COMPACTNESS ARGUMENTS WITH EFFECTIVELY CLOSED SETS FOR THE STUDY OF RELATIVE RANDOMNESS.

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Abstract. We present a variety of compactness arguments with Π^0_1 classes which yield results about relative randomness, and in particular properties of the LR degrees. Recall that two sets A, B have the same LR degree if Martin-Löf randomness relative to A coincides with Martin-Löf randomness relative to B. It is remarkable that in some cases, these arguments currently seem to be the only way to prove certain facts about the LR degrees. Hence they seem to play a more important role than in the context of the Turing degrees, where they were originally applied by Jockusch and Soare in their study of Π^0_1 classes and degrees of theories.

1. Introduction

The systematic use of compactness arguments for the study of the Turing degrees was initiated by Jockusch and Soare in their study of Π^0_1 classes and degrees of theories. For example, they showed that every non-empty Π^0_1 class contains a pair of paths whose Turing degrees have limit infimum 0. Of course, this was an alternative approach to building minimal pairs of Turing degrees, since the original argument was a forcing construction by Kleene and Post [KP54].

In the context of the LR degrees, an important measure for studying relative randomness, the approach of compactness arguments and the use of Π^0_1 classes seems to be more than an alternative methodology. For some problems were solved using this approach, although no other approach has been shown to work. As an example, we mention the construction of a minimal pair of LR degrees LR below the halting problem which was given in [BLN10]. This problem has resisted approaches that are more common in the Turing degrees, e.g. finite extension arguments or full approximation arguments. In fact, minimal pairs is a theme where the standard methods that are used in the Turing degrees are not effective in the LR degrees. This was demonstrated in [Bar10] where it was shown that there is no pair of Δ^0_2 sets which form a minimal pair in the LR degrees.

In this paper we use compactness arguments with effectively closed sets in order to derive a number of interesting results about the structure of the LR degrees.

We recall that a set is (Martin-Löf) random[1] if it does not belong in any 'effectively null' set in the Cantor space. Effectively null sets were defined to be those of the form $\cap_j E_j$ where (E_j) is a uniformly c.e. sequence of Σ^0_1 classes such that $\mu(E_j) < 2^{-j-1}$. We say that $A \leq_{LR} B$ (in words, A is LR reducible to B) if

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The results proved in this paper were mentioned—without proof—in [BLN10]. I am grateful to A.E.M. Lewis and K.-M. Ng for discussing these arguments.

[1]In the following when a set is said to be random, we mean Martin-Löf random.

every random sequence relative to B is also random relative to A. The induced equivalence relation $\equiv_{LR}$ identifies two oracles which correspond to the same notion of relative randomness. The equivalence classes are called LR degrees and form a partially ordered structure.

Recall that Martin-Löf randomness is equivalent to the so-called Chaitin-Levin randomness, which is based on Kolmogorov's idea of incompressibility of binary strings.[2] Let K denote the prefix-free complexity relative to a fixed universal prefix-free machine.[3] A set X is called Chaitin-Levin random if its initial segments all have high K-complexity, i.e. $K(X \upharpoonright n) \geq n - c$ for all $n \in \mathbb{N}$ and a constant c.

The least LR degree consists of the low for random sets (sets A such that every Martin-Löf random is also Martin-Löf random relative to A), which coincide with the low for K sets (sets A such that $K(\sigma) \leq^{+} K^{A}(\sigma)$) or even the K-trivial sets (sets A whose prefix-free complexity is less than the prefix-free complexity of a computable sequence, modulo a constant). The equivalence of these three notions is one of the most important recent results in the area of algorithmic randomness and was shown in [Nie05].

In Section 2 we use the methodology of [JS72a, JS72b] to show that the LR degree spectrum of every non-empty Π^0_1 class with no K-trivial members (i.e. the class of LR degrees which contain members of the Π^0_1 class) contains an antichain of size $2^{\aleph_0}$. This extends a result in [BLS07] which referred to Π^0_1 classes of positive measure and implies that such antichains occur in many LR lower cones. We also show that the LR upper closure of a Π^0_1 class P which contains no K-trivials is meager, and that for every such class P there is another Π^0_1 class Q which consists of paths of effective packing dimension 1 such that $X \not\leq_{LR} Y$ for all $X \in P$, $Y \in Q$. The latter result can be seen as an LR analog of a result of Cole and Simpson in [CS07] which referred to the Turing degrees. It also implies that for every Π^0_1 class P with no K-trivials there exists a set A of packing dimension 1 such that $X \not\leq_{LR} A$ for all $X \in P$, however we indicate how to obtain this as an application of the Baire category theorem.

Section 3 is a digression to the related topic of LR bases for randomness. In [Kuč93] Kučera introduced and studied the lowness notion of a basis for Martin-Löf randomness, which is a set A that is computed by an A-random (Martin-Löf random relative to A) set. It was later shown in [HNS07] that this class coincides with the K-trivial sets. In [BLS08b] it was shown that there are sets A which are not K-trivial but $A \leq_{LR} X$ for some A-random set X. If $A \leq_{LR} X$ for some A-random set X we call A an LR basis for randomness. We show that despite the result in [BLS08b], all LR bases for randomness are generalized low and we pose some questions motivating a more thorough investigation.

In Section 4 we study the sets $\leq_{LR} \emptyset'$ using compactness arguments and Π^0_1 classes. This class contains the oracles relative to which Martin-Löf randomness is equivalent to 2-randomness (i.e. Martin-Löf randomness relative to $\emptyset'$) and has recently received considerable attention, e.g. see of [Nie09, Section 5.6] and [BMN].

