T_{AB}^{f,n}$ as follows:

$$T_{AB}^{f,n}(\emptyset) = A \upharpoonright f(n)$$
$$T_{AB}^{f,n}(\tau \ast 0) = T_{AB}^{f,n}(\tau) \ast \min\{\sigma_A(|\tau|), \sigma_B(|\tau|)\}$$
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• Then consider the subspace

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- $T^{f,n}$ is compact and $f$-effectively homeomorphic to the Cantor space.
- If $f$ is recursive, the trees in $T^{f,n}$ which are contained in the $\Pi^0_1$ class $P$ is itself a $\Pi^0_1$ class!
- The set

  $$\{ A \oplus B \mid A, B \text{ are piecewise } f\text{-different from level } n \text{ and } [T_{AB}^{f,n}] \subseteq P \}$$

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Theorem

There exists a computable function $f$ such that if $P$ is a $\Pi^0_1$ class and $X \in P$ is sufficiently random then for some $n \in \mathbb{N}$ and some $Y$ piecewise $f$-different to $X$ from level $n$ we have $[T_{XY}^{f,n}] \subseteq P$. 
Sketch of proof

- X can be $f$-switched after level $n$ inside $P$ if there is a $Y$ piecewise $f$-different to $X$ such that $[T_{XY}^f] \subseteq P$
- Let $D_n$ be the reals that cannot be $f$-switched from level $n$ inside $P$.
- $\hat{D}_n = D_n \cap P$
- We show by induction that $\mu(\hat{D}_n) \leq O(2^{-n})$
- ... for a recursively defined $f$

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\begin{align*}
    f(0) &= 1 \\
    f(n+1) &= 2f(n) + n + 2
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• A real can be \textit{f-switched at $[n, m]$ for all $m \in \mathbb{N}$ inside $P$} iff it can be \textit{f-switched from level $n$ inside $P$}.

• Let $D_{n,m}$ be the set of reals which cannot be \textit{f-switched at $[n, m]$ inside $P$}.
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More details

• Hence $D_n = \bigcup_m D_{n,m}$.
• the sequence $D_{n,m}$ is uniformly $\Sigma^0_1$ for recursive $f$.
• $\hat{D}_{n,m} = P \cap D_{n,m}$.
• It suffices to construct $f$ such that

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\mu(\hat{D}_{n,n}) \leq 2^{-n-1}
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\mu(\hat{D}_{n,m+1} - \hat{D}_{n,m}) \leq 2^{-n-m-2}
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for all $n$ and all $m \geq n$. 
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More details

• Hence $D_n = \bigcup_m D_{n,m}$.
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for an arbitrary increasing \( f \), and then choose a recursive \( f \) appropriately.
Arbitrary $f$

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Example:  $\mu(\hat{D}_{n,n}) \leq 2^{f(n-1)-f(n)}$

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- $M_{\sigma\tau}(n,n) \cap M_{\sigma\rho}(n,n) = \emptyset$ for any strings $\tau \neq \rho$ of length $f(n) - f(n-1)$.
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PA and random degrees

\(\Pi_1^0\) classes and LR degrees
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Equivalence classes of Relative Randomness

- The *LR degrees* are equivalent classes containing oracles which induce the same notion of relative randomness.
- An LR degree $x$ is less than another $y$ if the notion of randomness that it represents is weaker than the one that $y$ represents.
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Known results

Theorem (Miller, second part independently by Yu)

Any LR degree forms a minimal pair with almost all others. In particular every the LR degrees of two relatively 2-random sets form a minimal pair.

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A $\Delta^0_2$ real has uncountably many LR predecessors iff it is not low for random.

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There is no $\Delta^0_2$ minimal LR degree.
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Low for $\Omega$ basis theorem

Theorem (Downey, Hirschfeldt, Miller, Nies)

Every nonempty $\Pi^0_1$ class contains a path which is low for $\Omega$. 
Theorem (Barmpalias and Ng)

Given a countable sequence \((C_i)\) of sets such that \(C_i \not\leq_{LR} \emptyset\) for all \(i \in \mathbb{N}\), and a nonempty \(\Pi^0_1\) class \(P\), there exists \(B \in P\) such that \(C_i \not\leq_{LR} B\) for all \(i\).

The proof is a forcing argument with \(\Pi^0_1\) classes.
Upper cone avoidance and $\Pi^0_1$ classes

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- Assume there are no trivial paths in the class.
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*There is a minimal pair of LR degrees below $0'$.*

Proof:

- By Barmpalias, Lewis, Stephan there is a $\Pi^0_1$ class without K-trivial paths such that all of its paths are LR-below $0'$.
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LR bases for randomness

- $X$ is an LR basis for randomness if there is some $Y$ which is $X$-random and $X \leq_{LR} Y$
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More applications of $\Pi^0_1$ arguments to LR

**Theorem (Barmpalias and Ng)**

*There exists a properly $\Delta^0_{\omega^1+1}$ LR degree below $\emptyset'$."

- We need a $\emptyset^{(\omega)}$ oracle for such a construction.
- But how can we keep the constructed set $\leq_{LR} \emptyset'$?
- We use a $\Pi^0_1$ class with special coding properties, all of whose paths are LR below $\emptyset'$.
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\[ \Pi_1^0 \text{ classes and LR degrees} \]

\[ \Delta_{\omega+1}^0 \cdot \Delta_3^0 \cdot \Delta_2^0 \cdot \text{arithmetical} \]

\[ 0' \]

\[ 0 \]
Some questions

- Are the c.e. Turing degrees and c.e. LR degrees elementarily equivalent?
- Are there minimal pairs in the c.e. LR degrees?
- Characterize the LR bases for randomness
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