Available online at www.sciencedirect.com

ScienceDirect

Annals of Pure and Applied Logic 145 (2007) 252–257

ANNALS OF
PURE AND
APPLIED LOGIC

www.elsevier.com/locate/apal

Randomness and the linear degrees of computability

Andrew E.M. Lewis[a,*,1], George Barmpalias[b,2]

[a] Dipartimento di Scienze Matematiche ed Informatiche Roberto Magari, Pian dei Mantellini 44, 53100 SIENA, Italy
[b] Department of Pure Mathematics, Leeds University, Leeds, LS2 9JT, United Kingdom

Received 12 March 2006; received in revised form 14 August 2006; accepted 21 August 2006
Available online 18 October 2006
Communicated by R.I. Soare

Abstract

We show that there exists a real α such that, for all reals β, if α is linear reducible to β ($\alpha \leq_\ell \beta$, previously denoted as $\alpha \leq_{\mathrm{sw}} \beta$) then $\beta \leq_T \alpha$. In fact, every random real satisfies this quasi-maximality property. As a corollary we may conclude that there exists no ℓ-complete Δ_2 real. Upon realizing that quasi-maximality does not characterize the random reals – there exist reals which are not random but which are of quasi-maximal ℓ-degree – it is then natural to ask whether maximality could provide such a characterization. Such hopes, however, are in vain since no real is of maximal ℓ-degree.

Keywords: Computability; Randomness; Degree

---

1. Introduction

In the process of computing a real α given an oracle for β it is natural to consider the condition that for the computation of the first n bits of α we are only allowed to use the information in the first n bits of β. It is not difficult to see that this notion of oracle computation is complexity sensitive in many ways. We can then generalize this definition in a straightforward way by allowing that, in the computation of $\alpha \upharpoonright n$, access is permitted to $\beta \upharpoonright (n+c)$ for some fixed constant c.

The study of oracle computations of this kind and of the reducibility they induce on 2^ω was initiated by Downey, Hirschfeldt and LaForte [6,5], the motivation being that they might serve as a measure of relative randomness. They presented the induced reducibility as a restriction of the weak truth table reducibility and gave it the (perhaps unfortunate!) name *strong weak truth table* reducibility—or sw reducibility for short. After discussions with other researchers in the area we introduce here the terminology *linear* reducible in place of *strong weak truth table* reducible—while another reasonable contender for this title would certainly be the set of reductions in which the use on argument n is bounded by $an + c$ for some constants a and c it would seem that reductions of this type for

---

[*] Corresponding author.
E-mail address: thelewisboy@hotmail.com (A.E.M. Lewis).

[1] The first author was supported by EPSRC grant No. GR/S28730/01.

[2] The second author was supported by EPSRC grant No. EP/C001389/1.

0168-0072/$ - see front matter © 2006 Elsevier B.V. All rights reserved.
doi:10.1016/j.apal.2006.08.001

values of $a \neq 1$ are of small relevance in the study of computability theory. From a computational point of view, then, the linear reducibility can be seen as formalizing the notion of *length efficient oracle computation*.

Definition 1.1. We say α is linear reducible to β ($\alpha \leq_\ell \beta$) if there is a Turing functional Γ and a constant c such that $\Gamma^\beta = \alpha$ and the use of this computation on any argument n is bounded by $n + c$. The Turing functionals which have their use restricted in such a way are called ℓ-functionals.

The linear reducibility (in particular the case where $c = 0$) was used in the recent work of Soare, Nabutovsky and Weinberger on applications of computability theory to differential geometry (see Soare [10]). If we consider partial computable functionals as operators from $2^{<\omega}$ to itself, the ℓ-functionals are also closely related to the notion of Lipschitz continuous operators.

Definition 1.2. A partial operator Γ from a (pseudo-)metric space (X, d) to itself is Lipschitz continuous if there is a constant C such that

$$d(\Gamma(x), \Gamma(y)) \leq C \cdot d(x, y) \tag{1}$$

for all x, y in the domain of Γ.

