COMPACTNESS ARGUMENTS WITH EFFECTIVELY CLOSED SETS FOR THE STUDY OF RELATIVE RANDOMNESS.

GEORGE BARMPALIAS

Abstract. We present a variety of compactness arguments with $\Pi^0_1$ classes which yield results about relative randomness, and in particular properties of the LR degrees. Recall that two sets $A, B$ have the same LR degree if Martin-Löf randomness relative to $A$ coincides with Martin-Löf randomness relative to $B$. It is remarkable that in some cases, these arguments currently seem to be the only way to prove certain facts about the LR degrees. Hence they seem to play a more important role than in the context of the Turing degrees, where they were originally applied by Jockusch and Soare in their study of $\Pi^0_1$ classes and degrees of theories.

1. Introduction

The systematic use of compactness arguments for the study of the Turing degrees was initiated by Jockusch and Soare in their study of $\Pi^0_1$ classes and degrees of theories. For example, they showed that every non-empty $\Pi^0_1$ class contains a pair of paths whose Turing degrees have limit infimum 0. Of course, this was an alternative approach to building minimal pairs of Turing degrees, since the original argument was a forcing construction by Kleene and Post [KP54].

In the context of the LR degrees, an important measure for studying relative randomness, the approach of compactness arguments and the use of $\Pi^0_1$ classes seems to be more than an alternative methodology. For some problems were solved using this approach, although no other approach has been shown to work. As an example, we mention the construction of a minimal pair of LR degrees LR below the halting problem which was given in [BLN10]. This problem has resisted approaches that are more common in the Turing degrees, e.g. finite extension arguments or full approximation arguments. In fact, minimal pairs is a theme where the standard methods that are used in the Turing degrees are not effective in the LR degrees. This was demonstrated in [Bar10] where it was shown that there is no pair of $\Delta^0_2$ sets which form a minimal pair in the LR degrees.

In this paper we use compactness arguments with effectively closed sets in order to derive a number of interesting results about the structure of the LR degrees.

We recall that a set is (Martin-Löf ) random if it does not belong in any ‘effectively null’ set in the Cantor space. Effectively null sets were defined to be those of the form $\bigcap_j E_j$ where $(E_j)$ is a uniformly c.e. sequence of $\Sigma^0_1$ classes such that $\mu(E_j) < 2^{-j-1}$. We say that $A \leq_{LR} B$ (in words, $A$ is LR reducible to $B$) if
every random sequence relative to $B$ is also random relative to $A$. The induced equivalence relation $\equiv_{LR}$ identifies two oracles which correspond to the same notion of relative randomness. The equivalence classes are called LR degrees and form a partially ordered structure.

Recall that Martin-Löf randomness is equivalent to the so-called Chaitin-Levin randomness, which is based on Kolmogorov’s idea of incompressibility of binary strings. Let $K$ denote the prefix-free complexity relative to a fixed universal prefix-free machine $\mathcal{M}$. A set $X$ is called Chaitin-Levin random if its initial segments all have high $K$-complexity, i.e., $K(X \upharpoonright n) \geq n - c$ for all $n \in \mathbb{N}$ and a constant $c$.

The least LR degree consists of the low for random sets (sets $A$ such that every Martin-Löf random is also Martin-Löf random relative to $A$), which coincide with the low for $K$ sets (sets $A$ such that $K(\sigma) \leq^+ K^A(\sigma)$) or even the $K$-trivial sets (sets $A$ whose prefix-free complexity is less than the prefix-free complexity of a computable sequence, modulo a constant). The equivalence of these three notions is one of the most important recent results in the area of algorithmic randomness and was shown in [Nie05].

In Section 2 we use the methodology of [JS72a, JS72b] to show that the LR degree spectrum of every non-empty $\Pi^0_1$ class with no $K$-trivial members (i.e. the class of LR degrees which contain members of the $\Pi^0_1$ class) contains an antichain of size $2^{\aleph_0}$. This extends a result in [BLS07] which referred to $\Pi^0_1$ classes of positive measure and implies that such antichains occur in many LR lower cones. We also show that the LR upper closure of a $\Pi^0_1$ class $P$ which contains no $K$-trivials is meager, and that for every such class $P$ there is another $\Pi^0_1$ class $Q$ which consists of paths of effective packing dimension 1 such that $X \not\leq_{LR} Y$ for all $X \in P$, $Y \in Q$. The latter result can be seen as an LR analog of a result of Cole and Simpson in [CS07] which referred to the Turing degrees. It also implies that for every $\Pi^0_1$ class $P$ with no $K$-trivials there exists a set $A$ of packing dimension 1 such that $X \not\leq_{LR} A$ for all $X \in P$, however we indicate how to obtain this as an application of the Baire category theorem.

Section 3 is a digression to the related topic of LR bases for randomness. In [Kuc93] Kucera introduced and studied the lowness notion of a basis for Martin-Löf randomness, which is a set $A$ that is computed by an $A$-random (Martin-Löf random relative to $A$) set. It was later shown in [HNS07] that this class coincides with the $K$-trivial sets. In [BLS08b] it was shown that there are sets $A$ which are not $K$-trivial but $A \leq_{LR} X$ for some $A$-random set $X$. If $A \leq_{LR} X$ for some $A$-random set $X$ we call $A$ an LR basis for randomness. We show that despite the result in [BLS08b], all LR bases for randomness are generalized low and we pose some questions motivating a more thorough investigation.

In Section 4 we study the sets $\leq_{LR} \emptyset'$ using compactness arguments and $\Pi^0_1$ classes. This class contains the oracles relative to which Martin-Löf randomness is equivalent to 2-randomness (i.e. Martin-Löf randomness relative to $\emptyset'$) and has recently received considerable attention, e.g. see of [Nie09, Section 5.6] and [BMN].