[2]Given a prefix-free machine M (a Turing machine with prefix-free domain) the prefix-free complexity of a string σ relative to M is the length of the shortest string τ such that $M(\tau) = \sigma$. There is a universal prefix-free machine, i.e. one that gives optimal descriptions to every string, modulo a constant. For more background on prefix-free complexity, see [Nie09].

[3]Let K^Z denote the prefix-free complexity with respect to an oracle universal prefix-free machine with oracle Z.

We show that there is a proper hyperarithmetical hierarchy of LR degrees below the LR degree of the halting problem.

Notation. Recall that an oracle Σ^0_1 class can be identified with a c.e. set of 'axioms' $\langle\tau,\sigma\rangle$ where string τ refers to the oracle and σ refers to a clopen set in the output of the machine. Given an oracle Σ^0_1 class V, a string ρ and a set X we let

$$\begin{aligned} V^\rho &= \{\sigma \mid \exists\tau(\tau\subseteq\rho \ \wedge\ \langle\tau,\sigma\rangle\in V)\} \\ V^X &= \{\sigma \mid \exists\tau(\tau\subset X \ \wedge\ \langle\tau,\sigma\rangle\in V)\}. \end{aligned}$$

Notice that the members of an oracle Martin-Löf test are oracle Σ^0_1 classes. A Σ^0_1 class is called *bounded* it its Lebesgue measure is <1. An oracle Σ^0_1 class V is called *bounded* if there is some $q<1$ such that the measure of V^X is $<q$ for all oracles X. In other words, if there is a uniform bound on the measure of the class, with respect to all oracles. Throughout this paper the following characterization of the LR reducibility from Kjos-Hanssen [KH07] is be freely used: for all $A,B\in\{0,1\}^\omega$ the following are equivalent:

(a) $A\leq_{LR} B$

(b) Every bounded $\Sigma^0_1(A)$ class T^A is contained in a bounded $\Sigma^0_1(B)$ class.

(c) A member U^A of a universal Martin-Löf test relative to A is contained in a bounded $\Sigma^0_1(B)$ class.

In the following, U refers to a member of a universal oracle Martin-Löf test.

2. Jockusch-Soare arguments and randomness reducibilities

We start with a basic tool for avoiding upper cones in the LR degrees. The special case where W is Δ^0_2 was shown in [BLS08b] and was applied, giving some basic results on the structure of LR degrees containing Δ^0_2 sets. Although we are not going to use this result in this paper it is likely to be useful, in the same way that a special case of it was useful in [BLS08b].

Proposition 2.1. *If $W\not\leq_{LR}\emptyset$ then for every bounded oracle Σ^0_1 class V and every 1-generic $Z\leq_{LR}\emptyset$ there is $\sigma\subseteq Z$ such that $U^{W\restriction|\sigma|}\not\subseteq V^\tau$ for all $\tau\supseteq\sigma$.*

Proof. Consider the Σ^0_1 sets of strings $M_n=\{\sigma \mid U^{W\restriction n}\subseteq V^\sigma\}$. Since Z is 1-generic, for every n we either have $U^{W\restriction n}\subseteq V^Z$ or there is some $\sigma\subseteq Z$ such that no reals extending σ are in M_n. If the claim did not hold, only the first possibility can happen. But then $U^W\subseteq V^Z$, which contradicts the fact that $W\not\leq_{LR}\emptyset$. □

The following lemma is an atomic version of LR cone avoidance inside a Π^0_1 class, and is crucial to some of the central results in this section.

Lemma 2.2. *Also let P be a nonempty Π^0_1 class, V a bounded oracle Σ^0_1 class and $A\not\leq_{LR}\emptyset$. Then there exists some $B\in P$ such that $U^A\not\subseteq V^B$.*

Proof. Suppose that for all $B\in P$ we have $U^A\subseteq V^B$. We define a Σ^0_1 class E such that $\mu(E)<1$ and $U^A\subseteq E$, which shows that $A\leq_{LR}\emptyset$. Let

$$E=\{\sigma \mid [\sigma]\subseteq V^Z \text{ for all } Z\in P\}.$$

By hypothesis we have $U^A\subseteq E$ and by compactness E is a Σ^0_1 class. Now take $Z\in P$ which exists since $P\neq\emptyset$. Then $E\subseteq V^Z$ and hence $\mu(E)\leq\mu(V^Z)<1$. □

Recall that the degree spectrum (with respect to some notion of degrees) of a Π^0_1 class is the set of the degrees of its members. In the following we prove an analog of a theorem in [JS72b] about the degree spectrum of a Π^0_1 class for the LR degrees. Notice that a collection of reals that form an antichain of LR degrees also forms an antichain in the Turing degrees.

