THE APPROXIMATION STRUCTURE OF A COMPUTABLY APPROXIMABLE REAL

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ABSTRACT. A new approach for a uniform classification of the computably approximable real numbers is introduced. This is an important class of reals, consisting of the limits of computable sequences of rationals, and it coincides with the $0'$-computable reals. Unlike some of the existing approaches, this applies uniformly to all reals in this class: to each computably approximable real x we assign a degree structure, *the structure of all possible ways available to approximate x.* So the main criterion for such classification is the variety of the effective ways we have to approximate a real number. We exhibit extreme cases of such approximation structures and prove a number of related results.

1. INTRODUCTION

The real numbers which are limits of computable sequences of rationals, also called recursively approximable reals (r.a. for short) form one of the most important classes of non-computable reals. We prefer calling them *computably approximable* (c.a.) according to the change of terminology in computability theory (adopted by many researchers in the field). By a result of Ho[8] they coincide with the $0'$-computable numbers, i.e. those that can be computed with pre-assigned accuracy using the halting set as an oracle (see [8]). There has been a lot of effort in order to classify c.a. reals and the main criterion was the *difficulty* to approximate them. One of the most successful attempts for such classification is Solovay's structure of computably enumerable (c.e.) reals (an important subclass of c.a. reals) which really captures the notion of a c.e. real being more difficult to approximate from another. The maximal elements of this structure, intuitively being the hardest c.e. reals to approximate, turn out to be *random* reals (see [5]). However Solovay's approach is applied only to c.e. reals [1] although a number of more recent approaches (via reducibilities or hierarchies) deal with more general classes. For example Rettinger and Zheng[12], [11] define a dense hierarchy of c.a. numbers which transcends the c.e. reals (yet it does not exhaust the c.a.

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[1] of course it is dually applied to co-c.e. reals.

reals). The underlying idea of this classification is how 'slow' is the 'fastest' computable sequence with limit a particular real (see Zheng[15] for a survey of results in this direction). Other approaches have to do with reducibilities e.g. Downey, Hirschfeldt and LaForte[6] where particular reducibilities are introduced as a measure of relative randomness.

In this paper we present a different approach for classifying c.a. reals: our criterion is *the variety of the possible ways to approximate a real.* Using restricted oracle computations we make this statement precise: having a real x and an approximation $\lim_s z_s = x$ we consider the set

$$A_z = \{s \mid z_s < x\}$$

which we may assume is infinite and co-infinite. We regard these sets as a sort of 'representations' of x and we study their complexity (and how they relate to the complexity of x). In particular, we order the class $\mathcal{S}_x$ of all such sets (for possible approximations of x) with a strong reducibility $\leq_r$ (e.g. $\leq_{\mathrm{wtt}}$, $\leq_m$ etc.) and we get a degree structure $\mathcal{D}_x^{\mathrm{r}}$. Each element of $\mathcal{D}_x^{\mathrm{r}}$ represents a different way to approximate x in terms of the restricted oracle computation associated with $\leq_r$. Indeed, if $A_z \leq_r A_y$ for z, y approximations of x, then given restricted access to the oracle A_y (which contains the information of which terms of y lie on the left of x) we can extract the relevant information about the approximation z. So in a way, y is at least as good as z. [2] If $\mathcal{D}_x^{\mathrm{r}}$ has a maximum element, then there is a best approximation. And if it is trivial, i.e. consists of a single degree, then we could say that all ways available to approximate x are quite similar (with respect to $\leq_r$).

In the following sections we will exhibit a variety of structures $\mathcal{D}_x^{\mathrm{r}}$ (which turns out to be a substructure of the r-degrees inside the Turing degree of x). In fact, we construct a c.e. real x such that an infinite antichain is embedable in $\mathcal{D}_x^{\mathrm{wtt}}$. In this case the approximation structure is quite rich and intuitively there are a lot of different ways to approximate x. In the other extreme we construct a c.e. non-computable real x such that $\mathcal{D}_x^{\mathrm{m}}$ is trivial, i.e. it consists of a single element. Such constructions of structures $\mathcal{D}_x^{\mathrm{r}}$ with desired properties are done on a special framework for priority injury. In particular, the proof of theorem 3 has several interesting special features. A notion of 'links' is defined which is central in the actual construction; the links are actively involved in the priority list and they behave as negative requirements. But since they are created during the construction (in a way which is not predictable) one could say that negative requirements are

[2] We note that all A_z contain the same information about x for various z with $\lim z = x$ (see proposition 1). The difference may be that this information is *arranged in different way.* This is the case when $A_z \not\equiv_r A_w$ for $\lim z = \lim w = x$. If $A_z \not\leq_r A_w$, the information in A_z is so much rearranged from the point of view of A_w, that a strong oracle procedure (based on $\leq_r$) is not enough to decode A_z from A_w.

generated in the course of the construction and special care has been taken in order to control them.

In section 5 we note that some strong reducibilities coincide if we restrict ourselves to the class $\mathcal{S}_x$ for a real x; these are m, bounded tt with one query (also called btt(1)) and the positive reducibility. Finally in section 6 we are looking at the immunity properties of the sets in $\mathcal{S}_x$ for a given real x. The motivation for this is that when a real is e.g. c.e., then its complexity intuitively depends on how *rough* the right dedekind cut of it is. Theorem 1 says that no matter how complex x is, we can always produce infinitely many rationals in a very small area of x in the right cut of it. So one may want to see how the complexity of x depends on the complexity of $\mathbb{N} - A_z$ (in case the last is non-trivial, i.e. infinite). When a set A is (h or hh-) immune, intuitively it is difficult to make correct guesses about elements in that set (in case of immunity the output of a machine is viewed as a sequence of such guesses; in h-immunity an element of a strong array is a guess and it is correct if it intersects A; and in hh-immunity the notion of 'guess' is even weaker, corresponding to weak arrays). In this sense one may hope to get different classes of c.e. reals (with different 'complexity') by changing the immunity requirements on the set $\mathbb{N} - A_z$. We show that this is imposible; namely this set is either computable or h-immune and *not* hh-immune. Similar results are obtained for co-c.e. reals and non semi-computable ones.

Here is a list of conventions we adopt in the rest of the paper:

- The expression $\Phi(A) = B;\ \varphi$ means that the equality holds and the calls to the oracle A are bounded by φ.
- We assume a standard 1-1 pairing function $\langle .,. \rangle : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$.
- The mode $[s]$ after a parameter of a construction means that we consider the value of the parameter at (end of) the s-th stage of the construction. Also, parameters which are not explicitely re-defined at some stage of the construction are assumed to preserve the value they had in the previous stage.
- If $\varphi_e(n)[s] \downarrow$ then $n, e < s$.

All rational sequences in this paper are computable sequences of rational numbers. They are often represented by z, w (when their terms are z_s, w_s) and we usually drop the subscripts in their limits (e.g. we write $\lim z$ for $\lim_s z_s$). The sequence (Φ_e, φ_e) is an effective enumeration of the computable functionals/functions and the symbol $\downarrow$ in front of a requirement or a parameter in a construction means that it is satisfied or defined respectively (the symbol $\uparrow$ indicates the opposite situation). Many arguments are accompanied with illustrations in order to make them more comprehensible. For backround and basic definitions in computable analysis we refer to Zheng[15], Dunlop and Pour-El[7], while Odifreddi[9], [10] cover the computability theory used in this paper.

2. The approximation structure.

In this section we are going to give the definition of the approximation structure of a c.a. real x. Consider all computable sequences of rationals $z = \{z_s\}$ with $\lim z = x$ and for each of them, the sets

$$A_z = \{s \mid z_s < x\} \qquad (1)$$
$$B_z = \{s \mid z_s > x\}$$

In the following we always consider z so that A_z is infinite and co-infinite (the other case being trivial).

2.1. **Basic fact.** The following theorem shows that such sequences always exist. We note that this follows from the proof of theorem 5; however we give a direct proof since the more complicated argument in theorem 5 is based in the simple idea of the proof we are going to present now (so reading this proof will help understanding the latter).

Theorem 1. *If x is a c.a. real then there is a computable sequence of rationals $z = \{z_s\}$ with limit x and A_z infinite and co-infinite.*

For the proof, it is easy to see that if x is computable the result holds. And if x is not semi-computable then every (computable) sequence z with limit x has A_z infinite and co-infinite. So the only interesting case is when x is non-computable and semi-computable, say c.e. (the other case being dual). We will show how from an increasing computable sequence of rationals with limit x one can effectively obtain a sequence satisfying the requirements of the theorem.

Suppose that $\lim_s x_s = x$, $\{x_s\}$ is strictly increasing and $|x - x_s| < \frac{1}{f(n)}$ for a function $f : \mathbb{N} \to \mathbb{N} - \{0\}$ which is of course non-computable and $\lim_n f(n) = \infty$. The idea of the construction is that we are able to make guesses about rationals which lie on the right of x as the following figure shows

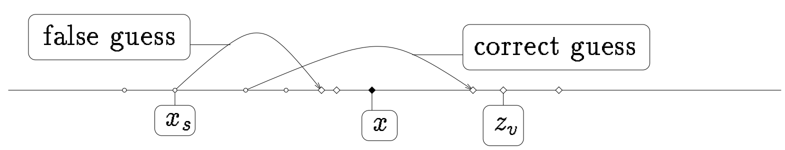

Figure 1: Guesses.

Suppose that at stage s we make a guess; if this is the n-th guess then we denote it (i.e. the rational which is proposed to be greater than x) by y^0_n. Because x is not co-c.e. (since it is not computable) it is hard to make correct (or, as we sometimes say, *successful*) guesses and some of the guesses will be false. The false guesses will be detected by the increasing sequence $\{x_s\}$; indeed, when at some stage s it is $x_s > y^0_n$ then we are sure that y^0_n is on the left of x and so the guess is false. And this will definitely happen if the guess is false. When at stage s we discover a false guess (as above) we propose a correction y_n^1: this is the 1-st correction of the n-th guess. Later we may find (in the same way) that the correction itself is false, in which case we propose another correction. So for each guess y_n^0 we get a sequence of corrections $y_n^1, y_n^2, \dots$ that will eventually reach a y_n^s which is indeed on the right of x. Moreover, we define the terms of a sequence $\{z_s\}$ to be the guesses and corrections produced during the construction and we ensure that $\lim_s z_s = x$ (by accumulating both the guesses and their corrections in smaller and smaller areas of x).

If at stage s the n-th guess has been made, we may also have some corrections by that time. So we define

$$y_n[s] = y_n^{t_0}$$

where $t_0 = \max\{t \mid \exists i \leq s \text{ with } z_i = y_n^t\}$. This is the most recent correction of the n-th guess (up to stage s) and we call it *the s-th version of the n-th guess.* We say that $y_n[s] \downarrow$ when $\{t \mid \exists i \leq s \text{ with } z_i = y_n^t\} \neq \emptyset$ (here by $z_i = y_n^t$ we don't mean just the equality but rather the *intentional* 'z_i was defined to be the t-th correction of the n-th guess'). Here is how we make the n-th guess at stage s:

$$y_n^0 = x_s + \frac{1}{s}$$

and any possible subsequent corrections of this guess will be

$$x_s + \frac{2}{s}, x_s + \frac{3}{s}, \dots$$

until we reach a rational greater than x. The last can be guaranteed if we choose x_0 such that $|x - x_0| < 1$. So for each term y_n^k we have

$$y_n^k = x_s + \frac{k+1}{s} \tag{2}$$

where s is the stage where y_n^0 was defined (i.e. the n-th guess was made). In the following when we say e.g. 'if $y_n[v] = x_s + \frac{k+1}{s} \dots$ ' (for some s, k) we don't mean just the arithmetical equation but rather 'if the v-th version of the n-th guess is its k-correction and the n-th guess was defined at stage s $\dots$ '. At stage s a unique term will be defined, namely z_s. If it is defined via step A of the construction, then it is going to be a (new) guess; otherwise it is a correction of a previously made guess.

2.1.1. *Construction.*

Stage 0. Define $z_0 = x_0$.
Stage $s+1$. Two steps:

step A See whether $x_{s+1} > y_n[s]$ for any n with $y_n[s]\downarrow$. If not, then define

$$z_{s+1} = y^0_{n_0} := x_{s+1} + \frac{1}{s+1} \tag{3}$$

where $n_0 = \mu t[y_t[s]\uparrow\ ]$, and go to stage $s+2$. Otherwise go to step B.

step B Suppose that

$$\{n \mid x_{s+1} > y_n[s] \ \wedge \ y_n[s]\downarrow\} = \{i_k \mid k < m\}$$

(i_k distinct) and that

$$y_{i_k}[s] = x_{n_k} + \frac{t_k}{n_k} = y^{t_k-1}_{i_k}.$$

for $k < m$. Then define

$$z_{s+k} = y^{t_k}_{i_k} := x_{n_k} + \frac{t_k+1}{n_k} \tag{4}$$

for all $k < m$ and go to stage $s+m$.

2.1.2. *About the construction.*

1. In the definition (4) in step B of the construction, we regard z_{s+k}, $y_{i_k}[s]$ to be defined at stage $s+k$, for $k < m$.
2. In (3) the definition of n_0 means that there have been made $n_0 - 1$ guesses up to stage $s-1$ (so the next one is the n_0-th guess in the construction).

2.1.3. *Verification.*

Lemma 1.1. *If y^0_m was defined at stage s then $m < s$.*

Proof. By induction on m. For y^0_0 it holds since at stage 0 no guess is made. If it holds for all $i < m$ and y^0_m is defined at stage s, then the $(m-1)$-th guess is already made by the end of stage $s-1$. So $m-1 < s-1$ which gives $m < s$. □

Lemma 1.2. $\mathbb{N} - A_z$ *is infinite.*

Proof. First we prove that every guess will eventually have a final correct version; formally

$$\forall n\ \exists s\ (y_n[s] > x).$$

Indeed, suppose otherwise. Then, according to the construction, there is an infinite sequence of corrections

$$y^0_n, y^1_n, y^2_n, \dots$$

such that $y_n^t < x$ for all t. But

$$y_n^t = x_s + \frac{t+1}{s}$$

(where s is the stage where y_n^0 was defined) and since $x_0 + 1 > x$ and $x_0 < x_s$ for all s, we have $y_n^s = x_s + \frac{s+1}{s} > x$, a contradiction.

To complete the proof of the lemma, we show that for any n there is $s > n$ such that $z_s > x$. Indeed, at each stage s, exactly one term of $\{z_i\}$ is defined, namely z_s. Choose n; it is

$$z_{n+1} = y_k^t$$

for some t, k. According to the above, consider $t_0 > t$ such that $y_k^{t_0} > x$ and the stage s where $y_k^{t_0}$ was defined. It is $z_s > x$ and $s > n$, i.e. what we where looking for. □

The following lemma finishes the proof of the theorem.

Lemma 1.3. $\lim_s z_s = x$.