---

2. Given a prefix-free machine $M$ (a Turing machine with prefix-free domain) the prefix-free complexity of a string $\sigma$ relative to $M$ is the length of the shortest string $\tau$ such that $M(\tau) = \sigma$. There is a universal prefix-free machine, i.e. one that gives optimal descriptions to every string, modulo a constant. For more background on prefix-free complexity, see [Nie09].

3. Let $K_Z^Z$ denote the prefix-free complexity with respect to an oracle universal prefix-free machine with oracle $Z$. 
We show that there is a proper hyperarithmetical hierarchy of LR degrees below the LR degree of the halting problem.

**Notation.** Recall that an oracle $\Sigma^0_1$ class can be identified with a c.e. set of ‘axioms’ $\langle \tau, \sigma \rangle$ where string $\tau$ refers to the oracle and $\sigma$ refers to a clopen set in the output of the machine. Given an oracle $\Sigma^0_1$ class $V$, a string $\rho$ and a set $X$ we let

$$V^\rho = \{ \sigma \mid \exists \tau (\tau \subseteq \rho \land \langle \tau, \sigma \rangle \in V) \}$$

$$V^X = \{ \sigma \mid \exists \tau (\tau \subseteq X \land \langle \tau, \sigma \rangle \in V) \}.$$  

Notice that the members of an oracle Martin-Löf test are oracle $\Sigma^0_1$ classes. A $\Sigma^0_1$ class is called *bounded* if its Lebesgue measure is $< 1$. An oracle $\Sigma^0_1$ class $V$ is called *bounded* if there is some $q < 1$ such that the measure of $V^X$ is $< q$ for all oracles $X$. In other words, if there is a uniform bound on the measure of the class, with respect to all oracles. Throughout this paper the following characterization of the LR reducibility from Kjos-Hanssen [KH07] is be freely used: for all $A, B \in \{0, 1\}^\omega$ the following are equivalent:

(a) $A \leq_{LR} B$

(b) Every bounded $\Sigma^0_1(A)$ class $T^A$ is contained in a bounded $\Sigma^0_1(B)$ class.

(c) A member $U^A$ of a universal Martin-Löf test relative to $A$ is contained in a bounded $\Sigma^0_1(B)$ class.

In the following, $U$ refers to a member of a universal oracle Martin-Löf test.

## 2. Jockusch-Soare Arguments and Randomness Reducibilities

We start with a basic tool for avoiding upper cones in the $LR$ degrees. The special case where $W$ is $\Delta^0_2$ was shown in [BLS08b] and was applied, giving some basic results on the structure of $LR$ degrees containing $\Delta^0_2$ sets. Although we are not going to use this result in this paper it is likely to be useful, in the same way that a special case of it was useful in [BLS08b].

**Proposition 2.1.** If $W \not\leq_{LR} \emptyset$ then for every bounded oracle $\Sigma^0_1$ class $V$ and every 1-generic $Z \leq_{LR} \emptyset$ there is $\sigma \subseteq Z$ such that $U^{W|\sigma} \not\leq V^\tau$ for all $\tau \supseteq \sigma$.

**Proof.** Consider the $\Sigma^0_1$ sets of strings $M_n = \{ \sigma \mid U^{W|n} \subseteq V^\sigma \}$. Since $Z$ is 1-generic, for every $n$ we either have $U^{W|n} \subseteq V^Z$ or there is some $\sigma \subseteq Z$ such that no reals extending $\sigma$ are in $M_n$. If the claim did not hold, only the first possibility can happen. But then $U^W \subseteq V^Z$, which contradicts the fact that $W \not\leq_{LR} \emptyset$. \qed

The following lemma is an atomic version of LR cone avoidance inside a $\Pi^1_1$ class, and is crucial to some of the central results in this section.

**Lemma 2.2.** Also let $P$ be a nonempty $\Pi^1_1$ class, $V$ a bounded oracle $\Sigma^0_1$ class and $A \not\leq_{LR} \emptyset$. Then there exists some $B \in P$ such that $U^A \not\leq V^B$.

**Proof.** Suppose that for all $B \in P$ we have $U^A \subseteq V^B$. We define a $\Sigma^0_1$ class $E$ such that $\mu(E) < 1$ and $U^A \subseteq E$, which shows that $A \leq_{LR} \emptyset$. Let

$$E = \{ \sigma \mid [\sigma] \subseteq V^Z \text{ for all } Z \in P \}.$$  

By hypothesis we have $U^A \subseteq E$ and by compactness $E$ is a $\Sigma^0_1$ class. Now take $Z \in P$ which exists since $P \neq \emptyset$. Then $E \subseteq V^Z$ and hence $\mu(E) \leq \mu(V^Z) < 1$. \qed
Recall that the degree spectrum (with respect to some notion of degrees) of a $\Pi_1^0$ class is the set of the degrees of its members. In the following we prove an analog of a theorem in [JS72b] about the degree spectrum of a $\Pi_1^0$ class for the LR degrees. Notice that a collection of reals that form an antichain of LR degrees also forms an antichain in the Turing degrees.

**Theorem 2.3.** The LR degree spectrum of a $\Pi_1^0$ class with no K-trivial members contains an antichain of size $2^{8\alpha}$. Moreover this antichain can be chosen disjoint from any given countable sequence of non-trivial LR upper cones.