Theorem 2.3. *The LR degree spectrum of a Π^0_1 class with no K-trivial members contains an antichain of size $2^{\aleph_0}$. Moreover this antichain can be chosen disjoint from any given countable sequence of non-trivial LR upper cones.*

Proof. Let P be a Π^0_1 class which does not contain K-trivial paths. We show the first part of the theorem, and then indicate a modification which gives the more general statement. We inductively define a perfect tree T inside P by repeatedly applying Lemma 2.2, in such a way that for any two paths A, B on T we have $A|_{LR}B$. Along with the nodes T_σ we define Π^0_1 classes $P_\sigma \subseteq P \cap [T_\sigma]$ for $\sigma \in 2^{<\omega}$ which force the condition we wish to meet. Given an effective sequence (V_e) of all bounded oracle Σ^0_1 classes, our requirements are:

$$\forall X, Y \in [T]\ (X \neq Y \Rightarrow U^X \not\subseteq V_e^Y). \tag{2.1}$$

Let $P_\emptyset = P$, $T_\emptyset = \emptyset$ and inductively suppose that $P_\sigma \downarrow, T_\sigma \downarrow$, $P_\sigma \subseteq [T_\sigma]$ for all $\sigma \in 2^e$ such that for all distinct $\sigma, \tau \in 2^e$ we have

$$A \in P_\sigma \Rightarrow U^{T_\tau} \not\subseteq V_e^A \tag{2.2}$$

and, of course, $P_\sigma \neq \emptyset$. Now we define T_ρ, P_ρ for each $\rho \in 2^{e+1}$ in k steps, where k is the number of ordered tuples (σ, τ) for two distinct $\sigma, \tau \in 2^{e+1}$. Let (σ_i, τ_i) denote the ith such tuple and let the parameters at step 0 be as follows:

$$T_{\rho^- * j}[0] = T_{\rho^-} * \tau^j \qquad\qquad P_{\rho^- * j}[0] = P_{\rho^-} \cap [\tau^j]$$

where τ^0, τ^1 are incompatible extensions of T_{ρ^-} which have infinite extensions in P_{ρ^-}. Now given $s < k$ and assuming that $T_\rho[s], P_\rho[s]$ are defined (and $P_\rho[s] \neq \emptyset$) for all $\rho \in 2^{e+1}$, we check if ρ equals σ_s or τ_s. If not, we let the parameters as in the previous stage, i.e.

$$T_\rho[s+1] = T_\rho[s] \qquad\qquad P_\rho[s+1] = P_\rho[s].$$

To define the parameters for σ_s, τ_s we use Lemma 2.2 to find some $A \in P_{\sigma_s}$ such that

$$\{X \in P_{\tau_s} \mid U^A \not\subseteq V_e^X\} \neq \emptyset.$$

Then there must exist some $\eta \subset A$, $\eta \supseteq T_{\sigma_s}[s]$ such that

$$Q = \{X \in P_{\tau_s} \mid U^\eta \not\subseteq V_e^X\}$$

is not empty. Now define

$$T_{\sigma_s}[s+1] = \eta \qquad\qquad P_{\sigma_s}[s+1] = P_{\sigma_s}[s] \cap [\eta]$$
$$T_{\tau_s}[s+1] = T_{\tau_s}[s] \qquad\qquad P_{\tau_s}[s+1] = Q.$$

When we finish all k steps for the eth level it is easy to verify that (2.2) holds. Therefore all requirements for the tree T are satisfied. For the second part of the theorem, suppose we are given a countable sequence of sets (C_i) which are not K-trivial and we wish to construct a tree T inside P such that (2.1) and $C_i \not\leq_{LR} X$

for all $i \in \mathbb{N}$ and $X \in [T]$. All we have to do is to augment the above construction with steps which deal with the requirements

$$\exists n\ [U^{C_i \restriction n} \not\subseteq V_e^X \text{ for all } X \in P_\rho,\ \rho \in 2^{\langle i,e\rangle}]$$

for all $i, e \in \mathbb{N}$. This step is a direct application of Lemma 2.2. $\square$

It is well known (see Sacks [Sac63]) that the Turing degree spectrum of every perfect set of reals contains an antichain of size $2^{\aleph_0}$. It is natural to ask whether the same holds for the LR degrees.

Question 2.4. Does the LR degree spectrum of every perfect set of reals contain an antichain of size $2^{\aleph_0}$?

Recall that the least LR degree consists of the K-trivial sets by [Nie05].

Theorem 2.5. *The LR upper closure of a Π_1^0 class which does not contain K-trivial sets is meager.*

Proof. Let P be a Π_1^0 class which does not contain K-trivial sets and for a contradiction assume that the LR upper closure of P is not meager. Then for some e the set

$$S = \{X \mid \{A \in P \mid U^A \subseteq V_e^X\} \neq \emptyset\}$$

is not meager. Hence there exists a string σ such that every string $\tau \supset \sigma$ extends to member of S. We will define a computable real B in stages, so let $B[s]$ be the segment of B that has been defined by stage s ($B[s]$ can be thought of as a movable marker which moves monotonically through the full binary tree). Define the following length of agreement:

$$\ell[s+1] = \max\{t\ |\exists Y \exists \rho (Y \in P[s+1]\ \wedge\ B[s] \subseteq \rho\ \wedge\ U^{Y \restriction t}[s+1] \subseteq V_e^\rho[s+1])\}.$$

A stage s is called *expansionary* if $\ell[s] > \ell[t]$ for all $t < s$. Let $B[0] = \sigma$ (with σ as above) and if s is an expansionary stage, move $B[s]$ to a string $\rho \supset B[s-1]$ such that

$$\{Y \in P[s] \mid U^{Y \restriction \ell[s]}[s] \subseteq V_e^\rho\} \neq \emptyset.$$

Since S is not meager, B is well defined. Also, if

$$M_s = \{Y \in P \mid U^{Y \restriction s} \subseteq V_e^B\}$$

we have $M_s \neq \emptyset$ for all $s \in \mathbb{N}$. Since these are clopen sets and $M_{s+1} \subseteq M_s$, by compactness $\cap_s M_s \neq \emptyset$. If we consider some $X \in \cap_s M_s$ then $X \in P$ and $U^X \subseteq V_e^B$ which is a contradiction since V_e^B is a bounded Σ_1^0 class but by hypothesis $X \not\leq_{LR} \emptyset$. $\square$