Proof. Choose $\epsilon > 0$; we will show that there is s_0 such that for all $s > s_0$ we have

$$|x - z_s| < \epsilon.$$

Choose n such that $\max\{\frac{1}{f(n)}, \frac{1}{n}\} < \epsilon$. Now choose s such that all i-guesses for $i \leq n$ have been successfully corrected. Formally, for all $s > s_0$ and $i \leq n$,

$$y_i[s] > x.$$

Consider $s > s_0$ and the term z_s. It will be

$$z_s = y_m[s] = y_m^{k-1} = x_t + \frac{k}{t}$$

for some t, k, m.

Claim. $t > n$.

Proof of claim. At stage s, z_s was defined either under step A or under step B. In the first case, $t = s > n$ (due to lemma 1.1). In the second case t is the stage where y_m^0 was defined and suppose that $t \leq n$ for a contradiction. Since $m < t$ by lemma 1.1, it is $m < n$. By the assumption we made about s_0, $y_m[s_0] > x$ and so the m-th guess (any version of it) is never considered in step B of any stage $v > s_0$, a contradiction. □

Claim. *If $z_s > x$ then we claim that*

$$|x - x_t - \frac{k}{t}| \leq \frac{1}{t}. \tag{5}$$

Proof of claim. Suppose otherwise for a contradiction, i.e. $|x - (x_t + \frac{k}{t})| > \frac{1}{t}$. Then

$$x < x_t + \frac{k-1}{t}$$

and since

$$y_m^{k-2} = x_t + \frac{k-1}{t}$$

the $(k-2)$-th version of the m-th guess (namely y_m^{k-2}) would be successful and y_m^{k-1} would never be defined, a contradiction. □

Since it is $t > n$, (5) gives $|x - z_s| < \frac{1}{n}$ and so $|x - z_s| < \epsilon$.

Now suppose that $z_s < x$ (it cannot be equal since x is not rational as it is non-computable). Then since $\{x_i\}$ is increasing and $t > n$ we have $x_t > x_n$ and so

$$|x - z_s| = |x - x_t - \frac{k}{t}| < |x - x_n - \frac{k}{t}| < |x - x_n| < \frac{1}{f(n)} < \epsilon.$$

which completes the proof. □

The theorem follows from the above lemmas.

2.2. **The definition.** One may want to consider the complement of the set A_z of (1); but since we assume it infinite and co-infinite, the two sets have roughly the same complexity which is directly related to the complexity of x (as we will see in the following). So it makes no difference which one we choose.

Each of these sets is a kind or 'representation' for x. We define a structure of all these representations (under a fixed reducibility) and we regard this as *the computability structure of the possible ways available to approximate the real x.* Fix a reducibility $\leq_r$ (e.g. T, wtt, tt, m etc.).

Definition 1. *Given a $0'$-computable real x, consider the class*

$$\mathcal{S}_x = \{A_z \mid \ z \textit{ computable and } \lim z = x\}$$

and the partially ordered set $\langle \mathcal{S}_x, \leq_r \rangle$. The elements of $\mathcal{S}_x$ are called x-sets. Also, consider the induced degree structure

$$\mathcal{D}_x^r = \{deg_r(A) \mid A \in \mathcal{S}_x\}.$$

This is called the approximation structure of x and its elements are called x-r-degrees.

Consider the case where $\lim z = \lim w = x$ for two computable sequences of rationals $z = \{z_s\}$, $w = \{w_s\}$. It is not difficult to prove that

Proposition 1. *If A_z, A_w are infinite and co-infinite then $A_z \equiv_T A_w \equiv_T x$.*

So $\mathcal{D}_x^{\mathrm{r}}$ is a substructure of the structure of r-degrees inside the Turing degree of x (see figure 2). Moreover, for any x, $\mathcal{D}_x^T$ is trivial, consisting of the Turing degree of x.

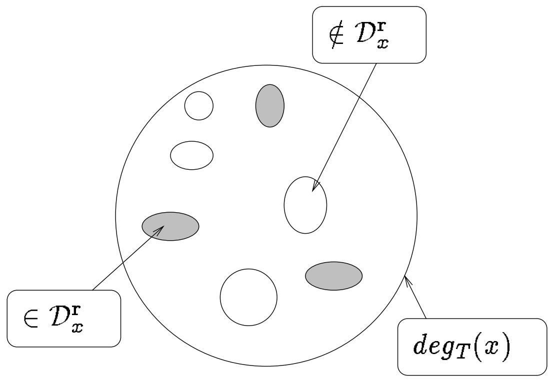

Figure 2: The approximation structure of x inside its Turing degree.

A natural question is whether this holds for stronger reducibilities. Not only this is not true, but we are able to construct reals with quite rich approximation structure (with respect to a strong reducibility). Due to our basic technique for such constructions, all reals constructed in this paper will be c.e. In Barmpalias[1] we attempted a full approximation argument for such a construction, but the proofs in this paper turn out to be simpler and establish much stronger results.

3. Antichain in $\mathcal{D}_x^{wtt}$.

We now construct a c.e. real x whose approximation structure is quite rich; namely an infinite antichain is embeddable in $\mathcal{D}_x^{\mathrm{r}}$.

Theorem 2. *There are c.e. reals x such that an antichain of wtt-degrees is embeddable in $\mathcal{D}_x^{wtt}$.*

For the proof, we are going to construct a sequence $\{z^n\}$ of computable sequences of rational numbers such that for each n, $\lim z^n = x$. Moreover, we will satisfy the following requirements:

$$R_{\langle e,i,j\rangle} : \neg[\Phi_e(A_{z^i}) = A_{z^j}\,;\ \varphi_e]$$

(where $i \neq j$). In particular, at each stage s of the construction the requirement $R_{\langle e,i,j\rangle}$ will have a current witness $x_{\langle e,i,j\rangle}[s]$ and eventually we will succeed

$$\neg[\Phi_e(A_{z^i}; x_{\langle e,i,j\rangle}) = A_{z^j}(x_{\langle e,i,j\rangle});\ \varphi_e]$$

where $x_{\langle e,i,j\rangle}$ is the final witness of the requirement $R_{\langle e,i,j\rangle}$.

At each stage s we want the first s terms of the sequences $z^0, \dots z^s$ defined. It does not hurt if for the sequence z^n we define only the terms z^n_t for $t \geq n$ (since we can assume that e.g. $\forall t < n,\ z^n_t = 0$). So, at stage s we define the terms $z^0_s, \dots, z^s_s$.

The positive actions for R_t will be implemented via a non-decreasing sequence y which tends to x, the real we want to construct. At any stage s, the interval covered by y (namely $[0, y_s]$) is called *the black area* (see figure 5); and if a term enters the black area at some point, we call it a *black term.* In particular, when we want to put i into A_{z^j} at stage s (for the satisfaction of some requirement) we define $y_s = z^j_i$ (in other words z^j_i *enters the black area*). If at stage s no definition of y_s is mentioned, then we mean that y preserves its last value, i.e. $y_s = y_{s-1}$ and otherwise we say that y is *redefined* (at stage s); so we treat y as a parameter of the construction which changes values in the course of stages.

3.1. **The definition of z^i_j.** The terms of the sequences z^i are defined during the construction; the first thing we do at the begining of a stage is to define some more terms z^i_j.

At stage $s+1$ we divide the interval (y_s, w_s) where

$$w_s = \min\{z^n_i, 1 \mid n \leq i < s+1 \wedge z^n_i > y_s\} \tag{6}$$

into $s+3$ equal parts and set $z^0_{s+1}, \dots z^{s+1}_{s+1}$ on the borders (e.g. such that $z^0_{s+1} > z^1_{s+1} > \cdots > z^{s+1}_{s+1}$); in other words

$$z^n_{s+1} = w_s - (n+1)\frac{w_s - y_s}{s+3} = y_s + (w_s - y_s)\frac{s+2-n}{s+3}$$

for all n with $0 \leq n \leq s+1$. So by stage s we have defined the terms z^i_j with $0 \leq i \leq j \leq s$, as figure 3 demonstrates.

Terms defined

Stages					
0	z_0^0				
1	z_1^0	z_1^1			
2	z_2^0	z_2^1	z_2^2		
$\vdots$	$\vdots$	$\vdots$	$\dots$	$\ddots$	
s	z_s^0	z_s^1	$\dots$	$\dots$	z_s^s

Figure 3: The terms defined by stage s.

3.2. **The satisfaction of R_t.** Now we describe the strategy for the satisfaction of R_t. This consists of two parts. So we break R_t into R_t^1 and R_t^2; and when the first part has been completed, we put $R_t^1\downarrow$; and when the second part is completed we put $R_t^2\downarrow$ and the requirement R_t is satisfied. A few explanatory words are appropriate here. Suppose that $t=\langle e,i,j\rangle$. What we really want to do is, having a current witness x_t for R_t, wait until $\Phi_e(A_{z^i};x_t)\downarrow$ and if it is 0, put x_t into A_{z^j}—this is the action of R_t^2. But one can see that this may injure the computation as some elements *below the use* of the computation may enter A_{z^i} in the course of this action (i.e. by the redefinition of y). For this reason we must act *in advance*—act under R_t^1.

We must also keep some priority on the injuries, so after any action motivated by some requirement, say R_t, we *initialise* all requirements of lower priority (i.e. $R_n, n>t$) according to the following

Definition 2. *To initialise all $R_n, n>t$ at stage s, means to set*

$$x_{t+k}=s+k$$

for $k=1,2,\dots$, and $R_n^1\uparrow$, $R_n^2\uparrow$ for all $n>t$.

We note the following

Fact 2.1. *At any stage s and for any terms $z_{j_1}^{i_1}, z_{j_2}^{i_2}$ (already defined at s) which do not lie in the black area (i.e. $>y_s$) it is*

$$z_{j_1}^{i_1}>z_{j_2}^{i_2} \iff j_1<j_2 \vee [j_1=j_2 \wedge i_1<i_2]$$

This follows from the way we define the terms z_j^i and will be proved in the verification as a lemma.

3.2.1. *The action of R_t^1.* We wait until $\varphi_e(x_t)\downarrow$ and suppose that this happens at stage s. If there are $z_k^i<z_{x_t}^j$ (so $k>x_t$, by fact 2.1) not in the black area (i.e. $z_k^i>y_{s-1}$), with $k<\varphi_e(x_t)$ then put

$$y_s = \max\{z^i_k \mid k \leq s \wedge z^i_k > y_{s-1} \wedge z^i_k < z^j_{x_t}\}$$

and $R^1_t \downarrow$ (remember that $t = \langle e, i, j \rangle$). Also we *initialise* all R_n, $n > t$. When such an action is performed we say that R^1_t *receives attention.*

3.2.2. *The action of R^2_t.* When we know that R^1_t has acted (that is when $R^1_t \downarrow$), then we draw our attention to the satisfaction of R^2_t: we wait until $\Phi_e(A_{z^i}; x_t) \downarrow$ and

1. If $\Phi_e(A_{z^i}; x_t) = 0$ *and* the use of the computation is below $\varphi_e(x_t)$ then we define

$$y_s = z^j_{x_t}$$

thus putting x_t into A_{z^j}.

2. Initialise all R_n for $n > t$ and set $R^2_t \downarrow$.

When this action is performed we say that R^2_t *receives attention.*

3.2.3. *More about the construction.* We say that R_t *requires attention* when one of the following holds

(i) $R^1_t \uparrow$, $R^2_t \uparrow$ and $\varphi_e(x_t) \downarrow$
(ii) $R^1_t \downarrow$, $R^2_t \uparrow$ and $\Phi_e(A_{z^i}; x_t) \downarrow$

And R_t *receives attention* when

- If (i) holds then R^1_t receives attention.
- If (ii) holds then R^2_t receives attention.

3.3. **Construction.**

- *stage* 0. Define $y_0 = 0$ and $z^0_0 = 0.9$.
- *stage* $s + 1$.

step A Define

$$z^n_{s+1} = w_s - (n+1)\frac{w_s - y_s}{s+3}$$

for all n with $0 \leq n \leq s + 1$.

step B Find the least $t < s+1$ such that R_t requires attention. R_t receives attention (and so, y_{s+1} is defined).

3.4. **Verification.** We start with the following basic

Lemma 2.1. *At any stage s and for any terms $z^{i_1}_{j_1}, z^{i_2}_{j_2}$ (already defined at s) which do not lie in the black area (i.e. $> y_s$) it is*

$$z^{i_1}_{j_1} > z^{i_2}_{j_2} \iff j_1 < j_2 \vee [j_1 = j_2 \wedge i_1 < i_2]$$

Proof. It follows from the way we define the terms z_t^k in *step A* of the construction by induction on the stages. Indeed, suppose that it holds at (the end of) stage s (it clearly holds at $s = 0$). The terms z_{s+1}^k (for $k \leq s+1$) will be defined less than all the existing terms which do not lie in the black area; so it holds after step A of stage $s+1$. And if there were $z_{j_1}^{i_1} > z_{j_2}^{i_2}$ in the non-black area at the end of $s+1$ (i.e. greater than y_{s+1}) with neither $j_1 < j_2$ nor $[j_1 = j_2 \wedge i_1 < i_2]$, then these two terms should be already defined at the end of step A of the same stage; but we saw that there are no such terms, a contradiction. So the lemma holds after stage $s+1$ and thus the induction step is proved. □

Note that from the above lemma it follows that

$$z_{j_1}^{i_1} = z_{j_2}^{i_2} \Rightarrow (i_1, j_1) = (i_2, j_2)$$

.

Lemma 2.2. *For every i it is $\lim y = \lim z^i$.*

Proof. By induction, all terms of z^i (for all i) belong in the unit interval. Also, y_s takes values of terms of z^i (for some i) during the construction; so the terms of y also lie in the unit interval and since y is non-decreasing and bounded, it is convergent, say $\lim y = x$. Now fix i. From the construction it follows that

Fact 2.2. *If $x_0 < x$ then there are only finitely many terms of z^i in $(0, x_0)$.*

Claim. *If there is $x_1 > x$ such that infinitely many terms of z^i are in $(x_1, 1)$ then no such term appears in (x, x_1).*

Proof of claim. Suppose otherwise and consider $x < z_j^i < x_1$. Then according to lemma 2.1, for all $k > i, j$

$$k \notin A_{z^i} \Rightarrow z_k^i \in (x, x_1)$$

This contradicts our assumption. □

By the claim we proved, it suffices to prove that for every $x_1 > x$ there are terms $z_j^i \in (x, x_1)$. Suppose that A_{z^i} is co-infinite. Let $j_1 < j_2 < \dots$ be an enumeration of the infinitely many elements of $\mathbb{N} - A_{z^i}$ (by lemma 2.1 we have $z_{j_1}^i > z_{j_2}^i > \dots$). We will show that $\lim_n z_{j_n}^i = x$, thus finishing the proof. Note that for all n, $x \leq z_{j_n}^i \leq z_{j_n}^0$, so that it is enough to prove $\lim_n z_{j_n}^0 = x$. From the construction it follows that if s_n is the stage where $z_{j_n}^0$ was defined then $s_1 < s_2 < \dots$ (actually $s_n = j_n$). At the beginning of stage s_1 we have the interval $I_{s_1} = (y_{s_1-1}, w_{s_1})$ of length ℓ as in the figure below

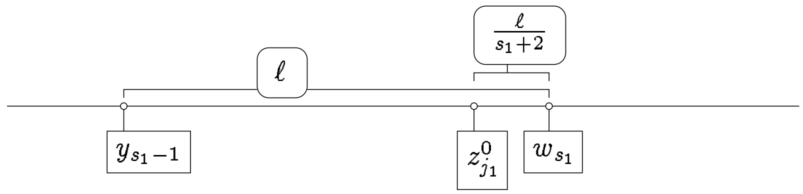

Figure 4: The step A of stage s_1 of the construction.

