On the Monotonic Computability of Semi-Computable Real Numbers

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Abstract. Let $h : \mathbb{N} \to \mathbb{Q}$ be a computable function. A real number x is h-monotonically computable (h-mc, for short) if there is a computable sequence (x_s) of rational numbers which converges to x in such a way that the ratios of the approximation errors are bounded by h. In this paper we discuss the h-monotonic computability of semi-computable real numbers which are limits of monotone computable sequences of rational numbers. Especially, we show a sufficient and necessary condition for the function h such that the h-monotonic computability is simply equivalent to the normal computability.

1 Introduction

According to Alan Turing [12], a real number $x \in [0; 1]$ [1] is called *computable* if its decimal expansion is computable, i.e., $x = \sum_{n\in\mathbb{N}} f(n) \cdot 10^{-n}$ for some computable function $f : \mathbb{N} \to \{0, 1, \cdots, 9\}$. Let **EC** denote the class of all computable real numbers. Equivalently (see [9, 5, 14]), x is computable if and only if its Dedekind cut is computable and if and only if there is a computable sequence (x_s) of rational numbers which converges to x effectively in the sense that $|x - x_n| \leq 2^{-n}$ for any natural number n. In other words, a computable real number can be effectively approximated with an effective error estimation. This effective error estimation is very essential for the computability of a real number because Specker [11] has shown that there is a computable increasing sequence of rational numbers which converges to a non-computable real number. The limit of a computable increasing sequence of rational numbers can be naturally called *left computable* [2]. The class of all left computable real numbers is denoted by **LC**. Since any effectively convergent computable sequence can be easily transferred to an increasing computable sequence, Specker's example shows in fact that the class **LC** is a proper superset of the class **EC**, i.e., $\mathbf{EC} \subsetneq \mathbf{LC}$.

Symmetrically, we will call the limit of a decreasing computable sequence of rational numbers a *right computable* real number. Left and right computable real

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[1] We consider in this paper only the real numbers in the unit interval $[0; 1]$. For any real number $y \notin [0; 1]$, there are an $x \in [0; 1]$ and an $n \in \mathbb{N}$ such that $y = x \pm n$ and x, y are considered to have the same type of effectiveness.

[2] Some authors use the notion *computably enumerable* (c.e. for short) instead of left computable. See e.g. [2, 4].

numbers are called *semi-computable*. The classes of all right and semi-computable real numbers is denoted by **RC** and **SC**, respectively. Notice that, for any semi-computable real number x, there is an effective approximation (x_s) to x such that the later approximation is always a better one, i.e., $|x - x_n| \geq |x - x_m|$ for any $n \leq m$. Unfortunately, the improvement of this approximation can be very small and can vary with the different index. Therefore, in general, we can not decide eventually how accurate the current approximation to x will be and hence x can be non-computable. However, an effective error estimation will be possible if we know in advance that there is a fixed lower bound for the improvements. Namely, if there is a constant c with $0 < c < 1$ such that

$$(\forall n, m \in \mathbb{N})(n < m \Longrightarrow c \cdot |x - x_n| \geq |x - x_m|). \tag{1}$$

Let k_0 be a natural number such that $c^{k_0} \leq 1/2$. Then the computable sequence (y_s) defined by $y_s := x_{k_0 s}$ converges effectively to x (remember $x, x_0 \in [0;1]$ and hence $|x - x_0| \leq 1$) and hence x is a computable real number. Calude and Hertling [3] discussed the condition (1) for more general case, namely, without the restriction of $0 < c < 1$. They call a sequence (x_s) *monotonically convergent* if there is a constant $c > 0$ such that (1) holds. Furthermore, they show in [3] that any computable sequence (x_s) which converges monotonically to a computable real number x converges also computably in the sense that there is a computable function $g : \mathbb{N} \to \mathbb{N}$ such that $|x - x_s| \leq 2^{-n}$ for any $s \geq g(n)$, although not every computable sequence which converges to a computable real number converges computably.