**Proof.** Let $P$ be a $\Pi_1^0$ class which does not contain K-trivial paths. We show the first part of the theorem, and then indicate a modification which gives the more general statement. We inductively define a perfect tree $T$ inside $P$ by repeatedly applying Lemma 2.2 in such a way that for any two paths $A,B$ on $T$ we have $A \mid_{LR} B$. Along with the nodes $T_s$ we define $\Pi_0^0$ classes $P_\sigma \subseteq P \cap [T_\sigma]$ for $\sigma \in 2^{<\omega}$ which force the condition we wish to meet. Given an effective sequence $(V_\epsilon)$ of all bounded oracle $\Sigma_0^0$ classes, our requirements are:

\[
\forall X,Y \in [T] \ (X \neq Y \Rightarrow U^X \not\subseteq V_\epsilon^Y).
\]

Let $P_0 = P$, $T_0 = \emptyset$ and inductively suppose that $P_\sigma \upharpoonright T, T_\sigma \downarrow$, $P_\sigma \subseteq [T_\sigma]$ for all $\sigma \in 2^\omega$ such that for all distinct $\sigma, \tau \in 2^\omega$ we have

\[
A \in P_\sigma \Rightarrow U^T_{A} \not\subseteq V_\epsilon^A
\]

and, of course, $P_\sigma \neq \emptyset$. Now we define $T_\rho$, $P_\rho$ for each $\rho \in 2^{s+1}$ in $k$ steps, where $k$ is the number of ordered tuples $(\sigma, \tau)$ for two distinct $\sigma, \tau \in 2^{s+1}$. Let $(\sigma_i, \tau_i)$ denote the $i$th such tuple and let the parameters at step $0$ be as follows:

\[
T_{\rho^{-}\ast j}[0] = T_{\rho^{-}} \ast \tau^j \\
P_{\rho^{-}\ast j}[0] = P_{\rho^{-}} \cap [\tau^j]
\]

where $\tau^0, \tau^1$ are incompatible extensions of $T_{\rho^{-}}$ which have infinite extensions in $P_{\rho^{-}}$. Now given $s < k$ and assuming that $T_{\rho}[s], P_{\rho}[s]$ are defined (and $P_{\rho}[s] \neq \emptyset$) for all $\rho \in 2^{s+1}$, we check if $\rho$ equals $\sigma_s$ or $\tau_s$. If not, we let the parameters as in the previous stage, i.e.

\[
T_\rho[s+1] = T_\rho[s] \\
P_\rho[s+1] = P_\rho[s].
\]

To define the parameters for $\sigma_s, \tau_s$ we use Lemma 2.2 to find some $A \in P_{\sigma_s}$ such that

\[
\{X \in P_{\tau_s} \mid U^A \not\subseteq V_e^X\} \neq \emptyset.
\]

Then there must exist some $\eta \subseteq A$, $\eta \supseteq T_{\sigma_s}[s]$ such that

\[
Q = \{X \in P_{\tau_s} \mid U^\eta \not\subseteq V_e^X\}
\]

is not empty. Now define

\[
T_{\sigma_s}[s+1] = \eta \\
P_{\sigma_s}[s+1] = P_{\sigma_s}[s] \cap [\eta]
\]

\[
T_{\tau_s}[s+1] = T_{\tau_s}[s] \\
P_{\tau_s}[s+1] = Q.
\]

When we finish all $k$ steps for the $e$th level it is easy to verify that (2.2) holds. Therefore all requirements for the tree $T$ are satisfied. For the second part of the theorem, suppose we are given a countable sequence of sets $(C_i)$ which are not K-trivial and we wish to construct a tree $T$ inside $P$ such that (2.1) and $C_i \not\subseteq_{LR} X$
for all $i \in \mathbb{N}$ and $X \in [T]$. All we have to do is to augment the above construction with steps which deal with the requirements
\[
\exists n \ [U^{C_i,n} \not\subseteq V^X_e \text{ for all } X \in P_\rho, \rho \in 2^{(i,e)}]
\]
for all $i, e \in \mathbb{N}$. This step is a direct application of Lemma 2.2.

It is well known (see Sacks [Sac63]) that the Turing degree spectrum of every perfect set of reals contains an antichain of size $2^{\aleph_0}$. It is natural to ask whether the same holds for the LR degrees.

**Question 2.4.** Does the LR degree spectrum of every perfect set of reals contain an antichain of size $2^{\aleph_0}$?

Recall that the least LR degree consists of the K-trivial sets by [Nie05].

**Theorem 2.5.** The LR upper closure of a $\Pi^0_1$ class which does not contain K-trivial sets is meager.

**Proof.** Let $P$ be a $\Pi^0_1$ class which does not contain K-trivial sets and for a contradiction assume that the LR upper closure of $P$ is not meager. Then for some $e$ the set
\[
S = \{ X \mid \{ A \in P \mid U^A \subseteq V^X_e \} \neq \emptyset \}
\]
is not meager. Hence there exists a string $\sigma$ such that every string $\tau \supseteq \sigma$ extends to a member of $S$. We will define a computable real $B$ in stages, so let $B[s]$ be the segment of $B$ that has been defined by stage $s$ ($B[s]$ can be thought of as a movable marker which moves monotonically through the full binary tree). Define the following length of agreement:
\[
\ell[s+1] = \max \{ t \mid \exists Y \exists \rho (Y \in P[s+1] \land B[s] \subseteq \rho \land U^Y(t[s+1] \subseteq V^\rho_e[s+1]) \}.
\]
A stage $s$ is called expansionary if $\ell[s] > \ell[t]$ for all $t < s$. Let $B[0] = \sigma$ (with $\sigma$ as above) and if $s$ is an expansionary stage, move $B[s]$ to a string $\rho \supseteq B[s-1]$ such that
\[
\{ Y \in P[s] \mid U^Y(\ell[s])[s] \subseteq V_e^\rho \} \neq \emptyset.
\]
Since $S$ is not meager, $B$ is well defined. Also, if
\[
M_s = \{ Y \in P \mid U^{Y[s]} \subseteq V^B_e \}
\]
we have $M_s \neq \emptyset$ for all $s \in \mathbb{N}$. Since these are clopen sets and $M_{s+1} \subseteq M_s$, by compactness $\cap_s M_s \neq \emptyset$. If we consider some $X \in \cap_s M_s$ then $X \in P$ and $U^X \subseteq V^B_e$ which is a contradiction since $V^B_e$ is a bounded $\Sigma^0_1$ class but by hypothesis $X \not\leq_{LR} 0$.