At step A of s_1 we divide I_{s_1} into s_1+2 equal intervals and set $z^0_{j_1}$ on the first border (and the rest $z^k_{j_1}$, $k \leq j_1$ successively on the other borders according to the construction). Notice that

$$|y_{s_1-1} - z^0_{j_1}| = \frac{s_1+1}{s_1+2}\,\ell$$

and so

$$\ell = \frac{s_1+2}{s_1+1}\,|y_{s_1-1} - z^0_{j_1}| \tag{7}$$

Now during the stages up to s_2, some $z^k_{j_1}$ $(k \leq j_1)$ may enter the black area—but not $z^0_{j_1}$, by the choice of s_1. At the beginning of stage s_2 the black area is up to y_{s_2-1} and suppose that $z^k_{j_1}$ is the least term defined at stage s_1 which now is not in the black area. The situation is pictured in the following figure

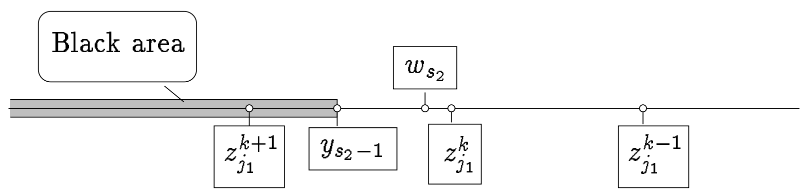

Figure 5: The step A of stage s_2 of the construction.

Now $w_{s_2} \leq z^k_{j_1}$ and $z^0_{j_2}$ is going to be defined according to step A of the construction in the interval (y_{s_2-1}, w_{s_2}) whose length is obviously $\leq \frac{\ell}{s_1+2}$ (the length according to the division done at stage s_1) and so, according to (7), $\leq \frac{|y_{s_1-1}-z^0_{j_1}|}{s_1+1}$. In the same way one can see that for all n,

$$|y_{s_{n+1}-1} - z^0_{j_{n+1}}| \leq \frac{|y_{s_n-1} - z^0_{j_n}|}{s_n+1}$$

so that,

$$|y_{s_{n+1}-1} - z^0_{j_{n+1}}| \leq |y_{s_1-1} - z^0_{j_1}|\,\prod_{i=1}^{n}\frac{1}{s_i+1}$$

But for all n, $y_{s_{n+1}-1} \leq x \leq z^0_{j_{n+1}}$, hence

$$|z^0_{j_{n+1}} - x| \leq |z^0_{j_{n+1}} - x| + |x - y_{s_{n+1}-1}| = |y_{s_{n+1}-1} - z^0_{j_{n+1}}|.$$

Since $\lim_n s_n = \infty$ it is $\lim_n z^0_{j_n} = x$.

Now the case is left where A_{z^i} is co-finite. This means that for almost all j there is s with $y_s > z^i_j$. This, together with fact 2.2 we stated earlier in this proof, gives $\lim_s y_s = \lim z^i$. □

We finish the proof with the following

Lemma 2.3. *All requirements R_t require attention finitely often and are eventually satisfied.*

Proof. We prove the lemma inductively. Suppose that it holds for all $t < t_0$ and $t_0 = \langle e_0, i_0, j_0 \rangle$. In the following for any t we suppose that $t = \langle e, i, j \rangle$. Choose the last stage s_0 where some R_t, $t < t_0$ received attention. Then R_{t_0} is not going to be initialised after s_0 because, according to the construction, this would mean that some R_t with $t < t_0$ receives attention. Also, at the end of s_0, R_{t_0} was assigned a new witness, say x_{t_0}, with $x_{t_0} > s_0$. So $z^j_{x_t}$ is going to be defined at a stage $s_1 > s_0$ and according to *step A* of the construction it is

$$y_{s_1} < z^{j_0}_{x_{t_0}}$$

and $R^1_{t_0} \uparrow, R^2_{t_0} \uparrow$. Now if $\varphi_e(x_{t_0}) \downarrow$ at some later stage $s_2 > s_1$ (the other case being trivial) then $R^1_{t_0}$ will receive attention since it has the priority. And the relevant action (see section 3.2.1) will be performed, so that

$$y_{s_2} = \max\{z^{i_0}_j \mid j \leq s_2 \wedge z^{i_0}_j > y_{s_2-1} \wedge z^{i_0}_j < z^{j_0}_{x_{t_0}}\}.$$

Note that before $z^{j_0}_{x_{t_0}}$ enters the black area (i.e. $y_s \geq z^j_{x_{t_0}}$, if ever) all subsequent (i.e. after s_2) terms of $z^{i_0}_j$ (that is $z^{i_0}_k$ with $k > s_2$) will appear in $(y_{s_2}, z^{j_0}_{x_{t_0}})$. And since all R_t, $t > t_0$ are assigned new witnesses greater than s_2 at stage s_2, all terms $z^j_{x_t}$ for $t > t_0$ will be in $(y_{s_2}, z^{i_0}_{t_0})$. This means that if some R_t acts (after s_2) before $R^2_{t_0}$ acts, then we will continue to have $y_s < z^{j_0}_{x_{t_0}}$ (i.e. $z^{j_0}_{x_{t_0}}$ outside the black area).

Suppose that it is not the case that $\Phi_{e_0}(A_{z^{i_0}}; x_{t_0}) = 0$ with use $< \varphi_{e_0}(x_{t_0})$ after s_2. Then one of the following happens:

1. $\Phi_{e_0}(A_{z^{i_0}}; x_{t_0}) \uparrow$
2. $\Phi_{e_0}(A_{z^{i_0}}; x_{t_0}) = 1$
3. $\Phi_{e_0}(A_{z^{i_0}}; x_{t_0}) \downarrow$ with use $\geq \varphi_{e_0}(x_{t_0})$.

In case 1 it is clear that R_{t_0} is not going to require attention from now on and it is trivially satisfied. Otherwise, suppose that the computation *halts* at stage $s_3 > s_2$. In case 2 we note that $z^{j_0}_{x_{t_0}}$ will continue to stay out of the black area for the same reason that it stayed out during the interval of stages between s_2 and s_3 (i.e. because at stage s_2 we initialised all R_t, $t > t_0$ and so the new witnesses will force the respective terms to be defined in $(y_{s_2}, z^{j_0}_{x_{t_0}})$). So for both of the last two cases it suffices to prove the following

Claim. *In the last two cases the computation is going to be preserved in the following stages.*

Proof of claim. By this we mean that no number *below* the use of the oracle $A_{z^{i_0}}$ in the computation is going to enter $A_{z^{i_0}}$ after stage s_3. Indeed, after the convergence at stage s_3, all R_t, $t > t_0$ will be initialised and assigned witnesses greater than s_3. So (according to lemma 2.1 and the fact that all currently defined terms z^k_r at stage s have $r \leq s$) at any forthcoming redefinition of y (say at stage s_4, caused by some R_t, $t > t_0$) we will still have y_{s_4} less than all terms existing (in the non-black area) at stage s_3. But the use of the computation $\Phi_{e_0}(A_{z^{i_0}}; x_{t_0}) \downarrow$ is less than s_3. So at any forthcoming stage s_4, y_{s_4} will be less than all non-black terms below the use. In other words, no element below the use is going to enter $A_{z^{i_0}}$ (and more generally $\cup_i A_{z^i}$) after s_3 and so the computation will be preserved forever. □

Indeed, now in case 2 the dissagreement will be preserved and in case 3 we will have a computation which is (and will remain) not appropriately bounded.

Now we left the case $\Phi_{e_0}(A_{z^{i_0}}; x_{t_0})[s_3] = 0$ with bound $\varphi_e(x_{t_0})$ which is the one where y is redefined for the sake of $R^2_{t_0}$. In that case, according to the construction, x_{t_0} enters $A_{z^{j_0}}$ (in particular $y_{s_3} = z^{i_0}_{x_{t_0}}$). It suffices to prove the following

Claim. *The computation will not be spoilt by such an action.*

Proof of claim. By the action perfomed at stage s_2, all terms $z^{i_0}_k$ with $z^{i_0}_k < z^{j_0}_{x_{t_0}}$ lying in the non-black area after stage s_2, have $k > s_2$ and thus $k > \varphi_e(x_{t_0})$. So (since $z^{i_0}_k \neq z^{j_0}_{x_{t_0}}$ for all k) all k which go into $A_{z^{i_0}}$ at stage s_3 are greater than the use of the computation and so the last is not spoilt. □

So at the end of stage s_3 we will have the desirable dissagreement and satisfaction of R_{t_0} which will be preserved in the later stages; the last is because the computation will be preserved by the same argument we used in the previous claim. □

3.5. **Further remarks.** Note that for the sequence $\{z^n\}$ we constructed above, it is

$$A_{z^n} \subseteq A_{z^{n+1}}$$

for all n. Also, by a modification of the definition of z^i_s ($i = 0, \ldots, s$) at stage s (namely we define them such that $z^0_s < z^1_s < \cdots < z^s_s$) we get

$$A_{z^n} \supseteq A_{z^{n+1}}$$

for all n. So we have

Corollary 1. *There is a Turing degree which contains an infinite antichain* $\{deg_{wtt}(D_n)\}_{n\in\mathbb{N}}$ *of wtt-degrees with*

$$D_n \subset D_{n+1}$$

for all n. Similar result holds with $D_n \supset D_{n+1}$ in place of $D_n \subset D_{n+1}$.

4. A TRIVIAL $\mathcal{D}_x^m$.

In the last section we exhibited a c.e. real x whose approximation structure is complicated; namely the distribution of the elements of $\mathcal{D}_x^{\mathrm{wtt}}$ in the Turing degree of x is quite sparse. It is natural to look for the other extreme: are there non-computable reals x such that $\mathcal{D}_x^{\mathrm{r}}$ is trivial (i.e. consisting of a unique element)? Well, if r = wtt then the existence of contiguous degrees, i.e. non-trivial Turing degrees which contain a unique wtt degree (a well known result of classical computability theory, see [10]) implies the existence of such reals (due to proposition 1). And this is a concrete example of how the nature of the Turing degree of x is related to the approximation structure of x. We will see however that this relation is not trivial. It is also well known that every non-trivial c.e. Turing degree contains not only infinitely many c.e, m-degrees, but also tt-degrees (again, see [10]). So the structure of m-degrees inside a non-trivial Turing degree is quite rich; and this makes a positive answer to the question whether there is a non-trivial x with $\mathcal{D}_x^{\mathrm{m}}$ trivial interesting. Before giving this answer, we would like to note what makes this fact possible. The reason is that strong reducibilities on the set $\mathcal{S}_x$ give oracle computations with special features. A demonstration of this fact is given in section 5 where we show that the positive and btt(1) reducibilities both coincide with the m-reducibility on $\mathcal{S}_x$.

Theorem 3. *There are non-computable c.e. reals x with the property*

$$\left.\begin{array}{l}\lim z = \lim w = x \\ A_z, A_w \textit{ co-infinite}\end{array}\right\} \Rightarrow A_z \equiv_m A_w$$

The proof is a finite injury priority argument with some special features which we are going to discuss in the following.

4.1. **Preliminaries.** Assume an effective enumeration of the rationals in the unit interval $(0,1)$, say $\{w_i\}$. Now define

$$w_i^e = w_{\varphi_e(i)}$$

where $\{\varphi_e\}$ is a standard enumeration of all partial computable functions. If $w^e = \{w_i^e\}_{i\in\mathbb{N}}$, then $\{w^e\}_{e\in\mathbb{N}}$ is an enumeration of all partial computable rational sequences in the unit interval. In the following, anything we consider on the real line (e.g. sequences, points etc.) are supposed to be in the unit interval (unless otherwise indicated).

We will construct a c.e. real $x = \lim y$ with y a non-decreasing sequence and a sequence z satisfying the following requirements

$$\begin{aligned}
Q :&\quad \lim z = x \\
P_e :&\quad \varphi_e \neq A_z \\
N_e^r :&\quad w^e \text{ total } \Rightarrow A_{w^e} \leq_m A_z \\
N_e^l :&\quad \left.\begin{array}{l} w^e \text{ total} \\ \lim_s w_s^e = x \\ A_{w^e} \text{ co-infinite} \end{array}\right\} \Rightarrow A_z \leq_m A_{w^e}
\end{aligned}$$

At stage s, y_s, z_s are defined. We need y non-decreasing in order to ensure that x is c.e. and also in order to control the enumeration in the various c.e. sets A_w *for any partial sequence* w *in the unit interval.* When we say that a number q at a particular stage of the construction is 'in the black area' we mean that it is $y_s \geq q$ (this terminology is motivated by the illustrations, e.g. figure 6). Also, $R(e,s) = max\{r(i,s) : i < e\}$; $r(e)$ are the restraints we impose on A_z (with respect to the various negative requirements) and $r(e,s)$ their approximation at stage s ($\lim_s r(e,s) = r(e)$, $\lim_s R(e,s) = R(e)$). We note that in many stages it will be $r(e,s) > s$. But we plan finite injury, so that eventually $r(e) = r(e,s) < s$. In the following, sentences like 'at stage s, $r(e)$ is ... ', any parameter considered (like $r(e)$) is supposed to have its current (s-) value. So $r(e,s)$ is sometimes referred to as $r(e)$. It will be clear from the context when $r(e)$ means $\lim_s r(e,s)$. The same applies for other parameters in the construction.

We satisfy P_e by choosing witnesses x_e from the whole pool $\mathbb{N}$ but we also keep priority on the witnesses; this means that at every stage s we have

$$i < j \iff x_i^s < x_j^s. \tag{8}$$

Finally we will arrange the construction so that

Fact 3.1. *At stage* s *we define* z_s*. If* z_k, z_j *are not in the black area at stage* s *then*

$$k < j \iff z_j < z_k.$$

Intuitively this means that there is a tendency to define the terms of z from right to left (i.e. succesively smaller). In contrast, the terms of y are defined from left to right (see e.g. figure 6).