More generally, Rettinger, Zheng, Gengler and von Braunmühl [8, 6] extended the condition (1) further to the following

$$(\forall n, m \in \mathbb{N})(n < m \Longrightarrow h(n) \cdot |x - x_n| \geq |x - x_m|), \tag{2}$$

where $h : \mathbb{N} \to \mathbb{Q}$ is a function. That is, the ratios of error estimations are bounded by the function h. In this case, we call the sequence (x_s) converges to x *h-monotonically*. A real number x is called *h-monotonically computable* (h-mc, for short) if there is a computable sequence (x_s) of rational numbers which converges to x h-monotonically. Furthermore, x is called *k-monotonically computable* (k-mc for short) if x is h-mc for the constant function $h \equiv k$ and x is *ω-monotonically computable* (ω-mc for short) if it is h-mc for some computable function h. The classes of all k-mc and ω-mc real numbers are denoted by k-**MC** and ω-**MC**, respectively.

In [8, 6], the classes h-**MC** for functions h with $h(n) \geq 1$ are mainly considered and they are compared with other classes of real numbers, for example, the classes **WC** of weakly computable real numbers and **DBC** of divergence bounded computable real numbers. Here a real number x called *weakly computable* (according to Ambos-Spies, Weihrauch and Zheng [1]) if there are left computable real numbers y, z such that $x = y - z$. That is, **WC** is the algebraic closure of the semi-computable real numbers. x is called *diverges bounded computable* if there is a computable sequence (x_s) of rational numbers which converges to x and a computable function $g : \mathbb{N} \to \mathbb{N}$ such that, for any $n \in \mathbb{N}$, the numbers of non-overlapped index pair (i, j) with $|x_i - x_j| \geq 2^{-n}$ is bounded by $g(n)$ (see [7, 14] for the details). Besides, a dense hierarchy for k-mc real numbers is also shown there. The main results of [8, 6] are summarized as follows.

Theorem 1.1 (Rettinger, Zheng, Gengler and von Braunmühl [8, 6]).

1. *A real number x is semi-computable if and only if it is 1-mc, i.e.,* $\mathbf{SC} = 1\text{-}\mathbf{MC}$*; And for any constant $0 < c < 1$, x is computable if and only if x is c-mc, i.e.,* $(\forall c)(0 < c < 1 \Longrightarrow \mathbf{EC} = c\text{-}\mathbf{MC})$*;*
2. *For any constants $c_2 > c_1 > 1$, there is a c_2-mc real number which is not c_1-mc, namely,* $c_1\text{-}\mathbf{MC} \subsetneq c_2\text{-}\mathbf{MC}$*;*
3. *For any constant c, if x is c-mc, then it is weakly computable. But there is a weakly computable real number which is not c-mc for any constant c. That is,* $\bigcup_{c\in\mathbb{R}} c\text{-}\mathbf{MC} \subsetneq \mathbf{WC}$*;*
4. *The class* $\omega\text{-}\mathbf{MC}$ *is incomparable with the classes* $\mathbf{WC}$ *and* $\mathbf{DBC}$*.*

Since $\mathbf{SC} \subseteq \omega\text{-}\mathbf{MC}$ and $\mathbf{WC}$ is an algebraic closure of $\mathbf{SC}$ under arithmetic operations $+, -, \times, \div$, the class $\omega\text{-}\mathbf{MC}$ is not closed under the arithmetic operations.

In this paper, we are interested mainly in the class $h\text{-}\mathbf{MC}$ which is contained in the class of semi-computable real numbers. Obviously, if a function $h : \mathbb{N} \to \mathbb{Q}$ satisfies $h(n) \leq 1$ for all $n \in \mathbb{N}$, then $h\text{-}\mathbf{MC} \subseteq \mathbf{SC}$. In fact, we can see in Section 3 that the condition $(\exists^\infty n \in \mathbb{N})(h(n) \leq 1)$ suffices for this conclusion. In Section 4, we will show a criterion on the function h under which a h-mc computable real number is in fact computable. Before we go to the technical details, let's explain some notions and notations more precisely in the next section at first.

2 Preliminaries

In this section, we explain some basic notions and notations which will are used in this paper. By $\mathbb{N}, \mathbb{Q}$ and $\mathbb{R}$ we denote the classes of natural numbers, rational numbers and real numbers, respectively. For any sets A, B, $f : A \to B$ denote a total function from A to B while $f :\subseteq A \to B$ is a partial function with $\mathrm{dom}(f) \subseteq A$ and $\mathrm{range}(f) \subseteq B$.