We consider the pseudo-metric d on $2^{<\omega}$ such that for incompatible strings τ and τ', $d(\tau, \tau') = 2^{-n}$ where n is the least position where τ, τ' differ, and such that $d(\tau, \tau') = 0$ if τ and τ' are compatible.

Proposition 1.1. *An ℓ-functional is a partial computable and Lipschitz continuous operator from $(2^{<\omega}, d)$ to itself. Conversely, every partial computable and Lipschitz continuous operator $\Gamma : (2^{<\omega}, d) \to (2^{<\omega}, d)$ equals an ℓ-functional on infinite strings.*

Proof. If Γ is an ℓ-functional, it is obviously partial computable but also Lipschitz continuous as a function on $2^{<\omega}$. Indeed, suppose we are given two finite binary strings τ, τ' such that $d(\Gamma^\tau, \Gamma^{\tau'}) = 2^{-t}$. If the use of Γ on n is $n + c$ for some fixed constant c, $d(\tau, \tau')$ must be at least $2^{-(t+c)}$. Hence

$$d(\Gamma^\tau, \Gamma^{\tau'}) \leq d(\tau, \tau') \cdot 2^c \tag{2}$$

and Γ is Lipschitz continuous. On the other hand, if Γ is partial computable and Lipschitz continuous (say with constant 2^c) we show that we can construct an ℓ-functional which is equal to Γ on infinite strings. To compute a total Γ^α on n knowing the first $n + c$ bits of α we effectively find an extension τ of $\alpha \upharpoonright (n + c)$ such that $\Gamma^\tau(n)\downarrow$. Since (2) holds, the distance between $\Gamma^\alpha \upharpoonright (n + 1)$ and $\Gamma^\tau \upharpoonright (n + 1)$ will be less than 2^{-n}. So $\Gamma^\alpha(n) = \Gamma^\tau(n)$. □

The following are some results from the literature on the ℓ-degrees (induced by $\leq_\ell$) which are relevant to our present considerations. For more background on this structure we refer the reader to [7,5].

Definition 1.3. A Solovay test is a c.e. set S of binary strings such that $\sum_{\sigma \in S} 2^{-|\sigma|} < \infty$. A real number α avoids S if for almost all $\sigma \in S$, $\sigma \not\subset \alpha$. A real α is (Martin-Löf) random if it avoids all Solovay tests.

Definition 1.4. A real number is *computably enumerable (c.e.)* if it is the limit of a computable increasing sequence of rationals.

The main justification for $\leq_\ell$ as a measure of relative randomness was the following:

Proposition 1.2 (*Downey et al. [6]*). *If $\alpha \leq_\ell \beta$ then for all n, the prefix-free complexity of $\alpha \upharpoonright n$ is less than or equal to that of $\beta \upharpoonright n$ (plus a constant).*

In particular, then, $\leq_\ell$ preserves randomness—if α is a random real and $\alpha \leq_\ell \beta$ then β is random, so that any ℓ-degree either contains only random or no random reals.

Yu and Ding proved the following:

Theorem 1.1 (*Yu and Ding [11]*). *There is no ℓ-complete c.e. real.*

By a 'uniformization' of their proof they got two c.e. reals which have no c.e. real ℓ-above them. Hence:

Corollary 1.1 (*Downey et al. [6]*). *The structure of ℓ-degrees is not an upper semi-lattice.*

The main idea of their proof of Theorem 1.1 can be applied for the case of c.e. sets in order to get an analogous result. Using different ideas Barmpalias [1] proved the following stronger result.

Theorem 1.2 (*Barmpalias [1]*). *There are no ℓ-maximal c.e. sets. That is, for every c.e. set A, there exists a c.e. set W such that $A <_\ell W$.*

Note that since the Solovay degrees and the ℓ-degrees coincide on the c.e. sets (see [5]) the following also holds.