Cole and Simpson showed in [CS07] that given any special $\Pi^0_1$ class $P$ (i.e. one containing no computable paths) we can find another special $\Pi^0_1$ class $Q$ such that $X \not\leq_T Y$ for all $X \in P$, $Y \in Q$. The analog of this result for the LR degrees is also true: given a $\Pi^0_1$ class containing no K-trivals we can find another $\Pi^0_1$ class $Q$ containing no K-trivals, such that $X \not\leq_{LR} Y$ for all $X \in P$, $Y \in Q$. It is natural to ask how complex the members of $Q$ could be made in the previous statement. For example, $Q$ cannot be required to contain only random members or, indeed, only members of positive effective dimension. Recall from [May02] [AHJE04] (or see [DH09]) that a real $X$ has effective dimension 1 if $\lim_n \frac{K(X|n)}{n} = 1$ and it has effective packing dimension 1 if $\lim \sup_n \frac{K(X|n)}{n} = 1$. The claim above holds because by a theorem of Terwijn [Ter98] every set which has positive effective dimension
computes a fixed point free function. Now Arslanov’s completeness criterion (e.g. see \cite{Soa87}) says that if a set of c.e. degree computes a fixed point free function then it also computes the halting problem. So given any class $P$, any constructed class $Q$ which only contains paths of positive effective dimension will contain a Turing complete path (namely the leftmost path) which shows that it has to have a member computing a member of $P$.

Notice that the class of random sets, and indeed the class of sets of effective dimension 1 is meager. On the other hand, the class of sets of packing dimension 1 is comeager. These observations follow if we view the category of a set as the outcome of a Banach-Mazur game, as detailed in \cite{Odi89}. In a game of extending initials segments of a set $X$ that we are building, it is easy to ensure that $X$ is not of effective dimension 1 (no matter what our opponent plays) by adding long strings of 0s and it is easy to ensure that $X$ is of effective packing dimension 1 (no matter what our opponent plays) by extending with segments of high prefix-free complexity. We show that $Q$ can be required to contain only paths of effective packing dimension 1.

**Theorem 2.6.** For any $\Pi_1^0$ class $P$ which does not have $K$-trivial members there is a $\Pi_1^0$ class $Q$ which contains only members of effective packing dimension 1, such that $X \not\subseteq_{LR} Y$ for all $X \in P$, $Y \in Q$.

**Proof.** Given $P$ as above it suffices to define the approximation to a $\Pi_1^0$ class $Q$ such that
\begin{align}
R_{2e} : & \quad X \in Q \Rightarrow \exists n > e \left[K(X \upharpoonright n) > n \cdot (1 - 2^{-e})\right] \\
R_{2e+1} : & \quad X \in Q \Rightarrow \forall Y \in P \left[U^Y \not\subseteq V_e^X\right]
\end{align}
where $U$ is a member of the universal oracle Martin-Löf test and $(V_e)$ is an effective list of all bounded $\Sigma_1^0$ classes. To define $Q$ we start from the full binary tree $T[0]$ and we start chopping particular branches, by enumerating the corresponding strings (as clopen sets) into the complement of $Q$. At every stage $s$ the current approximation to $Q$ is the set of all infinite paths through $T[s]$. Level $\ell$ of a tree

is the set of all strings of length $\ell$ which are extendible in the tree (i.e. which are extended by an infinite path through the tree). Each strategy $R_{2e}, R_{2e+1}$ will operate on a particular level, which may change as the construction progresses. Strategy $R_{2e}$ will operate on level $\ell_{2e}$ and strategy $R_{2e+1}$ on level $\ell_{2e+1}$. We will have $\ell_{i[s]} < \ell_{i+1[s]}$ for all $i, s \in \mathbb{N}$.

$R_{2e+1}$ strategy. We will use a version of the Sacks preservation strategy at level $\ell_{2e+1}$ of the tree. In order to describe the strategy let us assume inductively that $\ell_{2e}$ has reached a limit and that level $\ell_{2e}$ of $T$ has settled, i.e. the strings of this level that are currently extendible in $T$ will remain so. Let us denote the set of strings of level $n$ of $T$ by $L(n)$. For each string $\sigma$ and $e \in \mathbb{N}$ we define the following length of agreement with respect to the $e$th LR reduction:
\[d_e(\sigma)[s] = \max\{t \mid \exists Y \exists \rho(Y \in P[s] \land \rho \in T[s-1] \land \sigma \subseteq \rho \land U^Y \upharpoonright [s] \subseteq V_{e[t]}[s]\}\].
A stage $s$ is called $\sigma$-expansionary if $\sigma$ has length $\ell_{2e}[s-1]$ for some $e \in \mathbb{N}$ and $d_e(\sigma)[s] > d_e(\sigma)[t]$ for all $t < s$. A stage $s$ is called $e$-expansionary if it is $\sigma$-expansionary for some $\sigma \in L(\ell_{2e})[s-1]$. The strategy for $R_{2e+1}$ is to wait for an $e$-expansionary stage $s+1$, choose some $\sigma \in L(\ell_{2e})[s]$ such that $s+1$ is $\sigma$-expansionary, consider a string $\rho \supset \sigma$ in $T[s]$ of length greater than $\ell_{2e+1}[s]$ such
that
\[(2.5) \quad \{Y \in P[s + 1] \mid U^Y[d_\ell]_e[s + 1] \subseteq V^\rho_e[s + 1]\} \neq \emptyset\]
and chop from \(T\) all paths which extend \(\sigma\) and are incompatible with \(\rho\). We let \(\ell_{2e+1}[s+1] := |\rho|\) and for uniformity we also make sure that each string in \(L(\ell_{2e})[s+1]\) has exactly one extension in \(L(\ell_{2e+1})[s+1]\) (by arbitrarily chopping superfluous branches).