Cole and Simpson showed in [CS07] that given any special Π_1^0 class P (i.e. one containing no computable paths) we can find another special Π_1^0 class Q such that $X \not\leq_T Y$ for all $X \in P$, $Y \in Q$. The analog of this result for the LR degrees is also true: given a Π_1^0 class containing no K-trivials we can find another Π_1^0 class Q containing no K-trivials, such that $X \not\leq_{LR} Y$ for all $X \in P$, $Y \in Q$. It is natural to ask how complex the members of Q could be made in the previous statement. For example, Q cannot be required to contain only random members or, indeed, only members of positive effective dimension. Recall from [May02, AHJE04] (or see [DH09]) that a real X has effective dimension 1 iff $\lim_n \frac{K(X \restriction n)}{n} = 1$ and it has effective packing dimension 1 iff $\limsup_n \frac{K(X \restriction n)}{n} = 1$. The claim above holds because by a theorem of Terwijn [Ter98] every set which has positive effective dimension computes a fixed point free function. Now Arslanov's completeness criterion (e.g. see [Soa87]) says that if a set of c.e. degree computes a fixed point free function then it also computes the halting problem. So given any class P, any constructed class Q which only contains paths of positive effective dimension will contain a Turing complete path (namely the leftmost path) which shows that it has to have a member computing a member of P.

Notice that the class of random sets, and indeed the class of sets of effective dimension 1 is meager. On the other hand, the class of sets of packing dimension 1 is comeager. These observations follow if we view the category of a set as the outcome of a Banach-Mazur game, as detailed in [Odi89]. In a game of extending initials segments of a set X that we are building, it is easy to ensure that X is not of effective dimension 1 (no matter what our opponent plays) by adding long strings of 0s and it is easy to ensure that X is of effective packing dimension 1 (no matter what our opponent plays) by extending with segments of high prefix-free complexity. We show that Q can be required to contain only paths of effective packing dimension 1.

Theorem 2.6. *For any Π^0_1 class P which does not have K-trivial members there is a Π^0_1 class Q which contains only members of effective packing dimension 1, such that $X \not\leq_{LR} Y$ for all $X \in P$, $Y \in Q$.*

Proof. Given P as above it suffices to define the approximation to a Π^0_1 class Q such that

$$R_{2e}: \ X \in Q \Rightarrow \exists n > e\ [K(X \restriction n) > n \cdot (1 - 2^{-e})] \tag{2.3}$$

$$R_{2e+1}: \ X \in Q \Rightarrow \forall Y \in P\ [U^Y \not\subseteq V_e^X] \tag{2.4}$$

where U is a member of the universal oracle Martin-Löf test and (V_e) is an effective list of all bounded Σ^0_1 classes. To define Q we start from the full binary tree $T[0]$ and we start chopping particular branches, by enumerating the corresponding strings (as clopen sets) into the complement of Q. At every stage s the current approximation to Q is the set of all infinite paths through $T[s]$. Level ℓ of a tree is the set of all strings of length ℓ which are extendible in the tree (i.e. which are extended by an infinite path through the tree). Each strategy R_{2e}, R_{2e+1} will operate on a particular level, which may change as the construction progresses. Strategy R_{2e} will operate on level ℓ_{2e} and strategy R_{2e+1} on level ℓ_{2e+1}. We will have $\ell_i[s] < \ell_{i+1}[s]$ for all $i, s \in \mathbb{N}$.

R_{2e+1} *strategy.* We will use a version of the Sacks preservation strategy at level ℓ_{2e+1} of the tree. In order to describe the strategy let us assume inductively that ℓ_{2e} has reached a limit and that level ℓ_{2e} of T has settled, i.e. the strings of this level that are currently extendible in T will remain so. Let us denote the set of strings of level n of T by $L(n)$. For each string σ and $e \in \mathbb{N}$ we define the following *length of agreement* with respect to the eth LR reduction:

$$d_e(\sigma)[s] = \max\{t \mid \exists Y \exists \rho (Y \in P[s] \ \wedge\ \rho \in T[s-1] \ \wedge\ \sigma \subseteq \rho \ \wedge\ U^{Y\restriction t}[s] \subseteq V_e^{\rho}[s])\}.$$

A stage s is called *σ-expansionary* if σ has length $\ell_{2e}[s-1]$ for some $e \in \mathbb{N}$ and $d_e(\sigma)[s] > d_e(\sigma)[t]$ for all $t < s$. A stage s is called *e-expansionary* if it is σ-expansionary for some $\sigma \in L(\ell_{2e})[s-1]$. The strategy for R_{2e+1} is to wait for an e-expansionary stage $s+1$, choose some $\sigma \in L(\ell_{2e})[s]$ such that $s+1$ is σ-expansionary, consider a string $\rho \supset \sigma$ in $T[s]$ of length greater than $\ell_{2e+1}[s]$ such that

$$\{Y \in P[s+1] \mid U^{Y\restriction d_e(\sigma)}[s+1] \subseteq V_e^{\rho}[s+1]\} \neq \emptyset \tag{2.5}$$

and chop from T all paths which extend σ and are incompatible with ρ. We let $\ell_{2e+1}[s+1] := |\rho|$ and for uniformity we also make sure that each string in $L(\ell_{2e})[s+1]$ has exactly one extension in $L(\ell_{2e+1})[s+1]$ (by arbitrarily chopping superfluous branches).

