Degrees of unsolvability and degrees of compressibility

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Plan of the talk

- Introduction
- Triviality
- Relativization
- Structure of LK degrees
- Main result and applications
- Further questions
- References
The theory of computable sets and numbers (Turing 1936) naturally led to the theory of relative computation and unsolvability (Turing 1939, Post 1943)

In the same way...

- the study of the ‘descriptive’ complexity of strings and streams has naturally lead to the study of relativized complexity
Oracle machines

- Relative computability aims at providing measures to compare and study objects according to their information content.

- In the same way, relative randomness aims at comparing mathematical objects with respect to the randomness-related properties they might have.

- In this transition from the effective to the relativized theory, Turing machines get equipped with a source of external information.
The notion of a computable set is central in computability theory.

Recent work on effective randomness suggests that the notion of K-triviality is of analogous importance in this area.

A is K-trivial if its initial segments have trivial complexity: $K(A \upharpoonright n) \leq^+ K(0^n)$, for all $n \in \mathbb{N}$. 

Let $K^X$ be the prefix-free complexity relative to oracle $X$

- $K^X$ is based on a machine which can use external information $X$ for compressing.

- Therefore it may compress more effectively than a machine without an oracle.
More triviality

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- $K^X$ is based on a machine which can use external information $X$ for compressing.
- Therefore it may compress more effectively than a machine without an oracle.

A is low for $K$ if it does not have useful information for compressing strings: $K^A(\sigma) \leq^+ K(\sigma)$, for all strings $\sigma$. 

Nies/Hirschfeldt showed that \( \mathit{K-trivials = low for K} \)

Downey/Hirschfeldt/Laforte introduced the following measure: \( A \leq_K B \) if \( K(A \upharpoonright n) \leq^+ K(B \upharpoonright n) \) for all \( n \).

Similarly, Nies defined: \( A \leq_{LK} B \) if \( K_B^A(\sigma) \leq^+ K_A(\sigma) \) for all strings \( \sigma \).

Then \( A \equiv_{LK} B \) means that \( A, B \) have the same power for compression

**Compare:** \( A \equiv_T B \) means that \( A, B \) have the same power for computation, same information.
Facts

- Miller showed that $A \equiv_{LK} B$ iff $A$-randoms $= B$-randoms.

- The LK measure is a natural extension of the relative Turing measure.

**Remarkable fact:** There is a non-computable $A$ such that $A \equiv_{LK} \emptyset$. 
Structure theory

- The relation $\leq_{LK}$ and its connections with $\leq_T$ have been investigated in the last 5 years.
- A lot of the popular open problems in randomness today are about the relationship between $\leq_{LK}$, $\leq_T$.
- $\leq_{LK}$ has uncountable lower cones!
- However locally, the algebraic structure of $\leq_{LK}$ and $\leq_T$ looked the same.
- ...the techniques of the c.e. degrees and degrees in general seemed to have natural counterparts in the $LK$ degrees.
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. . . until now
Main result

Theorem

If $X, Y$ are $\Delta^0_2$ and not low for $K$, then there exists a c.e. set $A$ which is not low for $K$, such that $A \leq_{LK} X$ and $A \leq_{LK} Y$. 
Corollary

If $X$, $Y$ are relatively 1-random, this does not imply that the degrees $L^K$-below both of them are $K$-trivial.

Contrast: This holds for $L^K$ replaced by Turing.
Corollary: $\Sigma^0_1$, $\Delta^0_2$ structures

The $\Sigma^0_1$, $\Delta^0_2$ structures of the LK degrees and the Turing degrees are not elementarily equivalent.
Are there minimal pairs for $LK$?

Miller (2007) proved... There are minimal pairs of $LK$, even in $\Delta^0_3$. 
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There are minimal pairs of $L K$, even in $\Delta^0_3$. 
Structure $\leq_{LK} \emptyset'$

Theorem (Barmpalias/Lewis/Ng 2008)

There is a minimal pair for $\leq_{LK}$ which is $\leq_{LK} \emptyset'$.

Proof:

- There is a $\Pi^0_1$ class with no low for $K$ members, with all members $\leq_{LK} \emptyset'$ (Barmpalias/Lewis/Stephan 2007)

- Every $\Pi^0_1$ class contains a path with countably many $LK$-predecessors (Miller, Reimann/Slaman)

- Hence, by Jockusch-Soare methods, every $\Pi^0_1$ class contains a minimal pair for $\leq_{LK}$. 
Structure $\leq_{LK} \emptyset'$

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Corollary

The structure of \( \leq_{LK} \) restricted to \( \leq_{LK} \emptyset' \) is not elementarily equivalent to \( \leq_T \) restricted in \( \Delta_2^0 \) or \( \Sigma_1^0 \).
Many questions remain

Much of the basic machinery for the study of this structure has been developed.

▶ Are the LK degrees an upper semi-lattice?

▶ Are the c.e. LK degrees dense?

▶ Is there a minimal LK degree?

▶ Characterize the LK degrees with countable lower cones.

▶ . . . and so on . . . see literature.
References

- Barmpalias, Elementary differences between the degrees of unsolvability and degrees of compressibility.


- Papers by Barmpalias/Lewis/Ng/Soskova/Stephan

- Nies, Computability and Randomness, Oxford Press 2009


- Webpage: http://www.mcs.vuw.ac.nz/~georgeb/
Thank you!