4.2. **Strategies.**

4.2.1. *The strategy for N^r_e.* This is how to make A_z of maximum x-m-degree. This strategy is not difficult and it is easily compatible with other requirements: if $w^e_i[s]\downarrow$ and it is not in the *black area* (i.e. less than y_s, see figure 6) then in our considerations in defining z_s we take into account w^e_i: we define $z_s < w^e_i$ (for all $i \in \mathbb{N}$ with $w^e_i[s]\downarrow$ which have appeared in the non-black area). The situation is pictured in figure 6 where it is seen that we define the current term z_s to be less than all the w^e-terms existing at the time.

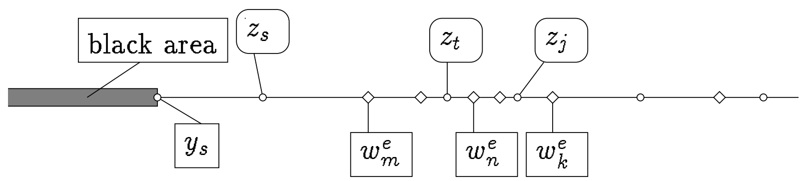

Figure 6: The configuration at stage s from the point of view of N^r_e: we define z_s in an interval (y_s, q) which (currently) contains no terms of w^e.

It is clear that we can put all the N^r_e-strategies together. Now the algorithm for $A_{w^e} \leq_m A_z$ (with the hypothesis that w^e is total) is as follows: to answer '$i \in A_{w^e}$?' we wait until a stage s_0 such that $w^e_i[s_0]\downarrow$ and suppose that i is not already in A_{w^e}. We assume the following

Fact 3.2. *At any stage s, $y_s = z_t$ for some $t \leq s$.*

Then, since no z_t with $t > s_0$ will be greater than w^e_i (before the last enters the black area), the only reason why at some stage s it might happen that $y_s \geq w^e_i$ (i.e. $i \in A^s_{w^e}$) is because some term z_t already existing at s_0 and greater or equal to w^e_i enters the black area. So if z_{t_0} is the least such z_t, we have

$$i \in A_{w^e} \iff t_0 \in A_z.$$

4.2.2. *The basic strategy for P_e.* This is simply wait until $\varphi_e(x_e)\downarrow$ and if it is 0 then put x_e into A_z (i.e. define $y_s \geq z_{x_e}$); otherwise keep x_e out of A_z. Of course, when we choose a witness x_e, it must be $x_e \notin A_z$. One way to do this is: at stage s, choose $x^s_e > s$ *and* set $r(e,s) = x^s_e$ (since we want to continue keeping x^s_e out of A_z until, if ever, it enters A_z under the action of P_e). So, according to (8), $r(e,s)$ is increasing in e and non-decreasing in s. Later we will present a more detailed description of the variation of this strategy we are going to use.

4.2.3. *The strategy for N^l_e.* This is the most important strategy. Viewing the construction from the point of view of N^l_e, the idea is the following: for a particular z_j which is not (yet) in the black area we wait until some w^e_i appears with

$$y_s < w^e_i \leq z_j$$

as in the figure below

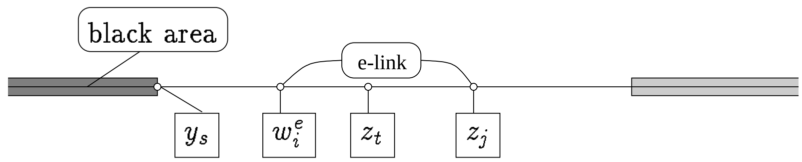

Figure 7: Linking z_j for the sake of N_e^l

Then we create a virtual *link* between w_i^e and z_j (called also *e-link* since it is created for the sake of N_e^l), in symbols (w_i^e, z_j), which indicates that (from the point of view of N_e^l) both w_i^e and z_j and any element (e.g. z_t) of the construction appearing between these two should be treated as one and unique point. By this we mean that if at some point we need to put an element of the interval $[w_i^e, z_j]$ into the black area, then *every* element of the interval must enter the black area. Later we are going to involve a different kind of links (the so-called *back links*) in the construction; to avoid confusion, we call the kind of links we just described *front links*; and a *link* is a front or back link. We note that in the following we identify a link (q,p) with the open (unless otherwise indicated) interval of the real line between q and p (so we may say that a real number 'belongs to a link ℓ'); the context will specify the exact meaning of the word. Moreover, we write e.g. $(\ell]$ for the interval $(q,p]$ (where ℓ is the link (q,p)) and if $q > y_s$ at a stage s, we say that the link is outside the black area.

For reference we give the following

Definition 3. *If at a particular stage we have $y_s < w_i^e \leq z_j$ then we say that z_j can be front e-linked.*

By this strategy we can argue $A_z \leq_m A_{w^e}$ roughly as follows: to answer '$j \in A_z$?' we wait until $z_j \downarrow$ and either z_j enters the black area (forever!) or an e-link (w_i^e, z_j) is created. In the last case the link will be present forever and thus $j \in A_z \iff i \in A_{w^e}$. Of course we will have injuries but we plan to have them finitely many, so that we can start the above procedure after a stage beyond which we have no injury of the higher priority requirements.

It will help if we describe the construction intuitively before we state it formally. As we said, at any stage s we have a black area in the unit interval which is the area which the sequence y_s has covered. The black area expands in the course of stages and approaches x. Also, it is universal in its nature i.e. *it does not depend on the way we look at the construction.* This means that with respect to any negative requirement the black area is the same. In the other direction we have

Definition 4. *Suppose that we are at a particular stage s of the construction. We call $r(e)$-white area the (least) upper part of the unit interval which contains all (currently defined) terms of z that are restrained by $r(e)$; that* *is* $[z_{j_0}, 1)$ *where* z_{j_0} *is the least term* z_j *with* $j \leq s$, $j \notin A_z$ *and* $j \leq r(e)$*; by fact 3.1,* $j_0 = max\{t : t \leq s \wedge t \notin A_z \wedge t \leq r(e)\}$*. Moreover, the e-white area is the union of all* $r(i)$*-white areas for* $i < e$*.*

Of course, if $\{t : t \leq s \wedge t \notin A_z \wedge t \leq r(e)\} = \emptyset$, then we define the $r(e)$-white area to be the empty interval (of reals). We notice that the $r(e)$-white area may expand during the stages, although $r(e)$ remains constant (this is because more terms z_j with $j \leq r(e)$ could be defined at later stages). But in this case, after finitely many stages, it will reach the limit

$$[z_{r(e)}, 1)$$

and will remain such unless $r(e)$ changes or its current value enters A_z. Also, we will take care so that $r(e, s) \notin A_z[s]$ at (the beginning of) every stage s. So, it follows that

Fact 3.3. *At every stage s and every e, if* $r(e, s) \leq s$ *then the* $r(e)$*-white area is* $[z_{r(e,s)}, 1)$.

Of course the black and the white area are subject to the particular stage s of the construction. The white area is also dependent on the particular 'priority level' from which we view the construction (i.e. the number e). More specifically, our priority list is the following:

$$P_0 > 0\text{-links} > N_0 > P_1 > 1\text{-links} > N_1 > \dots \tag{9}$$

where N_e means N_e^l (since we don't have any restraints or positive action for N_e^r). A priority level is a number e and we say that we view the construction from this level when we observe (in the flow of the stages) only the development (actions and restraint modifications) of N_i, P_i for $i \leq e$. In the next section we will see how exactly links are involved in the priority list; in particular, we emphasise that locally we have the following order

$$P_e > e\text{-links} > N_e.$$

4.2.4. *The restraints towards the links.* Suppose that (q, p) is a link outside the black area at some stage of the construction. Now the negative side of this approach of creating links is that, since all elements in $[q, p]$ are treated as one, this happens also when we define the restraints. In particular, if we install the e-link (q, p) at stage $s + 1$ and for some $e_0 \geq e$, p is in the $r(e_0)$-white area and the last does not cover the whole link (see figure 8) then we put

$$r(e_0, s + 1) = max\{t : q \leq z_t < p\}$$

(assuming that there are such t) and 'initialise' all P_i for $i > e_0$ (see definition 15) in order to make them respect the modified restraint.

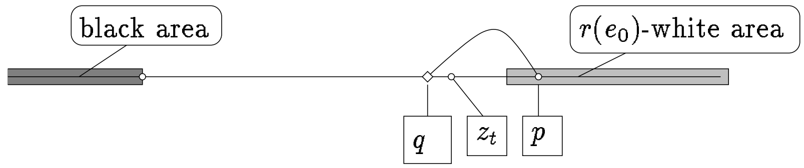

Figure 8: Modification of $r(e_0)$ according to the present e-links with $e \leq e_0$.

This means that in figure 8 we expand the $r(e_0)$-white area up to the smallest term z_t lying *on* the link. This should happen more generally, when we have a 'chain of links' instead of just one link, as in figure 9.

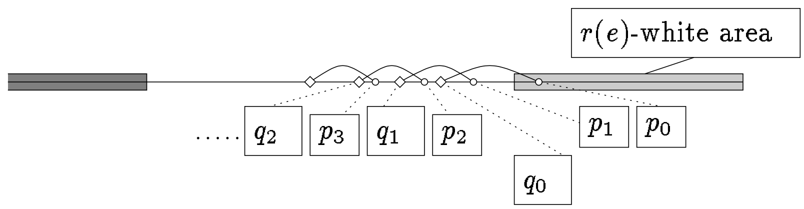

Figure 9: A chain of links.

In order to be more precise, we give the following definitions.

Definition 5. *A (finite) chain is a finite sequence of links* $(q_0, p_0), \ldots, (q_n, p_n)$ *existing at a given stage* s*, which are currently not in the black area (i.e.* $\forall i \leq n,\ q_i, p_i > y_s$*) and such that* $q_n < q_{n-1} \leq p_n < q_{n-2} \leq p_{n-1} < q_{n-3} \leq \cdots \leq p_2 < q_0 \leq p_1 < p_0$.

A chain looks as in the following figure

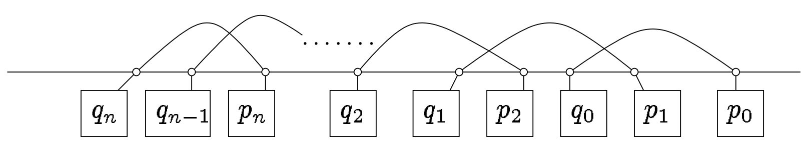

Figure 10: The chain of definition 5.

Note that in the above definition, not all links must be front links; in other words links can be 'back links' (a kind of link we are going to define later). Now the following definition makes precise how restraints 'travel' through links or chains of links.

Definition 6. *Suppose that a link* (q, p) *exists at a certain stage* s *outside the black area and for some* e_0, t *it is* $r(e_0) < t \leq s$, $z_t \in [q, p)$ *and* $z_{r(e_0)} \leq p$ *(see figure 8). Then we say that the restraint* $r(e_0)$ *can travel through the link; when we say that we travel* $r(e_0)$ *through that link we mean that we put*

$$r(e_0) = max\{t : q \leq z_t < p\} \tag{10}$$

Moreover $r(e)$ can travel through a set of links $\mathcal{S}$ which lie outside the black area, if it can travel through at least one of the links in $\mathcal{S}$. When we say that we travel $r(e)$ through the set of links $\mathcal{S}$ we mean that we successively travel $r(e)$ through a sequence of links $\ell_0, \ell_1, \dots, \ell_n$ in $\mathcal{S}$ such that

1. *$r(e)$ can travel through ℓ_0 and for all $i < n$, after it has travelled ℓ_i it can travel ℓ_{i+1}.*
2. *After the last trip through ℓ_n, $r(e)$ cannot travel through any of the links in $\mathcal{S}$.*

This sequence of links is called a path.

Lemma 3.1. *If $r(e)$ travels through a set of links $\mathcal{S}$ (at a stage s) then the path used (say $\ell_0, \dots, \ell_n$) forms a maximal chain in $\mathcal{S}$ (i.e. $\forall \ell \in \mathcal{S}$, the sequence $\ell_0, \dots, \ell_n, \ell$ is not a chain). Moreover, if it travels through different paths then the final value of $r(e)$ will be the same.*

Proof. Suppose that the path used (say $\ell_0, \dots, \ell_n$) is not a chain. Then there is a least i_0 such that $\ell_0, \dots, \ell_{i_0}$ is a chain but $\ell_0, \dots, \ell_{i_0+1}$ is not. Now by definitions 6 and 5 it follows that after travelling through ℓ_{i_0}, $r(e)$ could not travel through ℓ_{i_0+1}, a contradiction.

Now if this path is not maximal as a chain, there would be a link $\ell_{n+1} \in \mathcal{S}$ such that $\ell_0, \dots, \ell_{n+1}$ is a chain. But this would imply that $r(e)$, after travelling through ℓ_{i_n}, can travel through another link of $\mathcal{S}$ (links in a chain are distinct by definition) which contradicts definition 6.

Now suppose that when $r(e)$ travels through the paths $\ell_0, \dots, \ell_n$ and $h_0, \dots, h_m$ it yields different values (e.g. the value with the first is less than the one with the second trip). The value of $r(e)$ after the first trip is

$$[r(e)]_1 = max\{t : z_t \in \ell_n\}$$

(where ℓ_n is considered as a closed interval here). Find the maximum link of the second path (in the ordering $h_0, \dots, h_m$) say (q, p), such that $p \geq z_{[r(e)]_1}$. We claim that there is $z_{t_0} \in [q, p)$ with $z_{t_0} < z_{[r(e)]_1}$. Indeed, if not then $r(e)$ would have not reason to travel the successor link of (q, p) (according to the second trip). But since there is a link $(q, p) \in \mathcal{S}$ with the above properties, the first path is not a maximal chain! This contradicts the first part of the above proposition. So we must have $[r(e)]_1 = [r(e)]_2$. □

By the above proposition when we state the construction we can say e.g. 'travel $r(e)$ through the existing links' (and the new value of $r(e)$ will be uniquely determined).

The travelling of restraints we described in this section *will prevent a link from allowing a positive action to injure a higher priority restraint.* Indeed, suppose that P_{e_1} with $e_1 > e$ would like i in A_{w^e} but N_{e_0} with $e_0 < e_1$ would like j out of A_z (and the link (q, p) of figure 11 exists).

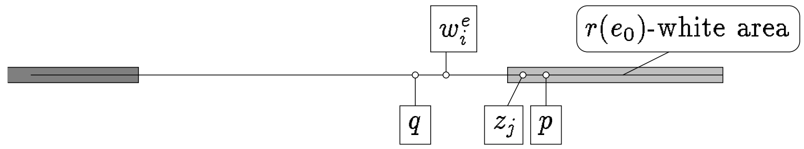

Figure 11: The priority amongst P_e, links and N_e.