We claim that if we follow this strategy there can only be finitely many e-expansionary stages, so that ℓ_{2e+1} reaches a limit and the same happens for $d_e(\sigma)$ for all $\sigma \in L(\ell_{2e})$. In this case it is clear that no real extending a node in $L(\ell_{2e+1})$ (hence no path through T) can be LR-above a member of P via the eth LR reduction, i.e. R_{2e+1} is satisfied. If there were infinitely many e-expansionary stages then for some (say, leftmost) $\sigma \in L(\ell_{2e})$ there would be infinitely many *σ-expansionary* stages. But then $\ell_{2e+1} \to \infty$ and the construction would define an infinite computable sequence G, namely the unique path through T which extends σ, and if

$$M_s = \{Y \in P \mid U^{Y\restriction s} \subseteq V_e^G\}$$

we would have $M_s \neq \emptyset$ for all $s \in \mathbb{N}$. Since these are clopen sets and $M_{s+1} \subseteq M_s$, by compactness $\cap_s M_s \neq \emptyset$. If we consider some $X \in \cap_s M_s$ then $X \in P$ and $U^X \subseteq V_e^G$ which is a contradiction since V_e^G is a bounded Σ^0_1 class but by hypothesis $X \not\leq_{LR} \emptyset$.

R_{2e} strategy. For (2.3) we will use a computable function f such that for every $m, n \in \mathbb{N}$ and every string σ of length m, there exists $\tau \supset \sigma$ of length $f(m,n)$ such that $K(\tau) > |\tau| \cdot (1 - 2^{-n})$. The strategy for R_{2e} does the following for each σ of level ℓ_{2e-1}. First it chooses an extension τ of σ such that $[\tau] \subseteq Q[s-1]$ (i.e. no paths extending τ have been thrown out of Q so far) and removes all paths extending σ which are incompatible with τ from Q. Then all extensions of τ of length $f(|\tau|, e)$ are currently extendible in Q. It defines $\ell_{2e} = f(|\tau|, e)$ and in the following stages s, whenever

$$K_s(\rho) < |\rho| \cdot (1 - 2^e) \tag{2.6}$$

for some ρ of length ℓ_{2e}, we enumerate $[\rho]$ into the complement of Q, thus removing ρ from $L(\ell_{2e})$. The choice of f ensures that $L(\ell_{2e})$ will remain non-empty.

Construction. We say that R_{2e} *requires attention at stage* $s+1$ if either $\ell_{2e}[s]\uparrow$ or (2.6) holds for some ρ in $L(\ell_{2e})[s]$. We say that R_{2e+1} *requires attention at stage* $s+1$ if either $\ell_{2e+1}[s]\uparrow$ or $s+1$ is an e-expansionary stage. At stage $s+1$ proceed as follows for the least $i < s$ such that R_i requires attention. Suppose that $i = 2e$ for some $e \in \mathbb{N}$. If $\ell_{2e}[s]\uparrow$ then pick a *large* number t, for each $\sigma \in L(\ell_{2e-1})[s]$ remove all but one extensions of σ of length t and set $\ell_{2e}[s+1] = f(t,e)$. If $\ell_{2e}[s]\downarrow$, for each $\sigma \in L(\ell_{2e-1})[s]$ (putting $\ell_{-1} = 0$) remove any extensions ρ of σ of length $\ell_{2e}[s]$ which satisfy (2.6). Now suppose that $i = 2e+1$ for some $e \in \mathbb{N}$. If $\ell_{2e+1}[s]\uparrow$ define $\ell_{2e+1}[s+1]$ to be a *large number* and for each $\sigma \in L(\ell_{2e})[s]$ remove all but one extensions of σ of length $\ell_{2e+1}[s+1]$. If $\ell_{2e+1}[s]\downarrow$ choose some $\sigma \in L(\ell_{2e})[s]$ such that $s+1$ is σ-expansionary, consider a string $\rho \supset \sigma$ in $T[s]$ of length greater than $\ell_{2e+1}[s]$ such that (2.5) holds and chop from T all paths which extend σ and are incompatible with ρ. Let $\ell_{2e+1}[s+1] := |\rho|$ and ensure that each string in $L(\ell_{2e})[s+1]$ has exactly one extension in $L(\ell_{2e+1})[s+1]$ by chopping superfluous branches. When any strategy acts, all lower priority strategies are initialized.

Verification. The construction computably enumerates clopen sets into the complement of Q, so the set Q of reals that are not prefixed by any strings enumerated by the construction is a Π^0_1 set. By induction we show that the parameters ℓ_n, $L(\ell_n)$ reach a limit and that the corresponding requirements are satisfied. Suppose that this holds for all $i < n$. If n is even, when ℓ_{n-1}, $L(\ell_{n-1})$ reach a limit strategy R_n will define ℓ_n and after a finite number of stages $L(\ell_n)$ will stabilize, containing at least one extension for each $\rho \in L(\ell_{n-1})$ with the property that $\neg$(2.6) (see the analysis of strategy R_{2e}). If n is odd, say $2e+1$, by the analysis of R_{2e+1} this strategy will only act finitely many times with respect to each string in $L(\ell_{n-1})$, thus finitely many times over all. It will define final values for ℓ_n, $L(\ell_n)$ at some stage s and R_n will be satisfied as there will be no more e-expansionary stages after stage s. □

As discussed above, there are no incomplete c.e. Turing degrees of positive effective dimension. However this no longer holds for effective packing dimension. In fact [DG08, Corollary 1.6] asserts that a c.e. degree computes a real with positive effective packing dimension iff it is array non-computable. Since every Π^0_1 class contains a path of c.e. Turing degree, Theorem 2.6 implies the following.