Corollary 1.2 (*Barmpalias [1]*). *The substructure of the Solovay degrees consisting of the ones with c.e. members (i.e. containing c.e. sets) has no maximal elements.*

In Barmpalias and Lewis [2] it was shown that there are c.e. reals α that cannot be ℓ-computed by any random c.e. real. That is, for any c.e. real $\beta \geq_\ell \alpha$, β is not random. Also, in Barmpalias and Lewis [3] it was shown that strictly below every random ℓ-degree there is another random ℓ-degree. The first aim of this paper is to prove the following (perhaps rather surprising) result.

Theorem 1.3. *There exists a (globally) quasi-maximal ℓ-degree, i.e. there exists a real α such that, for all reals β, if $\alpha \leq_\ell \beta$ then $\beta \leq_T \alpha$. In fact every random real satisfies this quasi-maximality property.*

The fascination of this result lies in the fact that we are generally not used to degree structures possessing anything like maximal elements in the global sense (where we consider the degrees of all reals).

2. Random reals are quasi-maximal

Let Ψ_i, the ith ℓ-functional, satisfy the condition that the use in computing argument n is $n + c_i + 1$ (should this computation converge).

Definition 2.1. For $\sigma \in 2^{<\omega}$ let $\Pi(\sigma, i)$ be the number of strings τ of length $|\sigma| + c_i$ such that $\sigma = \Psi_i^\tau$.

Lemma 2.1. *For any σ, i we have $\Pi(\sigma 0, i) + \Pi(\sigma 1, i) \leq 2\Pi(\sigma, i)$.*

Proof. Consider the set of all one bit extensions of those strings τ of length $|\sigma| + c_i$ such that $\Psi_i^\tau = \sigma$. There are $2\Pi(\sigma, i)$ strings in this set. □

The key to analyzing the relationship between Martin-Löf randomness and quasi-maximality lies in a theorem of Schnorr's on effective super-martingales.

Definition 2.2. A super-martingale is a function $f : 2^{<\omega} \mapsto \mathbb{R}^+ \cup \{0\}$ such that for all σ, $2f(\sigma) \geq f(\sigma 0) + f(\sigma 1)$. We say that the super-martingale succeeds on a real α if $\limsup_n f(\alpha \upharpoonright n) \to \infty$.

Definition 2.3. We say that the super-martingale f is effective if (i) for all σ, $f(\sigma)$ is a c.e. real and (ii) there is a computable function f' such that, for all σ, $\{f'(\sigma, s)\}_{s\in\omega}$ is an increasing sequence of rationals with limit $f(\sigma)$.

Theorem 2.1 (*Schnorr [9]*). *A real α is Martin-Löf random iff no effective super-martingale succeeds on α.*

The proof of Theorem 1.3. For all σ, i let $\Pi_i(\sigma) = \Pi(\sigma, i)$. Since each function Π_i can be effectively approximated from below, Lemma 2.1 says precisely that every Π_i is an effective super-martingale.

So suppose given α and β such that α is a random real and $\Psi_i^\beta = \alpha$. By Schnorr's theorem we may define $m^\star$ to be the maximum m such that there exist an infinite number of n, $\Pi_i(\alpha \upharpoonright n) = m$. Let T_n be all those strings τ of length $n + c_i$ such that Ψ_i^τ is the initial segment of α of length n and let $T = \bigcup_n T_n$. We say that a real lies on T if all but finitely many initial segments are in T. Since there are a finite number of β' lying on T there exists $\tau_0 \subset \beta$ such that if $\beta' \neq \beta$ lies on T then $\tau_0 \not\subset \beta'$. Now suppose we are given $\tau \supset \tau_0$ which is not an initial segment of β. Using an oracle for α we can enumerate all tuples $(n, \tau_1, .., \tau_{m^\star})$ such that $T_n = \{\tau_1, .., \tau_{m^\star}\}$ until we find such a tuple with no τ_m compatible with τ—whereupon we can deduce that τ is not an initial segment of β.