We assume only very basic background on the classical computability theory (cf. e.g. [10, 13]). A function $f :\subseteq \mathbb{N} \to \mathbb{N}$ is called (partial) computable if there is a Turing machine which computes f. Suppose that (M_e) is an effective enumeration of all Turing machines. Let $\varphi_e :\subseteq \mathbb{N} \to \mathbb{N}$ be the function computed by the Turing machine M_e and $\varphi_{e,s} :\subseteq \mathbb{N} \to \mathbb{N}$ an effective approximation of φ_e up to step s. Namely, $\varphi_{e,s}(n) = m$ if the machine M_e with the input n outputs m in s steps and $\varphi_{e,s}(n)$ is undefined otherwise. Thus, (φ_e) is an effective enumeration of all partial computable functions $\varphi_e :\subseteq \mathbb{N} \to \mathbb{N}$ and $(\varphi_{e,s})$ a uniformly effective approximation of (φ_e). One of the most important properties of $\varphi_{e,s}$ is that the predicate $\varphi_{e,s}(n) = m$ is effectively decidable and hence in an effective construction we can use $\varphi_{e,s}$ instead of φ_e. The computability notions on other countable sets can be defined by some effective coding. For example, let $\sigma : \mathbb{N} \to \mathbb{Q}$ be an effective coding of rational numbers. Then a function $f :\subseteq \mathbb{Q} \to \mathbb{Q}$ is called computable if and only if there is a computable function $g :\subseteq \mathbb{N} \to \mathbb{N}$ such that $f \circ \sigma(n) = \sigma \circ g(n)$ for any $n \in \mathrm{dom}(f \circ \sigma)$. Other type of computable functions can be defined similarly. Of course, the computability notion on $\mathbb{Q}$ can also be defined directly based on the Turing machine. For the simplicity, we use (φ_e) to denote the effective enumeration of partial computable functions $\varphi_e :\subseteq \mathbb{N} \to \mathbb{Q}$ in this paper too. This should not cause confuses from the context.

A sequence (x_s) of rational numbers is called *computable* if there is a computable function $f : \mathbb{N} \to \mathbb{Q}$ such that $x_s = f(s)$ for all $s \in \mathbb{N}$. It is easy to see that, (x_s) is computable if and only if there are computable functions $a, b, c : \mathbb{N} \to \mathbb{N}$ such that $x_s = (a(s) - b(s))/(b(s) + 1)$.

In this paper, we consider only the h-monotonic computability for the computable functions $h : \mathbb{N} \to \mathbb{Q}$. Because of the density of $\mathbb{Q}$ in $\mathbb{R}$, all results can be extended to the cases of the computable functions $h : \mathbb{N} \to \mathbb{R}$. These results are omitted here for the technical simplicity.

3 Monotonic Computability vs Semi-Computability

In this section we discuss the semi-computability of h-monotonically computable real numbers. For the constant function $h \equiv c$, the situation is very simple. Namely, the situation looks like the following:

$$\begin{cases} c\text{-}\mathbf{MC} = \mathbf{EC}, & \text{if } 0 < c < 1; \\ c\text{-}\mathbf{MC} = \mathbf{SC}, & \text{if } c = 1; \\ c\text{-}\mathbf{MC} \supsetneq \mathbf{SC}, & \text{if } c > 1. \end{cases} \tag{3}$$

If h is not a constant function but $h(n) \leq 1$ for all $n \in \mathbb{N}$, then any h-mc real number is also semi-computable. Of course, this is not a necessary condition. For example, if h takes the values larger than 1 only at finitely many places, then h-mc real numbers are still semi-computable. In fact, it suffices if h takes some values not larger than 1 at infinitely many places.