Corollary 2.7. *If P is a Π^0_1 class with no K-trivial members then there exists a set A of packing dimension 1 and c.e. degree, such that $X \not\leq_{LR} A$ for all $X \in P$.*

Note that if we do not require the set A to be of c.e. degree in corollary 2.7 then the result would follow from the Baire category theorem, given Theorem 2.5 and the fact that the class of sets of packing dimension 1 is comeager. Using similar techniques as in Theorem 2.6 we can show the following, which can be seen as a strengthening of [JS72b, Theorem 4.7] (also, compare with Theorem 2.3).

Theorem 2.8. *There is a perfect Π^0_1 class such that any two distinct members of it are LR incomparable.*

We only give a brief sketch of the proof, as it does not involve new ideas. According to the above mentioned characterization of $\leq_{LR}$ in [KH07], it suffices to build a perfect Π^0_1 class P and an oracle Σ^0_1 class $U_\star$ such that $\mu(U_\star^Z) < 1$ for each $Z \in 2^\omega$ and

$$\forall\, X, Y \in P\ (X \neq Y \Rightarrow U_\star^X \not\subseteq V_e^Y)$$

for all $e \in \mathbb{N}$, where (V_i) is an effective list of all bounded oracle Σ^0_1 classes. Given two extendible in P strings σ, τ we describe a strategy which ensures that $U_\star^X \not\subseteq V_e^Y$ for all $X \in P \cap [\sigma]$, $Y \in P \cap [\tau]$ and succeeds by adding arbitrarily small (say, at most 2^{-t}) measure in $U_\star^Z$, for each $Z \in 2^\omega$. Once this is clear, the full construction can be induced as a finite injury argument using these atomic strategies, where every time a strategy is injured the measure quota 2^{-t} that it holds becomes half of its present value (this ensures that $U_\star$ is a *bounded* Σ^0_1 class). The strategy is to pick 2^t incomparable strings ρ_i extending σ, which are currently extendible in P (such strings will exist since σ is currently extendible in P), split 2^ω into 2^t equal intervals of size 2^{-t} and enumerate each of them in exactly one $U_\star^{\rho_i}$, $i < 2^t$. It also removes from P all reals extending σ which do not extend one of ρ_i. From this point on, every time that some string $\eta \supseteq \tau$ is found with $U_\star^{\rho_j} \subseteq V_e^\eta$ for some $j < 2^t$, the strategy removes from P all reals extending τ which are incomparable with η and all reals extending ρ_j, and so on. Since V_e is a bounded oracle Σ^0_1 class, it follows that this procedure will terminate, leaving at least one ρ_i extendible in P and satisfying the requirement $U_\star^X \not\subseteq V_e^Y$ for all $X \in P \cap [\sigma]$, $Y \in P \cap [\tau]$. Moreover notice that the strategy has added at most 2^{-t} measure in $U_\star^Z$ for any $Z \in 2^\omega$.

P is built through successive applications of this atomic strategy in such a way that we approximate levels $l_0 < l_1 < \cdots$, where by level l_e we have acted to ensure that for all distinct extendible strings σ, τ of this level, $U_\star^X \not\subseteq V_i^Y$ for all $X \in P \cap [\sigma]$, $Y \in P \cap [\tau]$ and $i \leq e$. Suppose that l_e and the set of extendible strings of length l_e has stabilized. Now we must consider each ordered pair of strings (σ, τ) of length l_e in turn, and diagonalize with respect to V_e. The action we take for the first pair involves specifying 2^t extensions of σ (for some t), any number of which may have to remain extendible in P. If we begin to act next for the pair (τ, σ), for example, then we shall actually have to act separately with respect to each of the extensions of σ which the previous diagonalization step specified and which remain extendible in P. All of this, however, can easily be prioritized, in such a way that the action we take for any pair is only injured a finite number of times, and the action we have to take for each pair of strings of level l_e, only involves carrying out the atomic strategy for a finite number of pairs of strings.

3. LR bases for randomness

In [Kuč93] a set A was called a *basis for randomness* if it is computed by a set which is random relative to A. Every computable set is trivially a basis for randomness but in the same paper it was shown that there are noncomputable sets A with this property and that all of them have to be GL_1, i.e. $A' \leq_T A \oplus \emptyset'$. In [HNS07, Nie05] it was shown that they coincide with the low for Martin-Löf random sets, and in [BLS08b] the following variant of this notion was introduced.

Definition 3.1. A set A is an LR basis for randomness if there is an A-random set B such that $A \leq_{LR} B$.

Every low for random set is trivially an LR basis for randomness, but in [BLS08b] an LR basis for randomness was constructed which is not low for random.[4] That is, there is some $A \not\leq_{LR} \emptyset$ and a Z which is Martin-Löf random relative to A and $A \leq_{LR} Z$. The following theorem says that the same is not true if we replace Martin-Löf randomness with weak 2-randomness, much like the Turing degrees case.