According to the priority we set out in (9) we have:

Principle 1. *The e-links are visible (taken into account) only by the requirements P_t, N_k^l, with $t > e$ and $k \geq e$. In particular, $r(e_0)$ can only travel e-links with $e \leq e_0$. Similarly, when y is redefined for the sake of P_t (see definition 13), it can only 'travel' (see definition 13) e-links with $e < t$.*

The link in the above case creates conflict amongst two requirements which otherwise (i.e. if links where not involved in the construction) would not exist. The link should definitely be taken into account when P_{e_1} acts, since it is visible from that requirement (see principle 1). But of course, for priority reasons, the negative requirement N_{e_0} should also be taken into account by P_{e_1}. This means that *in this case we decide not to act*, thus respecting the priority of the requirements as usual: that is why we define (10); by this modification of the restraints, P_{e_1} will be assigned new witness and prevented from injuring higher priority requirements.

Now if we had $e_1 \leq e$ in the situation described above, then the e-link is not visible by P_{e_1} and thus the last need not take it into account at all. In this case i will enter A_{w^e} as P_{e_1} wants it but j will stay out of A_z: so the link (q, p) will be *cancelled.*

Definition 7. *Suppose that during a particular stage s there is a link (q, p) such that q is in the black area but p is not. Then we say that the link is half-black.*

We plan to cancel any *half-black* links at the very stage they appear:

Principle 2. *If at some stage there is a link (q, p) with $q \leq z_t \leq p$ and z_t enters the black area but p stays out of it, then the link is cancelled and never considered in the following stages.*

Note that all the above apply also for *back-links*, a kind of links we are going to define later.

Finally, if it was $e_1 > e$ and $e_1 \leq e_0$ then N_{e_0} should be injured by P_{e_1} (as in a typical priority argument) and the link will be, as we say, *travelled* (by y, see definition 13) (i.e. all t with $z_t \in [q, p]$ go into A_z).

4.2.5. *Problems with the restraints.* Now one can see that creating arbitrarily many links may cause a single restraint to go to infinity. A typical such situation is when the chain in figure 9 is infinite.

Definition 8. *An infinite chain is an infinite sequence of links (q_0, p_0), $(q_1, p_1), \ldots$ created in the course of the construction, such that*

- *the links never go into the black area.*
- *for all k, $p_0 > p_{k+1} \geq q_k > p_{k+2}$.*
- *none of the links is cancelled during the construction.*

Here the vicious situation starts from a term p_0 (see figure 9) which happens to be *in* an $r(e)$-white area. Within a link of this term (q_0, p_0) there is a term of z which causes the $r(e)$-white area (which represents the e-th restraint for A_z) to expand up to p_1. Now the same happens with another term of z and a p_2 in an e_2-link (q_1, p_1) and by creating links in such a fashion indefinitely during the construction, we make the $r(e)$-white area moving towards x without becoming eventually constant. This behaviour of the restraint $r(e,s)$ which now goes to infinity in the course of stages s, may prevent a positive requirement from fulfilling its purpose.

4.2.6. *Bounding the restraints.* To prevent the situation $\lim_s r(e,s) = \infty$ (arising from the existence of *infinite chains* travelled by a single restraint) we will be more careful when we are installing links; namely, we will do that only when we *really* need it.

Links only for terms outside the white area. We remark that

Principle 3. *We need to e-link z_j only when $j \notin A_z[s]$ and $j > R(e,s)$.*

And this is because in the verification of $A_z \leq_m A_{w^e}$ (described above) we can assume we know $R(e)$ (i.e. the final value of $R(e,s)$) *a priori* so that when we are asked about '$j \in A_z$?' with $j \leq R(e)$ we will be able to answer directly (j is in A_z only if it is there by the time $R(e)$ takes its final value). In other words, even if we have the above restriction in installing links, we can still be sure that when we run the stages (after a stage where $R(e)$ becomes constant) *we either find j enumerated in A_z or $j \leq R(e)$* (in which case if it is out, it will stay out forever) *or we find z_j e-linked, in which case the link will stay there forever, thus giving us the answer depending on a unique term of w^e.*

What we want is to 'e-settle' every term of z, for every e such that the hypotheses of N_e^l are satisfied; when we say that z_j is e-settled at some stage, we mean that one of the following cases is realized (*e-linked* means front or back e-linked; back links are defined in the next section):

- z_j has entered the black area.
- z_j has entered the e-white area.
- z_j is e-linked.

Back links. However we need one more trick on the production of links in order to avoid restraints travelling indefinitely. The idea is that instead of creating a front link for z_j, we can alternatively create a so-called 'back link' for it; that is simply link z_j with a term w_i^e with $w_i^e \geq z_j$. And if we prefer back links rather than front ones in a chain travelled by $r(e)$, the restraint cannot travel indefinitely; this is because a restraint cannot travel links which are situated in its own (white) area. In particular, we will prefer to back e-link z_j when we think that creating a front e-link instead, will force some $r(t)$ for $t > e$ to travel it; that is when z_j is in the $r(t)$-white area for some $t > e$. Also, because we do not want to make higher priority restraints to travel, we will require that w_i^e is not in the t-white area.

Definition 9. *We say that z_j can be back e-linked at stage s when it is in the $r(t)$-white area for some $t > e$ and there are terms w_i^e ($w_i^e \downarrow$ by stage s), z_v ($v \leq s$), not lying in the t-white area, with $z_j < w_i^e \leq z_v$.*

The reason why we involved z_v in the above definition is not obvious; we did this in order to realize fact 3.2 which was assumed when we sketched why the strategy for N_e^r works, in section 4.2. Namely, if y ever travels that back link (say at stage s_1), we will arrange that $y_{s_1} = z_v$ instead of merely $y_{s_1} = w_i^e$ (see definitions 13, 14). Because we do not want to injure requirements $r(m)$ for $m < t$ by such an action, we require z_v to be outside of the t-white area. So we treat the elements in $[z_j, z_v]$ as one, and thus we rather say that we link z_j with z_v (instead of w_i^e) and the link is written as (z_j, z_v).

We are now able to argue that while enumerating the links in order to answer '$j \in A_z$?', our search will *halt* giving an answer based on an m-query to A_{w^e}. In fact, one of the following will happen:

(1) z_j enters the black area
(2) z_j is in the e-white area
(3) z_j becomes front e-linked
(4) z_j becomes back e-linked

In other words we will witness that z_j has been *e-settled* according to the following

Definition 10. *We say that z_j is e-settled at stage s when one of the following cases holds:*

- *z_j has entered the black area ($z_j \leq y_s$)*
- *z_j has entered the e-white area ($z_j > y_s \wedge j \leq R(e, s)$)*
- *z_j is front e-linked*
- *z_j is back e-linked*

We say that it is ready to be e-settled when it can be front or back e-linked at the current stage.

Definition 11. *We say that z_j is settled at stage s when it is e-settled for every e. We say that it is ready to be settled when it is ready to be e-settled for some e.*

Note that we will have $r(e, s)$ increasing in e; so, for every stage s and term z_j there is a unique t such that z_j is in the $r(t)$-white area but not in the t-white area.

Definition 12. *Suppose that z_j is ready to be e-settled and it belongs to the $r(t_0)$-white area but not in the t_0-white area. We say that we e-settle z_j when*

- *if it can be back e-linked, we back e-link it with the least z_v with $z_v \geq w_i^e > z_j$ (for some w_i^e), not lying in the t-white area.*
- *Otherwise we e-link it with the largest w_i^e (with $w_i^e \leq z_j$ and not lying in the black area) available.*

We note that we will not succeed in settling all terms z_j. What is needed is to e-settle all z_j just for the e such that the hypotheses of N_e^l hold. If we succeed this then we can give an m-oracle procedure to answer the question '$j \in A_z$?' with the help of A_{w^e} as it was described above.

Why $\lim_s r(e,s) < \infty$. We make the following remarks

Remark 1. *Consider a finite chain as in the following figure*

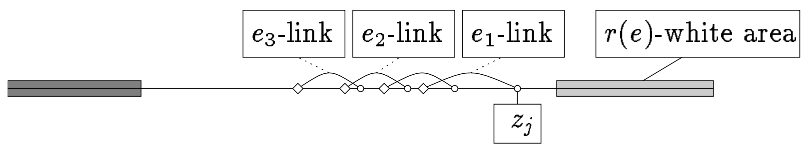

Figure 12: Travelling of $r(e)$ in a single stage.

in a stage where z_{j_0} is out of the $r(e)$-white area and $e_i \leq e$. We will arrange the construction so that the following holds: if at a later stage s_1, z_j enters the $r(e)$-white area (and none of the links in the chain is cancelled by that stage), at this very stage the $r(e)$-white area will be forced to cover all terms of z which lie on a link of the chain. So, when a link which belongs in such a finite chain is travelled by a restraint $r(e)$, then all the links in the chain are travelled at the same time.

Remark 2. *At any stage s only finitely many links are travelled by a restraint $r(e)$. This is because finitely many links exist at s.*

Now we explain why $\lim_s r(e,s) < \infty$; if this is not true, there is a least restraint $r(e)$ which travels through an infinite chain of links during the construction. There are infinitely many stages in which $r(e)$ travels front links and at a single such stage it will travel a finite chain of links (according to the above remarks). We can assume that this infinite chain is not in the e-white area (since e is the least with $\lim_s r(e,s) = \infty$). The situation is pictured in the following figure

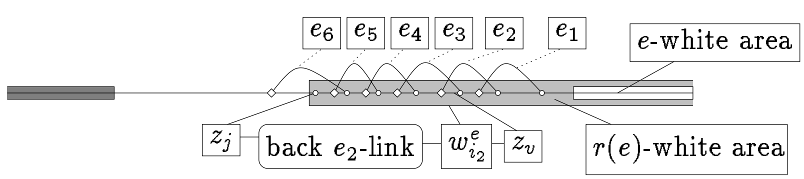

Figure 13: The role of back links.

By principle 3 we have that all the links of the infinite chain we consider are i-links with $i \leq e$. This means that after some stage there will be terms

$w_t^{e_k}$ in the $r(e)$-white area but not in the e-white area, for all k (see figure 13). And since we give priority to *back* linking for terms z_j like the one pictured above, we will stop issuing front e_k-links for such terms (the requests for e_k-linking will be satisfied with back e_k-linking); and $r(e)$ cannot travel any back e_k-link created in its own white area (i.e. the $r(e)$-white area). This is a rough explanation why this chain cannot expand forever; a more detailed proof of this fact is given in the verification.

4.2.7. *More about P_e.* For the satisfaction of P_e we act as usual: P_e can be in the state of 'satisfied' at a particular stage s ($P_e[s]\downarrow$) or in that of 'unsatisfied' ($P_e[s]\uparrow$). We say that it *requires attention at s* when $P_e[s]\uparrow$ and

$$\varphi_e(x_e)[s]\downarrow = 0 \tag{11}$$

As usual, the satisfaction of P_e may be achieved by putting x_e into A_z. This in turn is achieved by the definition of y_s, i.e. by expanding the black area. Because of the presence of links, this may take several steps which we view as substages of the stage s. At each of these intermediate steps we have an approximation y_s^n to y_s and after finitely many steps, this process of generating $y_s^0, y_s^1, \dots$ will come to an end, giving the value of y_s. For reference we give the following

Definition 13. *We say that y_s^n can travel through a (back or front) link (q,p) at a given stage s when $q \le y_s^n < p$. And y_s^n travels through that link when we have $y_s^{n+1} = p$.*

Definition 14. *Lets write $\mathcal{L}_e^s$ for the set of the e-links which exist at stage s (after the end of step B of the construction, see section 4.3). When we say that P_e receives attention at stage $s+1$, we mean that the following action is performed*

1. *Define*

$$\begin{aligned} y_{s+1}^0 &= z_{x_e^s} \\ y_{s+1}^{n+1} &= \max\{\sup \ell \mid \exists m[m < e \wedge \ell \in \mathcal{L}_m^{s+1}] \wedge y_{s+1}^n \in [\ell)\} \end{aligned}$$

where in the expression $\sup \ell$, *ℓ is considered as an interval. After some n, $y_{s+1}^n \uparrow$ (since the set to which the* max *applies will be empty). Now if $i_0 = \mu t[y_{s+1}^{t+1}\uparrow]$ define*

$$y_{s+1} = y_{s+1}^{i_0}.$$

This is a formal way to say that, we first put $y_{s+1}^0 = z_{x_e^s}$ (thus enumerating x_e^s into A_z) and then we travel successively $y_{s+1}^0, y_{s+1}^1, \dots$

through any m-links with $m < e$ that can travel them [3], *until we reach some $y^{i_0}_{s+1}$ which cannot travel through any existing link of this kind: this is the value of y_{s+1}.*

2. *Put $P_e[s+1]\downarrow$.*

Definition 15. *To initialise the requirements P_i for $i > e$ ($e \geq -1$) at stage s means to set*

- $x^s_{e+1} = \mu t[t > s+1 \wedge t > e+1]$ *and for* $i > e$, $x^s_{i+1} = \mu t[t > x^s_i]$.
- $r(i,s) = x^s_i$ *for* $i > e$.
- $P_i[s]\uparrow$.

4.3. **Construction.**

Stage 0 Initialise all P_e, $e > -1$ and set $y_0 = 0, z_0 = 0.9$.

Stage s+1

step A Define z_{s+1} to be in the middle of (y_s, w) where

$$w = min\{w^e_i, z_t, 1 : t \leq s \wedge e, i \in \mathbb{N} \wedge w^e_i[s+1]\downarrow \wedge w^e_i, z_t > y_s\} \tag{12}$$

step B B_1 Find the least j such that z_j is not settled and is ready to be settled. Then, find the least e, say e_0, such that z_j is not e-settled and is ready to be e-settled; e_0-settle z_j.

B_2 Now we travel the least restraint $r(e)$ that can travel through the existing i-links for $i \leq e$. Also, initialise all P_i, $i > e$.

step C Find the least P_e ($e \leq s$) which requires attention and

C_1 P_e receives attention : y_{s+1} is defined.

C_2 Initialise all P_i for $i > e$ and put $r(e, s+1) = s+2$.

C_3 Cancel any half-black links.

Note the redefinition of $r(e)$ at step C_2; this is done in order to realize $r(e,s) \notin A_z[s]$ which we promised just before stating fact 3.3.