Lemma 3.1. *Let $h : \mathbb{N} \to \mathbb{Q}$ be a computable function such that $h(n) \leq 1$ for infinitely many $n \in \mathbb{N}$. If x is a h-mc real number, then it is semi-computable , i.e., h-$\mathbf{MC} \subseteq \mathbf{SC}$.*

Proof. 1. Suppose that $h : \mathbb{N} \to \mathbb{Q}$ is a computable function with $(\exists^\infty n)(h(n) \leq 1)$ and x is a h-monotonically computable real number. Then there is a computable sequence (x_s) of rational numbers which converges to x h-monotonically. The sequence (x_s) can be speedup by choosing a subsequence. More precisely, we define a strictly increasing computable function $g : \mathbb{N} \to \mathbb{N}$ inductively by

$$\begin{cases} g(0) & := (\mu s)(h(s) \leq 1) \\ g(n+1) & := (\mu s)(s > g(n) \ \&\ h(s) \leq 1). \end{cases}$$

Then, the computable sequence (y_s) of rational numbers defined by $y_s := x_{g(s)}$ converges to x $h \circ g$-monotonically and the computable function $h \circ g$ satisfies obviously that $(\forall n \in \mathbb{N})(h \circ g(n) \leq 1)$. That is, x is $h \circ g$-mc. Therefore, by Theorem 1.1.1 and the fact $h \circ g$-**MC** $\subseteq$ 1-**MC**, x is a semi-computable real number.

On the other hand, the next lemma shows that, if $h : \mathbb{N} \to \mathbb{Q}$ is a computable function with $h(n) > 0$ for all n, then the class h-**MC** contains already all computable real numbers, no mater how small the values of $h(n)$'s could be or even if $\lim_{n\to\mathbb{N}} h(n) = 0$. This is not completely trivial, because only rational numbers can be h-mc if $h(n) = 0$ for some $n \in \mathbb{N}$ by the condition (2).

Lemma 3.2. *Let $h : \mathbb{N} \to \mathbb{Q}$ be a computable function such that $h(n) > 0$ for all $n \in \mathbb{N}$. If x is computable, then there is an increasing computable sequence which converges to x h-monotonically. Therefore,* **EC** $\subseteq$ h-**MC**.

Proof. Let x be a computable real number and (x_s) a computable sequence which converges to x and satisfies the condition that $|x - x_s| \leq 2^{-s}$ for any $s \in \mathbb{N}$. Let $y_s := x_s - 2^{-s+2}$. Then (y_s) is a strictly increasing computable sequence of rational numbers which converges to x and satisfies the following conditions

$$|x - y_s| \leq |x_s - x| + 2^{-s+2} \leq 2^{-s+3}, \quad \text{and}$$
$$|x - y_s| \geq 2^{-s+2} - |x_s - x| \geq 2^{-s+1},$$

for any $s \in \mathbb{N}$. Since $h(n) > 0$ for all $n \in \mathbb{N}$, we can define inductively a computable function $g : \mathbb{N} \to \mathbb{N}$ by $g(0) := 0$ and

$$g(n+1) := (\mu s)\left(s > g(n) \ \&\ h(n) \cdot 2^{-g(n)+1} > 2^{-s+3}\right).$$

Then, for any $n < m$, we have

$$h(n) \cdot |x - y_{g(n)}| \geq h(n) \cdot 2^{-g(n)+1} \geq 2^{-g(n+1)+3} \geq |x - y_{g(n+1)}| \geq |x - y_{g(m)}|.$$

That is, the computable sequence $(y_{g(s)})$ converges h-monotonically to x and hence x is h-mc.

Although any h-mc real number x is semi-computable by Lemma 3.1, if $(\exists^\infty n)(h(n) \leq 1)$. However, a computable sequence which h-monotonically converges to x is not necessarily monotone and a monotone sequence converging to x does not automatically converge h-monotonically. But the next result shows that, for any such h-mc real number x, there exists a monotone computable sequence which converges to x h-monotonically to x.

Lemma 3.3. *Let $h : \mathbb{N} \to \mathbb{Q}$ be a computable function such that $h(n) \leq 1$ for infinitely many $n \in \mathbb{N}$. If x is a h-mc real number, then there is a monotone computable sequence which converges to x h-monotonically.*