Theorem 3.2. *If $W \not\leq_{LR} \emptyset$ and A is weakly 2-random relative to W, then $W \not\leq_{LR} A$.*

Proof. Let V be an oracle Σ_1^0 class of bounded measure. Define the family of open sets $M_n = \{Z \mid U^{W\restriction n} \subseteq V^Z\}$ and $M = \cap_i M_i$ (we can assume here that $U^{W\restriction n}$ can be uniformly computed from W). Now M consists of the reals Z such that $U^W \subseteq V^Z$. In [BLS08b] it was shown that non-trivial LR upper cones have measure 0, hence $\mu(M) = 0$. Since $M_{n+1} \subseteq M_n$ we have $\lim_n \mu(M_n) = 0$ and so (M_n) is a test for weak 2-randomness relative to W. Since A is weakly 2-random relative to W, there must be some n such that $A \notin M_n$, so that $U^W \not\subseteq V^A$. Since this argument applies for all oracle Σ_1^0 classes of bounded measure V, we have $W \not\leq_{LR} A$. $\square$

[4]Notice that LR bases for randomness cannot be random. Indeed, if A is random and Z is random relative to A, then by van Lambalgen's theorem (see [Nie09, Theorem 3.4.6]) A is random relative to Z. Hence $A \not\leq_{LR} Z$.

We note that the LR basis for randomness constructed in [BLS08b] is not Δ^0_2. Given that this example happened to be a low for Ω real, it is natural to ask the following.

Question 3.3. What is the relation between the property *LR basis for randomness* and known lowness notions, like *low for* Ω and *hyperimmune-free*?

Using the methods of section 2 and of [Kuč93] we show that all LR bases for randomness are GL_1. We need the following.

Lemma 3.4. *For every set A which is not low for random and every oracle Σ^0_1 class V of bounded measure there exists a function $g \leq_T A\oplus\emptyset'$ such that $\mu\{M \mid U^{A\restriction g(n)} \subseteq V^M\} < 2^{-n}$.*

This lemma follows from the fact that LR upper cones have measure 0 (see [BLS08a]) and that $S = \cap_j S_j$ where

$$S = \{X \mid U^A \subseteq V^X\} \qquad\qquad S_j = \{Y \mid U^{A\restriction j} \subseteq V^Y\}.$$

Now $\lim_j \mu(S_j) = \mu(S) = 0$, so given $n \in \mathbb{N}$ the oracles A and $\emptyset'$ can find some j such that $\mu(S_j) < 2^{-n}$.

Theorem 3.5. *If A is an LR basis for randomness then $A' \equiv_T A \oplus \emptyset'$, i.e. A is generalized low.*

Proof. Let $f \leq_T A$ be a partial function which, given $e \in \mathbb{N}$, waits until a stage s where e is enumerated into A' and defines $f(e) = s$ for the least such s. Suppose that A is not low for random (otherwise A is anyway low) and $A \leq_{LR} X$ for some A-random set X. Then there is an oracle Σ^0_1 class V of bounded measure such that $U^A \subseteq V^X$. Consider the following recursive sequence of $\Sigma^0_1(A)$ classes

$$C_e = \begin{cases} \{M \mid U^{A\restriction f(e)} \subseteq V^M\}, & \text{if } n \in A' \\ \emptyset, & \text{if } n \notin A' \end{cases}$$

and define $B_e \subseteq C_e$ as follows: start enumerating C_e into B_e until $\mu(B_e) = 2^{-e}$, in which case stop the enumeration. Now it is clear that (B_e) is a Martin-Löf test relative to A and therefore, any A-random set has to avoid B_i for all but finitely $i \in \mathbb{N}$. In order to show that $A' \leq_T A \oplus \emptyset'$ it suffices to consider the function g of Lemma 3.4 and show that

$$f(e)\downarrow \Rightarrow f(e) \leq g(e) \tag{3.1}$$

for all but finitely many $e \in \mathbb{N}$. This is true because when 3.1 fails for some $e \in \mathbb{N}$ by definition of B_e we have $X \in B_e$. □

Given that the separation of the notions of K-triviality and LR-bases for Martin-Löf randomness was given in [BLS08b] through a set which was not Δ^0_2, it is natural to ask if the two notions differ inside Δ^0_2.

Question 3.6. Do the bases for Martin-Löf randomness and the LR-bases for Martin-Löf randomness coincide within Δ^0_2?

Recall that within the Δ^0_2 class, a set is low for Ω iff it is low for Martin-Löf random iff it is a basis for Martin-Löf randomness. See [Nie09, Section 5.1]. In light of this connection, Question 3.6 might have a positive answer.

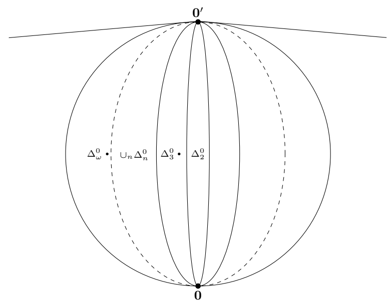

FIGURE 1. Illustration of Theorem 4.1.