We claim that if we follow this strategy there can only be finitely many \(e\)-expansionary stages, so that \(\ell_{2e+1}\) reaches a limit and the same happens for \(d_\ell(\sigma)\) for all \(\sigma \in L(\ell_{2e})\). In this case it is clear that no real extending a node in \(L(\ell_{2e+1})\) (hence no path through \(T\)) can be LR-above a member of \(P\) via the \(e\)th LR reduction, i.e. \(R_{2e+1}\) is satisfied. If there were infinitely many \(e\)-expansionary stages then for some (say, leftmost) \(\sigma \in L(\ell_{2e})\) there would be infinitely many \(\sigma\)-expansionary stages. But then \(\ell_{2e+1} \to \infty\) and the construction would define an infinite computable sequence \(G\), namely the unique path through \(T\) which extends \(\sigma\), and if
\[M_s = \{Y \in P \mid U^Y[s] \subseteq V^G_e\}\]
we would have \(M_s \neq \emptyset\) for all \(s \in \mathbb{N}\). Since these are clopen sets and \(M_{s+1} \subseteq M_s\), by compactness \(\bigcap_s M_s \neq \emptyset\). If we consider some \(X \in \bigcap_s M_s\) then \(X \in P\) and \(X^\sigma \subseteq V^G_e\) which is a contradiction since \(V^G_e\) is a bounded \(\Sigma_1^0\) class but by hypothesis \(X \notin \Sigma_1^0\).

**R_{2e} strategy.** For \((2.3)\), we will use a computable function \(f\) such that for every \(m,n \in \mathbb{N}\) and every string \(\sigma\) of length \(m\), there exists \(\tau \supset \sigma\) of length \(f(m,n)\) such that \(K(\tau) > |\tau| \cdot (1 - 2^{-n})\). The strategy for \(R_{2e}\) does the following for each \(\sigma\) of level \(\ell_{2e-1}\). First it chooses an extension \(\tau\) of \(\sigma\) such that \(|\tau| \leq Q[s - 1]\) (i.e. no paths extending \(\tau\) have been thrown out of \(Q\) so far) and removes all paths extending \(\sigma\) which are incompatible with \(\tau\) from \(Q\). Then all extensions of \(\tau\) of length \(f(|\tau|, e)\) are currently extendible in \(Q\). It defines \(\ell_{2e} = f(|\tau|, e)\) and in the following stages \(s\), whenever
\[(2.6) \quad K_s(\rho) < |\rho| \cdot (1 - 2^e)\]
for some \(\rho\) of length \(\ell_{2e}\), we enumerate \([\rho]\) into the complement of \(Q\), thus removing \(\rho\) from \(L(\ell_{2e})\). The choice of \(f\) ensures that \(L(\ell_{2e})\) will remain non-empty.

**Construction.** We say that \(R_{2e}\) requires attention at stage \(s + 1\) if either \(\ell_{2e}[s] \uparrow\) or \((2.6)\) holds for some \(\rho\) in \(L(\ell_{2e})[s]\). We say that \(R_{2e+1}\) requires attention at stage \(s + 1\) if either \(\ell_{2e+1}[s] \uparrow\) or \(s + 1\) is an \(e\)-expansionary stage. At stage \(s + 1\) proceed as follows for the least \(i < s\) such that \(R_i\) requires attention. Suppose that \(i = 2e\) for some \(e \in \mathbb{N}\). If \(\ell_{2e}[s] \uparrow\) then pick a large number \(t\), for each \(\sigma \in L(\ell_{2e-1})[s]\) remove all but one extensions of \(\sigma\) of length \(t\) and set \(\ell_{2e}[s + 1] = f(t, e)\). If \(\ell_{2e}[s] \downarrow\), for each \(\sigma \in L(\ell_{2e-1})[s]\) (putting \(\ell_{i-1} = 0\)) remove any extensions \(\rho\) of \(\sigma\) of length \(\ell_{2e}[s]\) which satisfy \((2.6)\). Now suppose that \(i = 2e + 1\) for some \(e \in \mathbb{N}\). If \(\ell_{2e+1}[s] \uparrow\) define \(\ell_{2e+1}[s + 1]\) to be a large number and for each \(\sigma \in L(\ell_{2e})[s]\) remove all but one extensions of \(\sigma\) of length \(\ell_{2e+1}[s + 1]\). If \(\ell_{2e+1}[s] \downarrow\) choose some \(\sigma \in L(\ell_{2e})[s]\) such that \(s + 1\) is \(\sigma\)-expansionary, consider a string \(\rho \supset \sigma\) in \(T[s]\) of length greater than \(\ell_{2e+1}[s]\) such that \((2.5)\) holds and chop from \(T\) all paths which extend \(\sigma\) and are incompatible with \(\rho\). Let \(\ell_{2e+1}[s + 1] := |\rho|\) and ensure that each string in \(L(\ell_{2e})[s + 1]\) has exactly one extension in \(L(\ell_{2e+1})[s + 1]\) by chopping superfluous branches. When any strategy acts, all lower priority strategies are initialized.
Verification. The construction computably enumerates clopen sets into the complement of $Q$, so the set $Q$ of reals that are not prefixed by any strings enumerated by the construction is a $\Pi^0_1$ set. By induction we show that the parameters $\ell_n$, $L(\ell_n)$ reach a limit and that the corresponding requirements are satisfied. Suppose that this holds for all $i < n$. If $n$ is even, when $\ell_{n-1}$, $L(\ell_{n-1})$ reach a limit strategy $R_n$ will define $\ell_n$ and after a finite number of stages $L(\ell_n)$ will stabilize, containing at least one extension for each $\rho \in L(\ell_{n-1})$ with the property that $\neg (2.6)$ (see the analysis of strategy $R_{2e}$). If $n$ is odd, say $2e + 1$, by the analysis of $R_{2e+1}$ this strategy will only act finitely many times with respect to each string in $L(\ell_{n-1})$, thus finitely many times over all. It will define final values for $\ell_n$, $L(\ell_n)$ at some stage $s$ and $R_n$ will be satisfied as there will be no more $e$-expansionary stages after stage $s$.