Proof. Let x be a h-mc real number and (x_s) a computable sequence of rational numbers which converges to x h-monotonically. By the proof of Lemma 3.1, we can assume without loss of generality that $h(n) \leq 1$ for all $n \in \mathbb{N}$. Notice that, if $x_s < x_{s+1}$, then $x_s < x$, otherwise , it $x \leq x_s$, then $h(s)|x - x_s| \leq x_s - x < x_{s+1} - x$ which contradicts the h-monotonic convergence. Similarly, if $x_s > x_{s+1}$, then $x_s > x$. If $(\forall^\infty s)(x_s < x_{s+1})$ or $(\forall^\infty s)(x_s > x_{s+1})$ hold, then the clam is obviously true. Here $(\forall^\infty n)$ means "for almost all n". Otherwise, if $(\exists^\infty s)(x_s < x_{s+1})$ and $(\exists^\infty s)(x_s > x_{s+1})$, then we can define an increasing and a decreasing computable sequences which converges to x. For example, the increasing computable sequence (y_s) can be defined by $y_s := x_{g(s)}$ where $g : \mathbb{N} \to \mathbb{N}$ is defined inductively by

$$\begin{cases} g(0) & := (\mu s)\,(x_s < x_{s+1}) \\ g(n+1) & := (\mu s)\,\big(s > g(n)\ \&\ x_{g(n)} < x_s < x_{s+1}\big). \end{cases}$$

But in this case, x is a computable real number and hence there is an increasing computable sequence which converges to x h-monotonically as shown Lemma 3.2.

As mentioned at the beginning of this section, if $c < 1$, then any c-mc real number is computable. It is natural to ask, for a function h with $0 < h(n) < 1$ for any $n \in \mathbb{N}$, is any h-mc real number computable? or is there any 1-mc real number which is not h-mc for any computable function h with $(\forall n)(h(n) < 1)$? Next theorem gives a negative answer to both of these questions.

Theorem 3.4. *Every semi-computable real number is h-mc for a computable function $h : \mathbb{N} \to \mathbb{Q}$ such that $0 < h(n) < 1$ for any $n \in \mathbb{N}$.*

Proof. Suppose that x is a left computable real number and (x_s) is a strictly increasing computable sequence of rational numbers which converges to x. Let a be a rational number which is greater than x. Define a computable function $h : \mathbb{N} \to \mathbb{Q}$ be

$$h(n) := \frac{a - x_{n+1}}{a - x_n}$$

for any $n \in \mathbb{N}$. Then $0 < h(n) < 1$ because $x_n < x_{n+1}$ for any $n \in \mathbb{N}$. Furthermore, the sequence (x_s) converges in fact to x h-monotonically because

$$\begin{aligned} h(n) \cdot |x - x_n| &= (x - x_n)\left(\frac{a - x_{n+1}}{a - x_n}\right) > (x - x_n)\left(\frac{x - x_{n+1}}{x - x_n}\right) \\ &= (x - x_{n+1}) \geq |x - x_m| \end{aligned}$$

for any natural numbers n and $m > n$.

Similarly, if x is a right computable real number and (x_s) a strictly decreasing sequence of rational numbers which converges to x, then we define a computable function $h : \mathbb{N} \to \mathbb{Q}$ by

$$h(n) := \frac{x_{n+1}}{x_n}$$

for any natural numbers n. Obviously, we have $0 < h(n) < 1$ for any $n \in \mathbb{N}$ and the sequence (x_s) converges to x h-monotonically because

$$\begin{aligned} h(n) \cdot |x - x_n| &= (x_n - x)\left(\frac{x_{n+1}}{x_n}\right) > (x_n - x)\left(\frac{x_{n+1} - x}{x_n - x}\right) \\ &= (x_{n+1} - x) \geq |x - x_m| \end{aligned}$$

for any natural numbers $m > n$.

4 Monotonic Computability vs Computability

In this section, we will discuss the computability of a h-mc real number for the computable function $h : \mathbb{N} \to \mathbb{Q}$ with $h(n) \leq 1$ for all $n \in \mathbb{N}$. By a simple argument similar to the proof of Lemma 3.1, it is easy to see that h-**MC** $\subseteq$ **EC** if there is a constant $c < 1$ such that $h(n) \leq c$ for infinitely many $n \in \mathbb{N}$. Therefore, it suffices to consider the computable functions $h : \mathbb{N} \to \mathbb{Q} \cap [0; 1]$ such that $\lim_{n\to\infty} h(n) = 1$. The next theorem gives a criterion for a computable function h such that any h-mc real number is computable.