4. ARITHMETICAL COMPLEXITY LR BELOW $\emptyset'$

Recently there has been a growing interest in the class of oracles $X \leq_{LR} \emptyset'$, see section 5.6 of [Nie09]. This can be seen as a generalization of the class of oracles $X \leq_T \emptyset'$ and it contains the oracles relative to which randomness is at most as strong as 2-randomness. In this sense it is quite natural. In [BLS08b] it was shown that it contains a set of hyperimmune-free Turing degree, while in [BMN] it was shown that it contains a weakly 2-random set. Hence a number of notions which do not have representatives in the class Δ^0_2 can be found $\leq_{LR} \emptyset'$. In this section we continue this line of investigation, showing that there is a proper hyperarithmetical hierarchy of LR degrees below the LR degree of the halting problem. An LR degree is Δ^0_2 if it contains a Δ^0_2 set. Similarly, it is Δ^0_α (where α is a computable ordinal) if it contains a set in Δ^0_α. For the definition of the hyperarithmetical hierarchy we refer the reader to [AK00]. Recall from [AK00] that given Kleene's $\mathcal{O}$ as a system of notations for the computable ordinals we can define the sets $H(a)$ for $a \in \mathcal{O}$ by recursion, in such way that $H(x) \equiv_T H(y)$ for notations $x, y \in \mathcal{O}$ representing the same ordinal. Given a computable ordinal α, let $\emptyset^{(\alpha)}$ be some $H(a)$ for a notation $a \in \mathcal{O}$ such that a represents α. For an infinite ordinal α we let Δ^0_α be the class of oracles which are computable from $\emptyset^{(\alpha)}$; for finite ordinals n let Δ^0_n be the usual arithmetical class $\Sigma^0_n \cap \Pi^0_n$ (notice the non-uniformity in the transition from the finite to the infinite case).

Theorem 4.1. *For each computable ordinal $\alpha \geq 2$ there is an LR degree below the LR degree of $\emptyset'$, which is Δ^0_α and is not Δ^0_γ for any $\gamma < \alpha$.*

Theorem 4.1 is illustrated in figure 1 for the first few levels of the hyperarithmetical hierarchy. For the proof we will use arguments with Π^0_1 classes (in contrast to the results in [BLS08b, BMN]) and in particular, we start with the Π^0_1 class P constructed in [BLS08b], which does not contain any K-trivials and $X \leq_{LR} \emptyset'$ for all $X \in P$.

First assume that $\alpha = \beta + 1$ is a successor computable ordinal and $\beta > 1$. It suffices to construct a set $X \leq_T \emptyset^{(\beta+1)}$ (the non-uniformity in notation in the transition from the finite to the infinite case means that, for the finite case, the following argument gives $X \in \Delta^0_{\alpha+1} - \Delta^0_\alpha$), such that the following requirements are satisfied.[5]

$$R_e : \text{If } \Phi_e^{\emptyset^{(\beta)}} = Z_e \text{ is total and } U^{Z_e} \subseteq V_e^X \text{ then } Z_e \leq_{LR} \emptyset$$

The construction will be computable in $\emptyset^{(\beta)}$ and X will be approximated effectively in $\emptyset^{(\beta)}$. Notice that since $\beta > 1$ we can ask Σ^0_1 questions during the construction. We have parameters O_e which will be defined (and revised) during the construction and will be either $\emptyset$ (if we believe that Z_e is either K-trivial or not total) or a non-empty Π^0_1 class (otherwise). Let

$$J_s = \{j \mid O_j[s] \neq \emptyset\}$$

and let X_s be the leftmost path of $(\cap_{j \in J_s} O_j) \cap P$ of length s (by convention the empty intersection equals the entire Cantor space). We set $O_e[0] = \emptyset$ for all $e \in \mathbb{N}$. At stage $s+1$ look for the least $i \leq s$ such that $i \notin J_s$ and for some $n < s$ such that the segment $Z_i \restriction n$ is defined and if

$$Q_{i,n} = \{Y \mid U^{Z_i \restriction n} \not\subseteq V_i^Y \text{ and } X[s] \restriction i \subset Y\}$$

the class $\cap\{O_j \mid j \in J_s \;\wedge\; j < i\}) \cap P \cap Q_{i,n}$ is nonempty. If there is such i, set $O_i = Q_{i,n}$ and $Q_j = \emptyset$ for all $j > i$. The sequence (X_s) is computable in $\emptyset^{(\beta)}$ and converges so $X \leq_T \emptyset^{(\beta+1)}$. By induction we show that for all $e \in \mathbb{N}$ parameter O_e reaches a limit and R_e is satisfied. Indeed, suppose that this holds for all $i < e$ and s_0 is a stage where no R_i, $i < e$ receives attention. If R_e receives attention after s_0, it will never receive attention again. Also, if $\Phi_e^{\emptyset^{(\beta)}} = Z_e$ is total and Z_e is not K-trivial then by Lemma 2.2 some k, s will be found such that $\cap\{O_i \mid i \in J_s \;\wedge\; i < e\}) \cap P \cap Q_{e,k} \neq \emptyset$, and O_e will permanently take the value $Q_{e,k}$. Since $X \in Q_{e,k}$, requirement R_e is satisfied.

Finally assume that α is a limit ordinal. But then $\emptyset^{(\alpha)}$ has computable access to the triple jumps of all oracles $\emptyset^{\beta}$ for $\beta < \alpha$. This means that it can instantly decide if they are K-trivial (given that K-triviality is a Σ^0_3 property) and hence, whether to act for R_e or not. If it decides to act, it defines $O_i = Q_{e,n}$ for a large enough n and proceeds with R_{e+1}. The construction now is just a forcing argument with Π^0_1 classes, in particular there is no injury in contrast with the successor ordinal case.

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[5] Notice that these requirements guarantee a stronger result: the LR degree that we construct is Δ^0_α and it *does not bound* any non-trivial LR degrees which are Δ^0_γ for some $\gamma < \alpha$. Thus Theorem 4.1 could be stated in this more general form. With a little bit more effort it is possible to construct (given any computable ordinal α) an LR degree which is Δ^0_α and it is incomparable with any non-trivial LR degree which is Δ^0_γ for some $\gamma < \alpha$.

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