As discussed above, there are no incomplete c.e. Turing degrees of positive effective dimension. However this no longer holds for effective packing dimension. In fact [DG08, Corollary 1.6] asserts that a c.e. degree computes a real with positive effective packing dimension iff it is array non-computable. Since every $\Pi^0_1$ class with no $K$-trivial members then there exists a set $A$ of packing dimension 1 and c.e. degree, such that $X \not\leq_{LR} A$ for all $X \in P$.

Note that if we do not require the set $A$ to be of c.e. degree in corollary 2.7 then the result would follow from the Baire category theorem, given Theorem 2.6 and the fact that the class of sets of packing dimension 1 is comeager. Using similar techniques as in Theorem 2.6 we can show the following, which can be seen as a strengthening of [JS72b, Theorem 4.7] (also, compare with Theorem 2.3).

**Theorem 2.8.** There is a perfect $\Pi^0_1$ class such that any two distinct members of it are LR incomparable.

We only give a brief sketch of the proof, as it does not involve new ideas. According to the above mentioned characterization of $\leq_{LR}$ in [KH07], it suffices to build a perfect $\Pi^0_1$ class $P$ and an oracle $\Sigma^0_1$ class $U_*$ such that $\mu(U^Z_*) < 1$ for each $Z \in 2^\omega$ and

$$\forall X,Y \in P \ (X \neq Y \Rightarrow U^X_* \nsubseteq V^Y_*)$$

for all $e \in \mathbb{N}$, where $(V_i)$ is an effective list of all bounded oracle $\Sigma^0_1$ classes. Given two extendible in $P$ strings $\sigma, \tau$ we describe a strategy which ensures that $U^X_* \nsubseteq V^Y_*$ for all $X \in P \cap [\sigma]$, $Y \in P \cap [\tau]$ and succeeds by adding arbitrarily small (say, at most $2^{-t}$) measure in $U^Z_*$, for each $Z \in 2^\omega$. Once this is clear, the full construction can be induced as a finite injury argument using these atomic strategies, where every time a strategy is injured the measure quota $2^{-t}$ that it holds becomes half of its present value (this ensures that $U_*$ is a bounded $\Sigma^0_1$ class). The strategy is to pick $2^t$ incomparable strings $\rho_i$ extending $\sigma$, which are currently extendible in $P$ (such strings will exist since $\sigma$ is currently extendible in $P$), split $2^t$ into $2^t$ equal intervals of size $2^{-t}$ and enumerate each of them in exactly one $U^i_*$, $i < 2^t$. It also removes from $P$ all reals extending $\sigma$ which do not extend one of $\rho_i$. From this point on, every time that some string $\eta \supseteq \tau$ is found with $U^\eta_* \subseteq V^\eta_*$ for some $j < 2^t$, the strategy removes from $P$ all reals extending $\tau$ which are incomparable with $\eta$ and all reals extending $\rho_j$, and so on. Since $V_*$ is a bounded oracle $\Sigma^0_1$ class,
it follows that this procedure will terminate, leaving at least one $\rho_i$ extendible in $P$ and satisfying the requirement $U^X \not\subseteq V^Y$ for all $X \in P \cap [\sigma], Y \in P \cap [\tau]$. Moreover notice that the strategy has added at most $2^{-t}$ measure in $U^Z$ for any $Z \in 2^\omega$.

$P$ is built through successive applications of this atomic strategy in such a way that we approximate levels $l_0 < l_1 < \cdots$, where by level $l_e$ we have acted to ensure that for all distinct extendible strings $\sigma, \tau$ of this level, $U^X_i \not\subseteq V^Y_i$ for all $X \in P \cap [\sigma]$, $Y \in P \cap [\tau]$ and $i \leq e$. Suppose that $l_e$ and the set of extendible strings of length $l_e$ has stabilized. Now we must consider each ordered pair of strings $(\sigma, \tau)$ of length $l_e$ in turn, and diagonalize with respect to $V_e$. The action we take for the first pair involves specifying $2^t$ extensions of $\sigma$ (for some $t$), any number of which may have to remain extendible in $P$. If we begin to act next for the pair $(\tau, \sigma)$, for example, then we shall actually have to act separately with respect to each of the extensions of $\sigma$ which the previous diagonalization step specified and which remain extendible in $P$. All of this, however, can easily be prioritized, in such a way that the action we take for any pair is only injured a finite number of times, and the action we have to take for each pair of strings of level $l_e$, only involves carrying out the atomic strategy for a finite number of pairs of strings.

3. LR bases for randomness

In [Kuč93] a set $A$ was called a basis for randomness if it is computed by a set which is random relative to $A$. Every computable set is trivially a basis for randomness but in the same paper it was shown that there are noncomputable sets $A$ with this property and that all of them have to be $GL_1$, i.e. $A' \leq_T A \oplus \emptyset'$. In [HNS07, Nie05] it was shown that they coincide with the low for Martin-Löf random sets, and in [BLS08] the following variant of this notion was introduced.

Definition 3.1. A set $A$ is an LR basis for randomness if there is an $A$-random set $B$ such that $A \leq_{LR} B$.

Every low for random set is trivially an LR basis for randomness, but in [BLS08] an LR basis for randomness was constructed which is not low for random. That is, there is some $A \not\leq_{LR} \emptyset$ and a $Z$ which is Martin-Löf random relative to $A$ and $A \leq_{LR} Z$. The following theorem says that the same is not true if we replace Martin-Löf randomness with weak $2$-randomness, much like the Turing degrees case.

Theorem 3.2. If $W \not\leq_{LR} \emptyset$ and $A$ is weakly $2$-random relative to $W$, then $W \not\leq_{LR} A$.