Theorem 4.1. *Let $h : \mathbb{N} \to \mathbb{Q}$ be a computable function such that $0 < h(n) < 1$ for any $n \in \mathbb{N}$. Then any h-mc real number is computable if and only if $\prod_{i=0}^{\infty} h(i) = 0$. Namely,*

$$h\text{-}\mathbf{MC} = \mathbf{EC} \iff \prod_{i=0}^{\infty} h(i) = 0.$$

Proof. "$\Leftarrow$": Suppose that $h : \mathbb{N} \to \mathbb{Q} \cap (0; 1)$ is a computable function such that $\prod_{i=0}^{\infty} h(i) = 0$. We are going to show that h-**MC** = **EC**. The inclusion **EC** $\subseteq$ h-**MC** follows from Lemma 3.2. We will show the another inclusion h-**MC** $\subseteq$ **EC**.

Let $x \in h$-**MC** and (x_s) be a computable sequence of rational numbers which converges to x h-monotonically. Suppose without loss of generality that $|x - x_0| \leq 1$. By the h-monotonic convergence, we have $h(n) \cdot |x - x_n| \geq |x - x_m|$ for all $m > n$. This implies that $|x - x_n| \leq \prod_{i=0}^{n} h(i) \cdot |x - x_0| \leq \prod_{i=0}^{n} h(i)$. Because $\lim_{n\to\infty} \prod_{i=0}^{n} h(i) = \prod_{i=0}^{\infty} h(i) = 0$, we can define a strictly increasing computable function $g : \mathbb{N} \to \mathbb{N}$ inductively as follows.

$$\begin{cases} g(0) := 0 \\ g(n+1) := (\mu s)\left(s > g(n) \ \& \ \prod_{i=0}^{s} h(i) < 2^{-(g(n)+1)}\right). \end{cases}$$

Then the computable sequence (y_s) of rational numbers defined by $y_s := x_{g(s)}$ converges to x effectively and hence $x \in$ **EC**.

"$\Rightarrow$": Suppose that $\prod_{i=0}^{\infty} h(i) = c > 0$. Fix a rational number q such that $0 < q < c$. We will construct an increasing computable sequence (x_s) of rational numbers from the unit interval $[0;1]$ which converges h-monotonically to a non-computable real number x. The requirements for the h-monotonic convergence are

$$h(n) \cdot |x - x_n| \geq |x - x_m|$$

for all $m > n$. But since the sequence is increasing, this is equivalent to

$$h(n)(x - x_n) \geq x - x_{n+1}.$$

for all $n \in \mathbb{N}$. Now adding the requirement that x is in the unit interval and hence $x \leq 1$, this requirement can be reduced further to $h(n)(1-x_n) \geq 1-x_{n+1}$ which is equivalent to the following

$$x_{n+1} \geq 1 - h(n)(1 - x_n). \tag{4}$$

Notice that, as shown in the proof of Lemma 3.3, for any computable real number y, there is an increasing computable sequence (y_s) which converges to y and satisfies the condition $|y_s - y_{s+1}| \leq 2^{-(s+1)}$ for any $s \in \mathbb{N}$. Therefore, the non-computability of x can be guaranteed by satisfying, for all $e \in \mathbb{N}$, the following requirements

$$R_e: \quad \left.\begin{array}{l} \varphi_e \text{ is an increasing total function,} \\ (\forall s)(|\varphi_e(s) - \varphi_e(s+1)| \leq 2^{-(s+1)}) \end{array}\right\} \Longrightarrow x \neq \lim_{s \to \infty} \varphi_e(s).$$