4.4. **Verification.**

Lemma 3.2. *If z_k, z_j are not in the black area at a particular stage s then*

$$k < j \iff z_j < z_k.$$

Proof. It follows from the way we define z_{s+1} in *step A* of the construction by induction on the stages. Indeed, suppose that it holds at (the end of) stage s (it clearly holds at $s = 0$). The term z_{s+1} will be defined less than all the existing terms of z which do not lie in the black area; so it holds after step A of stage $s+1$. And if there were $z_k < z_j$ with $k < j$ in the non-black area at the end of $s+1$ (i.e. greater than y_{s+1}), then these two terms should be already defined at the end of step A of the same stage; but we saw that

[3] for a y^n_{s+1} there may be more than one ways (links) to travel it. In the formal definition we choose the link which will make y^{n+1}_{s+1} maximum; but it is easy to see that any other choice would lead to the same definition of y_{s+1}.

there are no such terms, a contradiction. So the lemma holds after stage $s+1$ and thus the induction step is proved. □

Lemma 3.3. $\lim y = \lim z$

Proof. By induction, all terms of z belong in the unit interval. Also, y_s takes values of terms of z during the construction; so the terms of y also lie in the unit interval and since y is non-decreasing and bounded, it is convergent, say $\lim y = x$. Now from the construction it follows that

Fact 3.4. *If $x_0 < x$ then there are only finitely many terms of z in $(0, x_0)$.*

Claim. *If there is $x_1 > x$ such that infinitely many terms of z are in $(x_1, 1)$ then no term of z appears in (x, x_1).*

Proof of claim. Suppose otherwise and consider $x < z_{j_0} < x_1$. Then according to lemma 3.2, for all $j > j_0$, $j \notin A_z \Rightarrow z_j \in (x, x_1)$. This contradicts our assumption. □

By the claim we proved, it suffices to prove that for every $x_1 > x$ there are terms $z_j \in (x, x_1)$. Suppose that A_z is co-infinite. Let $j_1 < j_2 < \dots$ be an enumeration of the infinitely many elements of $\mathbb{N} - A_z$ (by lemma 3.2 we have $z_{j_1} > z_{j_2} > \dots$). We will show $\lim_s z_{j_s} = x$, thus finishing the proof. From the construction it follows that if s_n is the stage where z_{j_n} was defined then $s_1 < s_2 < \dots$ (actually $s_n = j_n$) and

$$z_{j_n} = y_{s_n - 1} + \frac{\lambda_{s_n} - y_{s_n-1}}{2}$$

where λ_{s_n} is the minimum of 1 and all z_j, w_i^e which have appeared by the end of stage $s_n - 1$ and are not yet in the black area (see (12) in the construction). Clearly we have $\lim y = x \geq y_n$ and $z_{j_n} \geq \lambda_{s_{n+1}}$ for all n. So, if

$$\begin{aligned} a_1 &= z_{j_1} \\ a_{n+1} &= x + \frac{a_n - x}{2} \end{aligned}$$

is a recursively defined sequence, then for all n

$$\begin{aligned} a_n &\geq z_{j_n} \\ a_n &> a_{n+1} \end{aligned}$$

So it is enough to prove that $\lim_n a_n = x$. But this is not difficult to do (we omit the proof).

Now the case is left where A_z is co-finite. This means that for almost all j there is s with $y_s > z_j$. This, together with fact 3.4 we stated earlier in this proof, gives $\lim y = \lim z$. □

Lemma 3.4. *N_e^r are satisfied.*

Proof. Suppose that the hypotheses of N_e^r hold. It is enough to prove $A_{w^e} \leq_m A_z$. To answer '$i \in A_{w^e}$?' wait until a stage s_0 with $w_i^e[s_0]\downarrow$ and suppose that $w_i^e > y_{s_0}$ (otherwise we answer positively). By the construction, every time y_s is redefined, it is $y_s = z_k$ for some $k \leq s$. Also,

$$s > s_0 \Rightarrow z_s < w_i^e$$

as long as w_i^e stays out of the black area. This means that, if $z_{j_0} = min\{z_t : t \leq s_0 \wedge z_t \geq w_i^e\}$ then

$$i \in A_{w^e} \iff j_0 \in A_z$$

Of course, if $\{z_t : t \leq s_0 \wedge z_t \geq w_i^e\} = \emptyset$ then $i \notin A_{w^e}$. □

Lemma 3.5. *If an e-link is cancelled at stage s then at the same stage some P_i with $i \leq e$ receives attention.*

Proof. Suppose that the link (q,p) is cancelled at s. From the construction it follows that some P_i receives attention at s and that (q,p) becomes half-black during part C_1 of the stage s. So $q \leq y_s < p$. But $y_s = y_s^n$ for some n such that y_s^n cannot travel through any of the existing m-links with $m < i$. Now if it was $i > e$, the link (q,p) would be visible from P_i and according to definition 14, y_s^n would travel it. But this would mean that this link is not half-black, a contradiction. □

Lemma 3.6. *A term z_j is settled at stage s $(s \geq j)$ iff it is e-settled for every $e < j$.*

Proof. By definition 11 if z_j is settled, then it is e-settled for all $e < j$. So it remains to prove the converse. By induction on the stages of the construction it follows that $r(e,s)$ is non-decreasing in s for every e. Also, by definition 15 and step 0 of the construction, we have $\forall e$, $r(e,0) > e$. So,

$$\forall e \forall s [r(e,s) > e \wedge R(e,s) \geq e]$$

This means that for all $e \geq j$, z_j is in the e-white area (if it is not in the black one) and thus it is automatically e-settled due to definition 10. So if it is e-settled for every $e < j$, then it is e-settled for all e, i.e. settled. □

Lemma 3.7. *A requirement P_e never injures a restraint with higher priority; i.e. if P_e acts at s, no number $m \leq R(e,s)$ enters A_z at that stage.*

Proof. Suppose that P_{e_0} acts at s_0 (under step C_1 of the construction) and $m \leq R(e_0, s_0)$ *enters* A_z.

Claim. *$r(e,s)$ is increasing in e at every s.*

Proof of claim. This holds at stage 0 by definition 15. Suppose that it holds at (some step of) stage s. At the next step, either all $r(e)$ remain the same or some $r(e)$ increases under step B_2 and all $P_i, i > e$ are initialised, or we just initialise all $P_i, i > e$ under step C_2. In any case the claim continues to hold due to definition 15. □

So we have $R(e,s) = r(e-1,s)$ and in particular $R(e_0,s_0) = r(e_0-1,s_0)$ and

$$m \leq r(e_0-1,s_0). \tag{13}$$

On the other hand we have

Claim. $\forall e,s,\ r(e,s) \geq x_e^s$

Proof of claim. At stage 0 it holds. If it holds at a particular step of a stage, then in the next step either $r(e)$ increases via step B_2 or all $P_i, i > t$ for some $t < e$ are initialised via step B_2 or C_2 (or both x_e, $r(e)$ remain the same). In any case, the claim continues to hold due to definition 15. □

Note that as long as $P_e \uparrow$, if $r(e,s) > x_e^s$ at some stage, then $r(e)$ has travelled some links; so, by definition 6, it is $r(e,s) \leq s$. This, along with the last claim gives

$$x_e^s \leq s \Rightarrow r(e,s) \leq s$$

for all e,s. Now since P_{e_0} acted at s_0, it must be $x_{e_0}^{s_0} \leq s_0$. So $r(e_0,s_0) \leq s_0$ and in particular $r(e_0-1,s_0) \leq s_0$; and because of (13) and lemma 3.2, $z_{r(e_0-1)} \leq z_m$. This means that by the action of P_{e_0}, $r(e_0-1)$ was also enumerated in A_z. Now by an induction on the stages (and substages) of the construction one can prove that

$$\forall e,s,\ x_e^s > R(e,s).$$

So we have $x_{e_0}^{s_0} > R(e_0,s_0)$ and according to the above,

$$z_{x_{e_0}} < z_{r(e_0-1)} \leq z_m \tag{14}$$

at s_0.

Claim. *At step C_1 of the construction a chain $\ell_1, \dots \ell_n$ of e-links with $e < e_0$ was travelled by y.*

Proof of claim. Suppose otherwise; then by construction, $y_{s_0} = z_{x_{e_0}}$. So, by 14, m stays out of A_z at s_0; a contradiction. □

For the chain of the above claim, it is $z_{x_{e_0}} \in [\ell_n)$ and $z_{r(e_0-1)} \in (\ell_i]$ for some $i \leq n$ (the last because m goes into A_z and (14)). Suppose that i is the maximum such that $z_{r(e_0-1)} \in (\ell_i]$. We have

Claim. *At step B_2 of s_0, $r(e_0-1)$ would travel ℓ_i.*

Proof of claim. Suppose that $i = n$. Then it follows from (14) and the fact $z_{x_{e_0}}, z_{r(e_0-1)} \in [\ell_n]$ that $r(e_0-1)$ will travel ℓ_n. Now suppose that $i < n$. Then, if $\ell_{i+1} = (q,p)$, it is $p \in [\ell_i)$ and $p < z_{r(e_0-1)}$ (otherwise i

would not be maximum such that $z_{r(e_0-1)} \in (\ell_i])$. And p is a term of z, so that $r(e_0 - 1)$ will travel ℓ_i. □

As a result, P_{e_0} would be initialised. This is a contradiction since the new witness will not satisfy $z_{x_{e_0}} \in [\ell_n)$. □

Lemma 3.8. *For all e, $\lim_s r(e,s) < \infty$ and P_e receives attention finitely often; also, if φ_e is total then P_e is satisfied.*

Proof. We prove this by induction on the priority list $P_0 > N_0 > P_1 > N_1 > \ldots$. That the lemma is true for P_0 it is easy to see. Now suppose that the lemma is true for all $i < i_0$; and choose the least stage s_0 after which no P_i with $i < i_0$ receives attention and no $r(i)$ changes its value. At s_0, P_{i_0} has a current witness, say x_{i_0}. P_{i_0} is not going to be initialised under step B_2 (after s_0) because otherwise some $r(e)$ with $e < i_0$ would travel and change its value. Also, it will not be initialised under C_2, because otherwise some P_i with $i < i_0$ receives attention. Now if P_{i_0} receives attention while it has x_{i_0} as a witness (the *last* witness), then it is $\varphi_{i_0}(x_{i_0}) \downarrow = 0$ and x_{i_0} will enter A_z (since it has the priority amongst the positive requirements) so P_{i_0} is satisfied for ever ($P_{i_0} \downarrow$ and it never requires attention again). Otherwise, suppose that $\varphi_{i_0}(x_{i_0}) \downarrow = 1$. Then x_{i_0} is restrained from A_z with priority i_0, *and it is going to stay restrained.* This is because the restraints $r(e,s)$ are non-decreasing in s; and since no P_i with $i < i_0$ is going to receive attention, by lemma 3.7 we have that x_{i_0} will stay out and so P_{i_0} is satisfied. The case when $\varphi_{i_0}(x_{i_0}) \uparrow$ is trivial.

To complete the proof it is enough to prove $\lim_s r(i_0, s) < \infty$. Choose a stage s_1 after which no P_i with $i \leq i_0$ receives attention and all $r(i)$ for $i < i_0$ have reached their limit. For a contradiction, suppose that $\lim_s r(i_0, s) = \infty$ (it cannot be otherwise since $r(i_0, s)$ is non-decreasing in s). We claim that step B_2 is performed infinitely many times for the sake of N_{i_0}; indeed, if not, then because of our assumptions about s_1, $r(i_0)$ would not change afterwards (since step C_1 would not be performed for P_i with $i \leq i_0$), a contradiction. It follows that there is an infinite chain of links (see definition 8) such that $r(i_0)$ travels through it at infinitely many stages. By construction we have that all links occurring in the chain are e-links for $e \leq i_0$ which never enter the black area. Indeed, when they were travelled by $r(i_0)$ they were outside the black area; and later they were *in* the $r(i_0)$-white area and thus protected with priority i_0. So, since no P_i with $i \leq i_0$ is going to act, they will never get injured (i.e. cancelled) or enter the black area.

So there is a finite set B which consists of all e such that infinitely many e-links occur in the infinite chain. Choose a stage $s_2 > s_1$ beyond which $r(i_0)$ travels only e-links with $e \in B$ and it has already travelled e-links for every $e \in B$ since stage s_1. At some $s_3 > s_2$, $r(i_0)$ is going to travel again a finite chain and at the last link of this chain it will stop because of the lack of a suitable link (only finitely many links exist at a particular stage). Note the following

Fact 3.5. *It is $r(i_0) \notin A_z$; so the $r(i_0)$-white area is $[z_{r(i_0)}, 1)$.*

In case P_{i_0} acted after s_0, this follows from the second action in step C_2 of the construction; also because, since $r(e)$ has started travelling and is not going to be initialised in later stages, it is $r(e,s) \leq s$ for all $s > s_2$. By the next (i.e. after s_3) stage where $r(i_0)$ is travelling again, a suitable link will have appeared; this is an e-link ℓ with $z_{r(i_0)} \in (\ell]$.

Claim. *This link cannot be back e-link.*

Proof of claim. Suppose otherwise. When $r(i_0)$ stopped travelling at stage s_3, ℓ did not exist. And such back link cannot be created in later stages by definition 9. Indeed, ℓ would have appeared for the back e-linking of some z_j which is situated in the $r(t)$-white area for some $t > i_0$. But then, by definition 9, $(\ell]$ does not intersect the t-white area and so the $r(i_0)$-white area (which is contained in the former). So $z_{r(i_0)} \notin (\ell]$, a contradiction. □

So ℓ is a *front* link. Now we claim that such a link should not appear, arriving thus to a contradiction; indeed, when some $z_j \geq z_{r(i_0)}$ asked for e-linking, according to the construction we first look whether we can create a back link (w.l.o.g. suppose that z_j is not in the i_0-white area). If this is not possible, i.e. there is no w_i^e, z_v with $z_j < w_i^e \leq z_v < z_t$ (where $[z_t, 1)$ is the i_0-white area) then by the choice of s_2 there must be a term w_i^e with $z_{r(i_0)} \leq w_i^e \leq z_j$ (otherwise $r(i_0)$ would not have travelled an e-link after stage s_1, a contradiction). So, since we front e-link z_j with *the least* w_i^e available, such a link should not appear, a contradiction. □

Lemma 3.9. *For every $j \in \mathbb{N}$ and e such that $x = \lim w^e$ and A_{w^e} is co-infinite, there is a stage s_0 in which z_j is settled and any links (q, z_j), (z_j, p) ($e \in \mathbb{N}$) are never cancelled (so it remains settled in later stages).*

Proof. Suppose not. Then there is a least j_0 for which the lemma does not hold. Choose a stage s_0 after which, for all $i \leq j_0$, P_i does not receive attention, $r(i)$ has reached its limit, and every z_j with $j < j_0$ which is to be settled, has already been settled. After this stage no $j < j_0$ will receive attention under step B_1 of the construction. Given j_0, choose e_0 to be the least e such that the lemma does not hold for j_0, e_0. Also, choose a stage $s_1 > s_0$ such that for every $i < e_0$, if z_{j_0} is to be i-settled, it is already so. Then, after that stage, z_{j_0} will not receive attention for i-linking with $i < e_0$.