Proof. Let $V$ be an oracle $\Sigma^0_1$ class of bounded measure. Define the family of open sets $M_n = \{Z \mid U^W \cap V^Z \subseteq V^Z\}$ and $M = \cap_n M_i$ (we can assume here that $U^W \cap V^Z$ can be uniformly computed from $W$). Now $M$ consists of the reals $Z$ such that $U^W \subseteq V^Z$. In [BLS08] it was shown that non-trivial LR upper cones have measure 0, hence $\mu(M) = 0$. Since $M_{n+1} \subseteq M_n$ we have $\lim_n \mu(M_n) = 0$ and so $(M_n)$ is a test for weak $2$-randomness relative to $W$. Since $A$ is weakly $2$-random relative to $W$, there must be some $n$ such that $A \not\leq M_n$, so that $U^W \not\subseteq V^A$. Since this argument applies for all oracle $\Sigma^0_1$ classes of bounded measure $V$, we have $W \not\leq_{LR} A$. \[\square\]

\[\text{Notice that LR bases for randomness cannot be random. Indeed, if } A \text{ is random and } Z \text{ is random relative to } A, \text{ then by van Lambalgen’s theorem (see [Nie05 Theorem 3.4.6]) } A \text{ is random relative to } Z. \text{ Hence } A \not\leq_{LR} Z.\]
We note that the LR basis for randomness constructed in \[BLS08b\] is not \(\Delta^0_2\). Given that this example happened to be a low for \(\Omega\) real, it is natural to ask the following.

**Question 3.3.** What is the relation between the property LR basis for randomness and known lowness notions, like low for \(\Omega\) and hyperimmune-free?

Using the methods of section 2 and of \[Kuc93\] we show that all LR bases for randomness are GL\(_1\). We need the following.

**Lemma 3.4.** For every set \(A\) which is not low for random and every oracle \(\Sigma^0_1\) class \(V\) of bounded measure there exists a function \(g \leq_T A \oplus 0'\) such that \(\mu\{M \mid U^A|g(n) \subseteq V^M\} < 2^{-n}\).

This lemma follows from the fact that LR upper cones have measure 0 (see \[BLS08a\]) and that \(S = \cap_j S_j\) where

\[
S = \{X \mid U^A \subseteq V^X\} \quad \quad \quad S_j = \{Y \mid U^{A|j} \subseteq V^Y\}.
\]

Now \(\lim_j \mu(S_j) = \mu(S) = 0\), so given \(n \in \mathbb{N}\) the oracles \(A\) and 0’ can find some \(j\) such that \(\mu(S_j) < 2^{-n}\).

**Theorem 3.5.** If \(A\) is an LR basis for randomness then \(A' \equiv_T A \oplus 0'\), i.e. \(A\) is generalized low.

**Proof.** Let \(f \leq_T A\) be a partial function which, given \(e \in \mathbb{N}\), waits until a stage \(s\) where \(e\) is enumerated into \(A'\) and defines \(f(e) = s\) for the least such \(s\). Suppose that \(A\) is not low for random (otherwise \(A\) is anyway low) and \(A \leq_{LR} X\) for some \(A\)-random set \(X\). Then there is an oracle \(\Sigma^0_1\) class \(V\) of bounded measure such that \(U^A \subseteq V^X\). Consider the following recursive sequence of \(\Sigma^0_1\) classes

\[
C_e = \begin{cases} 
\{M \mid U^A|f(e) \subseteq V^M\}, & \text{if } n \in A' \\
\emptyset, & \text{if } n \notin A'
\end{cases}
\]

and define \(B_e \subseteq C_e\) as follows: start enumerating \(C_e\) into \(B_e\) until \(\mu(B_e) = 2^{-e}\), in which case stop the enumeration. Now it is clear that \((B_e)\) is a Martin-Löf test relative to \(A\) and therefore, any \(A\)-random set has to avoid \(B_i\) for all but finitely \(i \in \mathbb{N}\). In order to show that \(A' \leq_T A \oplus 0'\) it suffices to consider the function \(g\) of Lemma 3.4 and show that

\[
(3.1) \quad f(e) \downarrow \Rightarrow f(e) \leq g(e)
\]

for all but finitely many \(e \in \mathbb{N}\). This is true because when \(3.1\) fails for some \(e \in \mathbb{N}\) by definition of \(B_e\) we have \(X \in B_e\).

Given that the separation of the notions of K-triviality and LR-bases for Martin-Löf randomness was given in \[BLS08b\] through a set which was not \(\Delta^0_2\), it is natural to ask if the two notions differ inside \(\Delta^0_2\).

**Question 3.6.** Do the bases for Martin-Löf randomness and the LR-bases for Martin-Löf randomness coincide within \(\Delta^0_2\)?

Recall that within the \(\Delta^0_2\) class, a set is low for \(\Omega\) iff it is low for Martin-Löf random if it is a basis for Martin-Löf randomness. See \[Nie09\] Section 5.1. In light of this connection, Question 3.6 might have a positive answer.
4. Arithmetical complexity LR below $\emptyset'$

Recently there has been a growing interest in the class of oracles $X \leq_{LR} \emptyset'$, see section 5.6 of [Nie09]. This can be seen as a generalization of the class of oracles $X \leq_T \emptyset'$ and it contains the oracles relative to which randomness is at most as strong as 2-randomness. In this sense it is quite natural. In [BLS08b] it was shown that it contains a set of hyperimmune-free Turing degree, while in [BMN] it was shown that it contains a weakly 2-random set. Hence a number of notions which do not have representatives in the class $\Delta_0^2$ can be found $\leq_{LR} \emptyset'$. In this section we continue this line of investigation, showing that there is a proper hyperarithmetical hierarchy of LR degrees below the LR degree of the halting problem. An LR degree is $\Delta_0^2$ if it contains a $\Delta_0^2$ set. Similarly, it is $\Delta_0^\alpha$ (where $\alpha$ is a computable ordinal) if it contains a set in $\Delta_0^\alpha$. For the definition of the hyperarithmetical hierarchy we refer the reader to [AK00]. Recall from [AK00] that given Kleene’s $\mathcal{O}$ as a system of notations for the computable ordinals we can define the sets $H(a)$ for $a \in \mathcal{O}$ by recursion, in such way that $H(x) \equiv_T H(y)$ for notations $x, y \in \mathcal{O}$ representing the same ordinal. Given a computable ordinal $\alpha$, let $H^{(\alpha)}$ be some $H(a)$ for a notation $a \in \mathcal{O}$ such that $a$ represents $\alpha$. For an infinite ordinal $\alpha$ we let $\Delta_0^\alpha$ be the class of oracles which are computable from $H^{(\alpha)}$; for finite ordinals $n$ let $\Delta_0^n$ be the usual arithmetical class $\Sigma_0^n \cap \Pi_0^n$ (notice the non-uniformity in the transition from the finite to the infinite case).