Let's explain our idea to construct the sequence (x_s) informally at first. Let's consider the computable sequence (y_s) defined inductively by $y_0 = 0$ and $y_{n+1} = 1 - h(n)(1 - y_n)$ as our first candidate. Obviously, we have $y_n := 1 - \prod_{i<n} h(i)$ for any $n \in \mathbb{N}$ and hence $\lim_{n\to\infty} y_n = 1 - c$. This sequence is an increasing computable sequence and satisfies the condition (4) too. In order to satisfy a single requirement R_e, it suffices for some y_s to make an extra increment of $2\delta_e$ if it is necessarily, where δ_e is a rational number with $0 < \delta_e < c$. Concretely, if the premises of R_e hold, then we can choose a natural number t such that $2^{-t} < \delta_e$. If there is a stage s such that $\varphi_e(t) < y_{s+1} + \delta_e = 1 - h(s)(1 - y_s) + \delta_e$, then we redefine $y_{s+1} := 1 - h(s)(1 - y_s) + 2\delta_e$. In this case, $\lim_{n\to\infty} \varphi_e(n) \leq \varphi_e(t) + 2^{-t} \leq \varphi_e(t) + \delta_e \leq 1 - h(s)(1 - y_s) + 2\delta_e = y_{s+1} < \lim_{n\to\infty} y_n$. If no such stage s exist, then $\lim_{n\to\infty} \varphi_e(n) \geq \varphi_e(t) - 2^{-t} > \varphi_e(t) - \delta_e \geq \lim_{n\to\infty} y_n$. In both cases, R_e is satisfied. This strategy can be implemented for each requirement independently. To guarantee that the sequence remains in the interval $[0;1]$, the δ_e's should be chosen in such a way that $\sum_{e\in\mathbb{N}} 2\delta_e \leq c$. Therefore, we can define simple $\delta_e := q \cdot 2^{-(e+1)}$.

The formal construction of the sequence (x_s):

Stage 0. Define $x_0 = 0$. All requirements are set into the state of "unsatisfied".

Stage $s + 1$. Given x_s. We say that a requirement R_e *requires attention* if $e \leq s$, R_e is in the state of "unsatisfied" and the following condition is satisfied

$$(\exists t \leq s) \left(2^{-t} < \delta_e \ \& \ \varphi_{e,s}(t) \leq 1 - h(s)(1 - x_s) + \delta_e\right) \tag{5}$$

If some requirements require attention, then let R_e be the requirement of the highest priority (i.e., of the minimal index e) which requires attention at this stage. Then define

$$x_{s+1} := 1 - h(s)(1 - x_s) + 2\delta_e$$

and set R_e to the state of "satisfied". In this case, we say that R_e *receives attention* at stage $s + 1$.

Otherwise, if no requirement requires attention at stage $s + 1$, then define simply

$$x_{s+1} := 1 - h(s)(1 - x_s).$$

To show that our construction successes, we prove the following sublemmas.

Sublemma 4.1.1 *For any $e \in \mathbb{N}$, the requirement R_e receives attention at most once and hence $\sum_{i=0}^{\infty} \sigma(i) \leq q$, where $\sigma(s) := 2\delta_e$ if the requirement R_e receives attention at stage s, and $\sigma(s) := 0$ otherwise.*

Proof of sublemma: By the construction, if a requirement R_e receives attention at stage s, then R_e is set into the state of "satisfied" and will never require attention again after stage s. That is, it receives attention at most once. This implies that, for any $e \in \mathbb{N}$, there is at most one $s \in \mathbb{N}$ such that $\sigma(s) = \delta_e$. Therefore, $q = \sum_{e=0}^{\infty} \delta_e \geq \sum_{i=0}^{\infty} \sigma(i)$.

Sublemma 4.1.2 *The sequence (x_s) is an increasing computable sequence of rational numbers from the interval $[0;1]$ and it converges h-monotonically to some $x \in [0;1]$.*

Proof of sublemma: At first we prove by induction on n the following claim

$$(\forall n \in \mathbb{N}) \left(x_n \leq 1 - \prod_{i<n} h(i) + \sum_{i<n} \sigma(i) \right). \tag{6}$$

For $n = 0$, the claim (6) holds trivially, because $\prod_{i\in\emptyset} \cdots = 1$ and $\sum_{i\in\emptyset} \cdots = 0$ by convention.