Now by the hypothesis that z_{j_0} does not satisfy the lemma, we have that $j \notin A_z$ and that there are arbitrarily close terms of w^{e_0} to x from the right. This means that at some stage after s_1, z_{j_0} will be ready to be e_0-settled (if it has not done so far) and it will be e_0-settled immediately since it has the priority; and the link by which it is settled will never be cancelled (by lemma 3.5 and the assumption that no P_i with $i \leq j_0$ receives attention). This is a contradiction. □

Lemma 3.10. *N_e^l are satisfied.*

Proof. Suppose that $\lim w^e = x$ and A_{w^e} co-infinite. Choose a stage s_0 after which no P_t with $t \leq e$ acts. To answer '$j \in A_z$?' look for a stage $s > s_0$ such that

(1) it has entered the black area; or
(2) it has entered the e-white area; or
(3) z_j is front e-linked with some w_i^e.
(4) z_j is back e-linked with some z_v ($z_j < w_i^e \leq z_v$ for some w_i^e).

From lemma 3.9 it follows that we will find such a stage. In case (1) answer $j \in A_z$ and in (2) negatively. In case (3) say that

$$j \in A_z \iff i \in A_{w^e}.$$

This is true; indeed, since $w_i^e < z_j$ (as we have a front link), we have $j \in A_z \Rightarrow i \in A_{w^e}$. For the other direction we note that this link is not going to be cancelled (by our assumptions about s_0 and lemma 3.5). So, if i enters A_{w^e} at some later stage, this will be due to a movement of y motivated by some P_t for $t > e$; this means that the link will be visible from P_t and y will follow it, thus pushing j into A_z.

A similar argument (but with z_j in place of w_i^e and vice-versa) shows that in case (4) we also have $j \in A_z \iff i \in A_{w^e}$. □

5. Strong reducibilities on $\mathcal{S}_x$.

In this section we demonstrate that strong reducibilities lose some of their generality when restricted to the class of x-sets $\mathcal{S}_x$. In other words the oracle computations are of more special nature, a fact which (as we noted before) allowed us to prove the strong result in section 4 (theorem 3) which is not true in the general case (i.e. the whole structure of m-degrees inside a Turing degree). In particular we show that the positive and btt(1) (or $m*$ as we call it, see definition 16 or [9]) coincide with the m-reducibility.

The m-reducibility consists of *one positive* query. We remind the reader of *btt*(1) reducibility, which consists of either one positive or *one negative* query, and which we prefer to call '$m*$ reducibility' (see [9] for more on that). Formally,

Definition 16. *For any two sets A, B we say that $A \leq_{m*} B$ (via f, g) when there are computable functions f, g such that for all n,*

$$n \in A \iff \begin{cases} f(n) \in B & \text{if } g(n) = 0 \\ f(n) \notin B & \text{otherwise.} \end{cases}$$

Proposition 2. *If x is non-computable then $\leq_{m*}\restriction \mathcal{S}_x = \leq_m \restriction \mathcal{S}_x$ and in particular, for every two sequences z, w with $\lim z = \lim w = x$ and $A_z \leq_{m*} A_w$ via f, g, $g(n) = 0$ for almost all n. In other words the $m*$-reduction is actually an m-reduction with finitely exceptions (i.e. negative queries).*

Proof. Suppose that for the A_z, A_w above, $A_z \leq_{m*} A_w$ and $g(n) > 0$ for infinitely many n. We will prove that x is computable. Indeed, we can effectively find the zeros of g, so that there is an increasing function h such that for all n, $g(h(n)) > 0$. But then we have

$$h(n) \in A_z \iff f(h(n)) \notin A_w$$

So for every n, $z_{h(n)}$, $w_{h(n)}$ are not on the same side of x. And since h is increasing, these are subsequences of z, w with the property

$$\lim_n |z_{h(n)} - w_{h(n)}| = 0$$

and for all n,

$$x \in (\min\{z_{h(n)}, w_{h(n)}\}, \max\{z_{h(n)}, w_{h(n)}\}).$$

But this means that x is computable, a contradiction. □

We remind that positive reducibility $\leq_p$ is like *tt* but we don't allow negative queries (formally, the p-formulas are constructed from the atoms via $\{\wedge, \vee\}$ instead of $\{\neg, \wedge, \vee\}$; for more details see [9]).

Proposition 3. *Suppose that $x = \lim z = \lim w$. If $A_z \leq_p A_w$ then $A_z \leq_m A_w$. Moreover, the second reduction is obtained effectively from the first one (given w).*

Proof. Suppose that $\{\sigma_n\}_{n\in\mathbb{N}}$ is an effective enumeration of the positive (p-) conditions (i.e. the propositional formulas built from the atoms $m \in X$ by applying $\vee$, $\wedge$ inductively, using the standard syntactical rules). For the proof of the proposition it is enough to define an algorithm g which takes a number n and (the program for) a computable sequence of rationals w as inputs, and outputs a number $g(n, w)$ such that

$$\sigma_n \vDash A_w \iff g(n, w) \in A_w. \tag{15}$$

Indeed, having defined such an effective procedure, suppose that we are given a p-reduction $A_z \leq_p A_w$ via a computable function f, i.e.

$$n \in A_z \iff \sigma_{f(n)} \vDash A_w.$$

Then for every n we have

$$\sigma_{f(n)} \vDash A_w \iff g(f(n), w) \in A_w$$

which gives

$$n \in A_z \iff g(f(n), w) \in A_w$$

i.e. an m-reduction (which we got effectively from f and w).

We define the program g by induction on the length[4] of the p-conditions.

[4]The length ℓ of a p-condition is defined by induction as usual: if σ_n is an atom then $\ell(n) = 0$; and if σ_n is $\sigma_k \vee \sigma_s$ or $\sigma_k \wedge \sigma_s$ then $\ell(n) = \max\{\ell(k), \ell(s)\} + 1$.

For all n and computable sequences of rationals w we define $g(n,w)$ as follows;

1. If $\ell(n)=0$ then σ_n is an atom, say '$t \in X$' (where t is a number we get effectively from n). For all w define $g(n,w)=t$.
2. Suppose $m>0$ and $g(t,w)\downarrow$ for all w and t with $\ell(t)<m$. If $\ell(n)=m$ for some formula σ_n, then σ_n is $\sigma_k \vee \sigma_s$ or $\sigma_k \wedge \sigma_s$ for k,s with $\ell(k),\ell(s)<m$. In the first case define

$$g(n,w)=\begin{cases} g(k,w) & \text{if } w_{g(k,w)} \leq w_{g(s,w)} \\ g(s,w) & \text{otherwise} \end{cases}$$

and in the second case define

$$g(n,w)=\begin{cases} g(s,w) & \text{if } w_{g(k,w)} \leq w_{g(s,w)} \\ g(k,w) & \text{otherwise.} \end{cases}$$

To finish the proof, we prove by induction that (15) holds for all n,w. If $\ell(n)=0$ it is obvious. If $\ell(n)>0$ and for all σ_t with $\ell(t)<\ell(n)$ it holds then two can happen;

1. If $\sigma_n=\sigma_k \vee \sigma_s$ then $\ell(k),\ell(s)<\ell(n)$ and by induction hypothesis

$$\begin{aligned} \sigma_k \models A_w &\iff g(k,w)\in A_w \\ \sigma_s \models A_w &\iff g(s,w)\in A_w \end{aligned}$$

for all w. But

$$\sigma_n \models A_w \iff \sigma_k \models A_w \vee \sigma_s \models A_w \iff g(k,w)\in A_w \vee g(s,w)\in A_w$$

for all w. And if for a particular w, $w_{g(k,w)} \leq w_{g(s,w)}$ then

$$g(k,w)\in A_w \vee g(s,w)\in A_w \iff g(k,w)\in A_w$$

(by definition of A_w) which means that (15) holds for this w (by definition of g). Also if $w_{g(k,w)} > w_{g(s,w)}$,

$$g(k,w)\in A_w \vee g(s,w)\in A_w \iff g(s,w)\in A_w$$

which means again that (15) is correct for this w.

2. If $\sigma_n=\sigma_k \wedge \sigma_s$ then $\ell(k),\ell(s)<\ell(n)$ and by induction hypothesis

$$\begin{aligned} \sigma_k \models A_w &\iff g(k,w)\in A_w \\ \sigma_s \models A_w &\iff g(s,w)\in A_w \end{aligned}$$

for all w. But

$$\sigma_n \models A_w \iff \sigma_k \models A_w \wedge \sigma_s \models A_w \iff g(k,w)\in A_w \wedge g(s,w)\in A_w.$$

And if $w_{g(k,w)} \leq w_{g(s,w)}$ then

$$g(k,w) \in A_w \wedge g(s,w) \in A_w \iff g(s,w) \in A_w$$

(by definition of A_w) which means that (15) holds for this w (by definition of g). Also if $w_{g(k,w)} > w_{g(s,w)}$,

$$g(k,w) \in A_w \wedge g(s,w) \in A_w \iff g(k,w) \in A_w$$

which means again that (15) is correct.

So in any case (15) holds for n and all w, and the induction step is proved. □

6. Immunity properties.

In this section we look at the immunity of the sets A_z given a c.a. real x and sequences z with limit x. As we explained in the introduction, one may think that the more immune the set A_z (or its complement) is, the more complicated the real x is (one can prove that the immunity of such a set does not depend on the choice of z); this is because, in a way, the more immune e.g. the set A_z is, the more difficult is to make correct 'guesses' about terms of z which are on the left of x (so, rationals in the left Dedekind cut of x). However we show that not only the immunity of A_z is independent of z, but it is in a sense independent of x itself! In particular, assuming that x is not computable, A_z cannot be (co-)hh-immune but it is always bi-h-immune or (co-)h-simple (the later when x is semi-computable). So we always have h-immunity and hh-immunity never occurs.

6.1. Hyperimmunity.

Suppose that $\lim z = x$ for a computable sequence of rationals $z = \{z_s\}$.

Proposition 4. *If $x = \lim z$ is not c.e. then A_z is h-immune and if it is not co-c.e., then B_z is h-immune. So, if x is not semi-computable, A_z, B_z are bi-immune. Also, if it is c.e. (co-c.e. resp.) non-computable then A_z (B_z resp.) is hypersimple.*

Proof. Suppose that x is not c.e. and A_z was not h-immune. Then there exists a disjoint strong array $D_{g(n)}$ such that for every n,

$$D_{g(n)} \cap A_z \neq \emptyset$$

Now consider the sequence

$$y_s = \min\{z_n \mid n \in D_{g(s)}\}$$

which is a computable sequence of rationals with the property for all s, $y_s < x$. Indeed, if that was not the case for some s, this would mean that $D_{g(s)} \cap A_z = \emptyset$. Moreover, since the array $D_{g(n)}$ is disjoint and $\lim z = x$, it follows that $\lim y = x$. But this is a contradiction since we assumed that x is not c.e. So A_z is in fact h-immune. The case of x co-c.e. can be treated by a dual proof and the rest of the statements in the proposition follow easily. □

After we sorting out h-immunity we would like to look at hh-immunity (and in particular prove that it never occurs). This is more difficult, and we consider separately the cases when x is semi-computable or not.

6.2. Non semi-computable reals and hh-immunity.

Theorem 4. *If x is not semi-computable and $z = \{z_s\}$ is a computable sequence of rationals with $\lim z = x$, then A_z, B_z are not hh-immune.*

Proof. Given a set A, define a tree $I(A) : \Sigma^* \to \mathcal{P}(A)$ (where $\mathcal{P}(A)$ is the powerset of A and $\Sigma = \{0, 1\}$). For all $w \in \Sigma^*$ define

$$\begin{aligned} I_\emptyset(A) &= A \\ I_{w0}(A) &= I_w(2A) \\ I_{w1}(A) &= I_w(2A+1) \end{aligned}$$

In this way we split A into the nodes of a tree (which represent subsets of A) such that, if two nodes $I_{w_1}(A)$, $I_{w_2}(A)$ lie in different branches (that is $w_1 \mid w_2$ i.e. they are incomparable w.r.t. the lexicographical ordering of binary strings) then $I_{w_1}(A) \cap I_{w_2}(A) = \emptyset$. We will only need the tree $I(\mathbb{N})$ which we are going to write simply as I in the following. It is easy to see that all nodes of this tree are infinite subsets of $\mathbb{N}$. Now we define a suitable disjoint weak array $W_{g(n)}$ ($n \in \mathbb{N}$) which indicates that A_z is not hh-immune. The computable function g is implicitly defined in the following. The enumeration of $W_{g(n)}$ (for a particular n) is associated with the node I_{1^n0} of the tree I. In particular, it is defined as follows: start enumerating all $s \in I_{1^n0}$ (for successively larger s) and when you come across an s_0 such that z_{s_0} is smaller that all z_s for the s enumerated so far (that is, for all $s \in I_{1^n0}$ with $s < s_0$), enumerate s_0 into $W_{g(n)}$; continue in the same way. Since each node I_{1^n0} is a c.e. set and x is not semi-computable, it is impossible to have $I_{1^n0} \subset A_z$ or $I_{1^n0} \subset B_z$. So, in our enumeration we will find some terms z_s lying on the left of x *and* some lying on the right of x. So, since $\lim z = x$ we have that for all n, $W_{g(n)}$ is finite and the last element s enumerated in it, is in A_s. So

$$W_{g(n)} \cap A_z \neq \emptyset$$

Finally, the array $W_{g(n)}$ is also disjoint since for all n, $W_{g(n)} \subset I_{1^n0}$ and

$$n \neq m \Rightarrow 1^n0 \mid 1^m0 \Rightarrow I_{1^n0} \cap I_{1^m0} = \emptyset$$

Now the array $W_{g(n)}$ with all the above properties witnesses that A_z is not hh-immune. The case for B_z is dual. □

Remark 3. *One other question is for which c.e. reals $x = \lim z$ the set A_z is promptly simple. If the degree of x is not promptly simple, then obviously there is no (computable) sequence z with limit x and A_z promptly simple. Also, it is easy to construct reals $x = \lim z$ such that A_z is promptly simple. The requirements for a basic such construction are:*

$$\begin{array}{ll} Q_e: & W_e \text{ infinite} \Rightarrow \exists x \exists s\ x \in W_{e,\ at\ s} \cap A_z[s] \\ P_e: & \qquad A_z \neq \varphi_e \end{array}$$

and they are satisfied as in a usual finite injury construction (we have restraints for the requirements P_e). Note that instead of requiring $\mathbb{N} - A_z$ to be infinite, we require A_z to be non-computable, which here amounts to the same thing. The requirements Q_e are easily compatible with a large range of other requirements. Also, in a construction where non-computability of A_z is guaranteed by other requirements, P_e may be omitted.

6.3. **Semi-computable reals and hh-immunity.** According to the above, if x is not semi-computable, both A_z, B_z are h-immune and not hh-immune. On the other hand, if x is c.e. non-computable, A_z is h-simple (if not co-finite); a dual result holds for co-c.e. reals. A natural question is whether A_z or B_z can be hh-immune for semi-computable reals (note that the proof for the case of non semi-computable x cannot be adapted for this case). We prove not only a negative answer to this, but also that A_z or B_z cannot be even finitely strongly h-immune (fsh-immune for short). Before presenting the result, we remind the definition of fsh-immunity.