**Theorem 4.1.** For each computable ordinal $\alpha \geq 2$ there is an LR degree below the LR degree of $\emptyset'$, which is $\Delta_0^\alpha$ and is not $\Delta_0^\gamma$ for any $\gamma < \alpha$.

Theorem 4.1 is illustrated in figure 1 for the first few levels of the hyperarithmetical hierarchy. For the proof we will use arguments with $\Pi_1^0$ classes (in contrast to the results in [BLS08b, BMN]) and in particular, we start with the $\Pi_1^0$ class $P$ constructed in [BLS08b], which does not contain any K-trivials and $X \leq_{LR} \emptyset'$ for all $X \in P$.

First assume that $\alpha = \beta + 1$ is a successor computable ordinal and $\beta > 1$. It suffices to construct a set $X \leq_T \emptyset^{(\beta+1)}$ (the non-uniformity in notation in the transition from the finite to the infinite case means that, for the finite case, the following argument gives $X \in \Delta_0^{\alpha+1} - \Delta_0^{\alpha}$), such that the following requirements
are satisfied\(^5\)

\[ R_e : \text{If } \Phi_e^{\langle \beta \rangle} = Z_e \text{ is total and } U_e^{Z_e} \subseteq V_e^X \text{ then } Z_e \leq_{LR} \emptyset \]

The construction will be computable in \(\emptyset^{\langle \beta \rangle}\) and \(X\) will be approximated effectively in \(\emptyset^{\langle \beta \rangle}\). Notice that since \(\beta > 1\) we can ask \(\Sigma_0^1\) questions during the construction. We have parameters \(O_e\) which will be defined (and revised) during the construction and will be either \(\emptyset\) (if we believe that \(Z_e\) is either K-trivial or not total) or a non-empty \(\Pi_1^0\) class (otherwise). Let

\[ J_s = \{ j \mid O_j[s] \neq \emptyset \} \]

and let \(X_s\) be the leftmost path of \((\cap j \in J_s, O_j) \cap P\) of length \(s\) (by convention the empty intersection equals the entire Cantor space). We set \(O_e[0] = \emptyset\) for all \(e \in \mathbb{N}\).

At stage \(s + 1\) look for the least \(i \leq s\) such that \(i \notin J_s\) and for some \(n < s\) such that the segment \(Z_i \upharpoonright n\) is defined and if

\[ Q_{i,n} = \{ Y \mid U_{Z_i[n]} \not\subseteq V^{Y}_i \text{ and } X[s] \mid i < Y \} \]

the class \(\cap\{O_j \mid j \in J_s \land j < i\}\) \(\cap P \cap Q_{i,n}\) is nonempty. If there is such \(i\), set \(O_i = Q_{i,n}\) and \(O_j = \emptyset\) for all \(j > i\). The sequence \((X_s)\) is computable in \(\emptyset^{\langle \beta \rangle}\) and converges so \(X \leq_T \emptyset^{\langle (\beta + 1) \rangle}\). By induction we show that for all \(e \in \mathbb{N}\) parameter \(O_e\) reaches a limit and \(R_e\) is satisfied. Indeed, suppose that this holds for all \(i < e\) and \(s_0\) is a stage where no \(R_i\), \(i < e\) receives attention. If \(R_e\) receives attention after \(s_0\), it will never receive attention again. Also, if \(\Phi_e^{\langle \beta \rangle} = Z_e\) is total and \(Z_e\) is not K-trivial then by Lemma\(^2\)\(\ref{lem:K-trivial}\) some \(k\), \(s\) will be found such that \(\cap\{O_i \mid i \in J_s \land i < e\}\) \(\cap P \cap Q_{e,k} \neq \emptyset\), and \(O_e\) will permanently take the value \(Q_{e,k}\). Since \(X \in Q_{e,k}\), requirement \(R_e\) is satisfied.

Finally assume that \(\alpha\) is a limit ordinal. But then \(\emptyset^{\langle \alpha \rangle}\) has computable access to the triple jumps of all oracles \(\emptyset^{\beta}\) for \(\beta < \alpha\). This means that it can instantly decide if they are K-trivial (given that K-triviality is a \(\Sigma_0^1\) property) and hence, whether to act for \(R_e\) or not. If it decides to act, it defines \(O_i = Q_{e,n}\) for a large enough \(n\) and proceeds with \(R_{e+1}\). The construction now is just a forcing argument with \(\Pi_1^0\) classes, in particular there is no injury in contrast with the successor ordinal case.

References


\(^5\)Notice that these requirements guarantee a stronger result: the LR degree that we construct is \(\Delta_0^0\) and it does not bound any non-trivial LR degrees which are \(\Delta_0^\gamma\) for some \(\gamma < \alpha\). Thus Theorem\(^4\) could be stated in this more general form. With a little bit more effort it is possible to construct (given any computable ordinal \(\alpha\)) an LR degree which is \(\Delta_0^\alpha\) and it is incomparable with any non-trivial LR degree which is \(\Delta_0^\gamma\) for some \(\gamma < \alpha\).


George Barmpalias Institute for Logic, Language and Computation, Universiteit van Amsterdam, P.O. Box 94242, 1090 GE Amsterdam, The Netherlands.

E-mail address: barmpalias@gmail.com
URL: http://www.barmpalias.net/