Assume by induction hypothesis that the claim holds for n. Then we have

$$\begin{aligned} x_{n+1} &= 1 - h(n)(1 - x_n) + \sigma(n+1) \\ &\leq 1 - h(n) \cdot \prod_{i<n} h(i) + h(n) \cdot \sum_{i<n} \sigma(n) + \sigma(n+1) \\ &\leq 1 - \prod_{i<n+1} h(i) + \sum_{i<n+1} \sigma(n) \end{aligned}$$

That is, the claim holds also for $n + 1$ and this completes the proof of the claim. Since $\prod_{i<n} h(i) \geq \prod_{i=0}^{\infty} h(i) > q \geq \sum_{i=0}^{\infty} \sigma(i) \geq \sum_{i<n} \sigma(i)$ for any $n \in \mathbb{N}$, it follows that $x_n < 1$ for any $n \in \mathbb{N}$. Furthermore, because of

$$\begin{aligned} x_{n+1} - x_n &= 1 - h(n)(1 - x_n) + \sigma(n+1) - x_n \\ &\geq 1 - h(n)(1 - x_n) - x_n = (1 - h(n))(1 - x_n) > 0 \end{aligned}$$

for any $n \in \mathbb{N}$, the sequence (x_s) is a strictly increasing computable sequence of rational numbers from $[0;1]$.

Besides, by the construction, the sequence (x_s) satisfies obviously the condition (4). Therefore, it converges to some $x \in [0;1]$ h-monotonically and hence x is a h-mc real number.

Sublemma 4.1.3 *For any $e \in \mathbb{N}$, the requirement R_e is eventually satisfied and hence x is not a computable real number.*

Proof of sublemma. For any $e \in \mathbb{N}$, suppose that the premisses of the requirement R_e are satisfied. Namely, φ_e is an increasing total function and satisfies the condition that $|\varphi_e(s) - \varphi_e(s+1)| \leq 2^{-(s+1)}$. Suppose that the limit $z_e := \lim_{t\to\infty} \varphi_e(t)$. Then $|z_e - \varphi_e(s)| \leq 2^{-s}$ holds for any $s \in \mathbb{N}$. We consider the following two cases:

Case 1. The requirement R_e receives attention at some stage $s+1$. According to the requiring condition (5), there is a natural number $t \leq s$ such that $2^{-t} < \delta_e$ and $\varphi_e(t) \leq 1 - h(s)(1-x_s) + \delta_e$. This implies that $\lim_{n\to\infty} \varphi_e(n) \leq \varphi_e(t) + 2^{-t} \leq 1 - h(s)(1-x_s) + 2\delta_e \leq x_{s+1} < x$. That is, $\lim_{t\to\infty} \varphi_e(t) \neq x$ and hence R_e is satisfied in this case.

Case 2. Suppose that the requirement R_e does not receive attention at all. We will show that R_e is satisfied too. By Sublemma 4.1.1, we can choose an s_0 large enough such that no requirement R_i for $i < e$ receives attention after stage s_0. Assume now by contradiction that $z_e = \lim_{t\to\infty} \varphi_e(t) = x$. From the hypothesis of R_e, φ_e is an increasing total function. Choose $t, s_1 > s_0$ such that $2^{-t} < \delta_e$ and $\varphi_{e,s_1}(t)$ is defined. Then there exists an $s_2 \geq \max\{e, t\}$ such that $\varphi_{e,s_2}(t) < x_{s_2}$. Since $0 < h(s_2), x_{s_2} < 1$, we have $x_{s_2} < 1 - h(s_2)(1 - x_{s_2})$. This implies that $\varphi_{e,s_2}(t) \leq 1 - h(s_2)(1-x_{s_2}) + \delta_e$. That is the requiring condition (5) is satisfied at stage $s_2 + 1$. Therefore, R_e will require and also receive attention at stage $s_2 + 1$, because no requirement R_i for $i < e$ requires attention at this stage. This is a contradiction.

In summary, x is a h-monotonically computable but not computable real number. This completes the proof of the theorem.

Corollary 4.2. *For any computable function $h : \mathbb{N} \to \mathbb{Q}$ with $\lim_{n\to\infty} h(n) = 1$, there is a computable increasing function $g : \mathbb{N} \to \mathbb{N}$ such that the class $h \circ g$-**MC** contains non-computable reals.*

Proof. It is known that $\prod_{n=1}^{\infty}(1 - 1/n^2) = 1/2$. So if we define a computable increasing function inductively by

$$\begin{cases} g(0) & := 0 \\ g(n+1) & := (\mu t)\left(t > g(n) \wedge h(t) > 1 - 1/n^2\right) \end{cases} \tag{7}$$

then $\prod_{i=0}^{\infty} h \circ g(i) > 0$ and by Theorem 4.1, the class $h \circ g$-**MC** contains non-computable reals.

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