Definition 17 (Soare[13]). *A set D is finitely strongly h-immune if (it is infinite and) there is no disjoint weak finite array $W_{g(n)}$ (g computable, $W_{g(n)}$ finite for all n, and $n \neq m \Rightarrow W_{g(n)} \cap W_{g(m)} = \emptyset$) all of its members intersecting it and $D \subset \cup_i W_{g(i)}$. In other words, if $W_{g(n)}$ is such an array then $\exists n[W_{g(n)} \cap D = \emptyset]$. D is finitely strongly h-simple (fsh-simple) if it is c.e. and $\mathbb{N} - D$ is fsh-immune.*

Theorem 5. *If x is c.e. then A_z is not fsh-simple for any computable sequence of rationals $z = \{z_s\}$ with $\lim z = x$.*

And according to the previous discussion we have

Corollary 2. *If $x = \lim z$ for $z = \{z_s\}$ computable sequence of rationals, then A_z, B_z are not hh-immune.*

Note that a dual version of theorem 5 holds for co-c.e. reals (by similar proof).

6.4. **Proof of theorem 5.** The proof is a kind of finite injury construction and is presented in the following sections. Suppose $x = \lim z$ where x is a non-computable c.e. real and A_z is infinite and co-infinite (the other case being trivial). W.l.o.g. we also assume that $z_n \in (0,1)$ for all n; $\{x_s\}$ is an increasing sequence with limit x.

6.4.1. *About the construction.* We want to define a weak array $W_{g(t)}$ which shows that B_z is not fsh-immune. The idea is to try to install a sequence of markers y_i and witnesses $w_k = z_{i_k}$ on the right hand side of x so that the following holds:

$$x < \cdots < w_1 < y_1 < w_0 < y_0$$
$$i_k \in W_{g(k)}$$

where g indicates a uniform enumeration of the (indices of the) terms of z into separate 'boxes' $W_{g(k)}$ (and is defined implicitly during the construction). We can have the weak array $W_{g(k)}$ disjoint by ensuring that any element enumerated in $W_{g(k)}$ at a particular stage has not been enumerated in any $W_{g(i)}$ during the earlier stages. Beyond some point, only numbers n with $y_{i+1} < z_n < y_i$ will be enumerated in $W_{g(i)}$ and so we will have $|W_{g(i)}| < \infty$ for all i (since $\lim z = x$); this also helps to succeed $\cup_{t\in\mathbb{N}} W_{g(t)} \supseteq B_z$. Moreover, the witness w_k will ensure that $W_{g(k)} \cap B_z \neq \emptyset$ (since $i_k \in W_{g(k)} \cap B_z$). That $\mathbb{N} - B_z = A_z$ is c.e. it is easy to see.

The difficulty is that since we assume that x is not computable (this case is trivial) it is not easy to find which terms of $\{z_i\}$ lie on the right of x. Also it is not easy to find rationals close to but greater than x. So we have to approximate y_i and w_i by making guesses y_i^s, w_i^s. After finitely many guesses, we will have a suitable y_i^s (and w_i^s) and the construction will not change it later. So we will have $\lim_s y_i^s = y_i, \lim_s w_i^s = w_i$ and the limits are finite (in the sense that after some point the sequence becomes constant). Some of the y_i^s's may lie on the left of x (we say that the *guess is false*) in which case the false guess will be *recovered* (at some later stage s_1) by a fixed (increasing) sequence $\{x_k\}$ which tends to x from the left (i.e. $x_{s_1} > y_i^s$). In this case we say that y_i^s (or y_i) is *injured* (at stage s_1). And according to the construction we leave it undefined; in symbols $y_i^{s+1} \uparrow$. Now we make w_i^s dependent on y_i^s.

Definition 18. *Define*

$$w_i^s = max\{z_t : t \leq s \wedge z_t < y_i^s \wedge t \notin \cup\{W_{g(j),s} : j < s, j \neq i\}\}.$$

If $y_i^s \uparrow$ then $w_i^s \uparrow$ and $max\{\emptyset\} \uparrow$.

Note that the s in $W_{g(j),s}$ denotes the enumeration into $W_{g(j)}$'s *defined in the construction* and *not* a general enumeration of all c.e. sets. So $t \notin \cup\{W_{g(j),s} : j < s, j \neq i\}$ means that t has not been enumerated in any of $W_{g(j)}$ for $j < s, j \neq i$ by the s-th stage of the construction.

During the construction, if $y_i^s \uparrow$ then this term 'wants to be defined' or, as we say, it requires attention. More generally

Definition 19. *At any stage $s+1$ we say that y_n^s requires attention if one of the following holds*

- $y_n^s \uparrow$

- $y_n^s \downarrow$ *and* $x_{s+1} \geq y_n^s$

Note that the second clause means that while y_n^s was defined, at stage $s+1$ it is injured, i.e. it is found to be a false guess (and thus it must be corrected). We say that y_n^s *receives attention* at stage s if action is being taken at the particular stage for its (re)definition (and this happens according to its priority). Unfortunately, in general we will not be able to (re)define it at once, and it may take several stages. So at the particular stage we *start taking action* for its (re)definition. In order to indicate this (so that later we know that we have started the (re)definition and continue and finish this procedure) we associate with y_i^s a parameter σ_i^s. This is undefined ($\sigma_i^s \uparrow$) when action is not being taken for the satisfaction of y_i^s; and when action is actually being taken, we store in σ_i^s a value relevant to the last stage of its (re)definition which will enable us to continue and eventually finish the procedure. Of course, the (re)definition of y_i^s may be interrupted by an injury of a y_j with higher priority (i.e. $j < i$). In this case we start from zero at a later stage. Finally, when an injury occurs, say y_i is injured, we *initialize* all y_j for $j > i$. This means that we set $y_i^s \uparrow, \sigma_i^s \uparrow$ for all $j > i$ (s is the current stage).

6.4.2. *Construction.*

Stage 0.

Initialize all y_i^s.

Stage $s+1$.

step A Satisfy the following

$$\left.\begin{array}{l} i,j < s+1;\ y_i^s, y_{i+1}^s \downarrow \\ y_{i+1}^s < z_j \leq y_i^s \\ j \notin \cup_{t<s+1} W_{g(t),s} \end{array}\right\} \Longrightarrow j \in W_{g(i),s+1} \text{ (enumerate } j \text{ into } W_{g(i)}\text{)}$$

step B Choose the least y_i with $i < s+1$ which requires attention.

$\rightsquigarrow$ *case 1:* $y_i^s \uparrow$ (so $y_j \uparrow$ for $j > i$) and $\sigma_i^s \uparrow$ (i.e. action is not being taken for the redefinition of y_i^s). If $w_{i-1}^{s+1} \uparrow$ do nothing. Otherwise *start taking action* for y_i: Check whether $x_{s+1} + \frac{1}{s+1} < w_{i-1}^{s+1}$. If it is, then define

$$y_i^{s+1} = x_{s+1} + \frac{1}{s+1};\ \sigma_i^{s+1} \uparrow$$

If not, put $y_i^{s+1} \uparrow$ and $\sigma_i^{s+1} = x_{s+1} + \frac{1}{s+1}$.

$\rightsquigarrow$ *case 2:* $y_i^s \downarrow$ and it is *injured* at stage $s+1$, i.e. $x_{s+1} > y_i^s$. First we *initialize* all y_j with $j > i$. If $y_i^s = x_{s_1} + \frac{t}{s_1+k}$, we try whether $x_{s_1} + \frac{t+1}{s_1+k} < w_{i-1}^{s+1}$. If yes, then put

$$y_i^{s+1} = x_{s_1} + \frac{t+1}{s_1+k};\ \sigma_i^{s+1} \uparrow$$

If not, then put $y_i^{s+1}\uparrow$ and $\sigma_i^{s+1} = x_{s_1} + \frac{t+1}{s_1+k}$.

$\rightsquigarrow$ *case 3:* $y_i^s \uparrow$ and $\sigma_i^s \downarrow$ (i.e. y_i^s is undefined *but* action is being taken for its (re)definition). σ_i^s will have the form $x_{s_1} + \frac{t}{k}$. We try whether $x_{s_1} + \frac{t}{k+1} < w_{i-1}^{s+1}$. If yes, then define

$$y_i^{s+1} = x_{s_1} + \frac{t}{k+1}$$

If not, then put $y_i^{s+1}\uparrow$, $\sigma_i^{s+1} = x_{s_1} + \frac{t}{k+1}$.

6.4.3. *Verification.* The first step is to prove that for all i the following limits exist

$$\lim_s w_i^s = w_i;\ \lim_s y_i^s = y_i;\ \lim_s \sigma_i^s = \uparrow$$

and

$$x < \cdots < y_i < w_{i-1} < y_{i-1} < \cdots < w_0 < y_0 \tag{16}$$

We will prove this by induction. Since it is $y_0^s = y_0$ for all s, its enough to prove the induction step described bellow. Our hypothesis is that for all $s > s_0$ and $i < n$ (for some fixed s_0, n) we have

$$w_i^s = w_i;\ y_i^s = y_i;\ \sigma_i^s = \uparrow$$
$$x < w_{n-1} < y_{n-1} < \cdots < w_0 < y_0$$

and we want to find $s_1 > s_0$ such that for all $s > s_1$

$$w_n^s = w_n;\ y_n^s = y_n;\ \sigma_n^s = \uparrow$$
$$x < w_n < y_n < \cdots < w_0 < y_0$$

The proof for y_n.

case 1 $y_n^{s_0}\uparrow$; $\sigma_n^{s_0}\uparrow$.

According to the construction, y_n receives attention at stage s_0+1 (it has the highest priority). So we check whether $x_{s_0+1} + \frac{1}{s_0+1} < w_{n-1}$.

1A. If true, then we define $y_n^{s_0+1} = x_{s_0+1} + \frac{1}{s_0+1}$.

$1A_1$. If $y_n^{s_0+1} > x$ then for all $s > s_0$, $y_n^{s_0+1} = y_n^s = y_n$.

$1A_2$. If the guess is false and $y_n^{s_0+1} < x$, at some stage $s > s_0+1$ the false guess will be recovered, i.e. we will have $x_s > y_n^{s_0+1}$. At that stage (according to the construction) a sequence of corrections will follow of the form

$$x_{s_0+1} + \frac{2}{s_0+k_2}, x_{s_0+1} + \frac{3}{s_0+k_3}, \ldots$$

where $k_t = \mu m[\frac{t}{s_0+m} < w_{n-1}]$.

Claim. *This sequence of corrections cannot be infinite and there will be* t, s_1 *such that* $\forall\ s > s_1\ [y_n^{s_1} = x_{s_0+1} + \frac{t}{s_0+k_t} = y_n^s]$.

Proof of claim Suppose not. We know that $x < w_{n-1}$ and $x_{s_0+1} < x < w_{n-1}$, so there are t, t_1 such that $x < x_{s_0+1} + \frac{t}{s_0+t_1} < w_{n-1}$ which implies $x < x_{s_0+1} + \frac{t}{s_0+k_t} < w_{n-1}$. So after trying $y_n^{s_1} = x_{s_0+1} + \frac{t}{s_0+k_t}$ we will have $\lim_s y_n^s = y_n^{s_1}$ (finite limit) which contradicts our hypothesis. □

$1B$. If false, as above, a sequence of corrections will follow of the form $x_{s_0+1} + \frac{t}{s_0+k_t}, t = 1, 2, \ldots$ which must come to an end, so it stops at some $x_{s_0+1} + \frac{t_0}{s_0+k_{t_0}}$. We have $\lim_s y_n^s = y_n = x_{s_0+1} + \frac{t_0}{s_0+k_{t_0}}$.

case 2 $y_n^s \uparrow$; $\sigma_n^{s_0} \downarrow$.
The case is similar to $1B$ above and $\lim_s y_n^s$ exists and is less than w_{i-1}.

case 3 $y_n^{s_0} \downarrow$ (so $\sigma_n^{s_0} \uparrow$) and it is injured at stage s_0 (or at a later stage). Then, in the same way as in $1A_2$ above, $\exists \lim_s y_n^s < w_{i-1}$. Of course if it is not injured later, the same result is obvious.

The proof for w_n**.** The crucial point is to prove $\exists \lim_s w_n^s = w_n$. Suppose that $\forall\ s > s_1\ y_n^{s_1} = y_n$ (s_1 is the least such). Then according to the construction $y_{n+1}^{s_1+1} \uparrow$ and it remains so all the time (i.e. stages $s > s_1$) that $w_n^s \uparrow$ (and of course $y_i^s \uparrow$ for $i > n, s > s_1$ such that $w_n^s \uparrow$).

First we notice that w_n^s will be defined at some stage $s > s_1$ because infinitely many terms z_j will appear at subsequent stages in a close area of x, which have not been enumerated in any $W_{g(i)}$.

if $w_n^s > x$ then $w_n^s < y_n$ and $\exists \lim_t w_n^t$ since there are finitely many z_i's between w_n^s and x (and w_n^s is non-decreasing on s).

if $w_n^s < x$ then we will show that at some point w_n^s will be redefined to $w_n^t > x$. Indeed, if not, at the stages $s > s_1$ such that $y_{n+1}^s \downarrow$ we have $y_{n+1}^s < w_n^s < x < y_n^s$ and so no z_i ($i > s_1$) with $x < z_i < y_n^s$ appears in such intervals (i.e. if $s_2 < i < s_3$ and $\forall s\ [s_2 < s < s_3 \Rightarrow y_{n+1}^s \downarrow]$ then $z_s \notin (x, y_n^s)$ for any $s \in (s_2, s_3)$). And at intervals (s_4, s_5) such that $\forall s \in (s_4, s_5)\ y_{n+1}^s \uparrow$ we have $z_s \notin (x, y_n^s)$ because otherwise w_n^s would be redefined to an element $> x$. So by induction it follows that $\forall s > s_1\ z_s < x$ which contradicts our hypothesis that $\exists^\infty s\ z_s > x$. So eventually for some $s > s_1$ we have $y_n^s > w_n^s > x$.

And from that point w_n^s will be non-decreasing, and so the sequence $\{w_n^s\}_s$ will become eventually constant, reaching a final $w_n < y_n$ (since $\lim z = x$).

The rest of the verification. Finally from the construction it is easy to see that $i \neq j \Rightarrow W_{g(i)} \cap W_{g(j)} = \emptyset$ and $\forall i\ |W_{g(i)}| < \infty$ (the last because after y_i, y_{i+1} are fixed, only finitely many terms of $\{z_j\}$ can appear in (y_{i+1}, y_i)). It is also clear that $w_j \in W_{g(j)}$ and $\exists \lim_s \sigma_i^s = \uparrow$. Finally $\cup_{i \in \mathbb{N}} W_{g(i)} \supseteq B_z$ since $\lim_s y_s = x$ (due to the convergence of $\{z_s\}$ and (16)) and for any i, after y_i, y_{i+1} are fixed, all terms of $\{z_s\}$ that will appear in (y_{i+1}, y_i) will have their indices in $W_{g(i)}$.

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