Corollary 2.1. *There exist low reals which are of quasi-maximal ℓ-degree.*

Proof. There exist low random reals [7]. □

Corollary 2.2 (*The equivalent of the Yu–Ding Theorem for the Δ_2 Reals*). *There exists no ℓ-complete Δ_2 real.*

Proof. This follows immediately from the previous corollary. □

Corollary 2.3. *Every Turing degree above $0'$ contains a set of quasi-maximal ℓ-degree.*

Proof. Every Turing degree above $0'$ contains a random real [8]. □

Corollary 2.4. *The ℓ-degrees are not an upper semi-lattice, in fact there exists a set of two ℓ-degrees with no upper bound.*

Proof. Just choose any α and β which are random and Turing incomparable. □

Theorem 2.3 below, however, tells us that quasi-maximality does not characterize the random reals.

Theorem 2.2 (*Chaitin [4]*). *Consider a total computable prediction function f which, given an arbitrary finite initial segment of a real α, returns either "no prediction", "the next bit is a 0", or "the next bit is a 1". If α is random and f predicts infinitely many bits of α then in the limit the proportion of correct predictions to total predictions made tends to $\frac{1}{2}$.*

Theorem 2.3. *There exists α of quasi-maximal ℓ-degree which is not random.*

We make the following definitions.

Definition 2.4. For $\sigma \in 2^{<\omega}$ let $\Upsilon(\sigma, i) = \min\{\Pi(\sigma', i) \mid \sigma' \supseteq \sigma\}$. Let $\Upsilon^\star(\sigma, i)$ be the least string $\sigma' \supseteq \sigma$ such that $\Pi(\sigma', i) = \Upsilon(\sigma, i)$.

Lemma 2.2. *Given σ_0, i, let $\sigma_1 = \Upsilon^\star(\sigma_0, i)$. For all $\sigma_2 \supseteq \sigma_1$ we have $\Pi(\sigma_2, i) = \Upsilon(\sigma_0, i)$.*

Proof. By induction on the length of σ_2. So suppose given $\sigma_2 \supseteq \sigma_1$ such that $\Pi(\sigma_2, i) = \Upsilon(\sigma_0, i)$. Now if $\Pi(\sigma_2 0, i) < \Upsilon(\sigma_0, i)$ or $\Pi(\sigma_2 1, i) < \Upsilon(\sigma_0, i)$ this would contradict the fact that $\sigma_1 = \Upsilon^\star(\sigma_0, i)$. Thus by Lemma 2.1 $\Pi(\sigma_2 0, i) = \Pi(\sigma_2 1, i) = \Upsilon(\sigma_0, i)$. □

Lemma 2.3. *Given σ_0, i, let $\sigma_1 = \Upsilon^\star(\sigma_0, i)$. For all $\alpha \supset \sigma_1$ and all β such that $\Psi_i^\beta = \alpha$ we have that $\beta \leq_T \alpha$.*

Proof. Given α and β as in the statement of the lemma, let T_n be all those strings τ of length $n + c_i$ such that Ψ_i^τ is the initial segment of α of length n and let $T = \bigcup_n T_n$. The following facts follow immediately from the fact that, by Lemma 2.2, there are precisely the same number of strings (actually $\Upsilon(\sigma_0, i)$) in T_n for all sufficiently large n.

(i) There are a finite number of reals lying on T (at most $\Upsilon(\sigma_0, i)$).
(ii) We can compute (not just enumerate) T using an oracle for α.

By (i) there exists $\tau_0 \subset \beta$ such that if $\beta' \neq \beta$ lies on T then $\tau_0 \not\subset \beta'$. If we are given $\tau_1 \supset \tau_0$ which is not an initial segment of β then using an oracle for α it follows by (ii) that we can find n such that there are no extensions of τ_1 in T_n. □

For all σ, define $f(\sigma) =\mid \{n : \sigma(n) \downarrow = 0\}\mid$. If α is a random real then, by Theorem 2.2:

$$(\dagger) \lim_n \frac{f(\alpha \restriction n)}{n} \downarrow = \frac{1}{2}.$$

The construction.

Let σ_0 be the empty string. Given σ_i let $\sigma_i' = \Upsilon^\star(\sigma_i, i)$ and then define σ_{i+1} to be σ_i' concatenated with $2|\sigma_i'|$ zeros. Define $\alpha = \bigcup_i \sigma_i$.

The verification.

Since $\alpha \supset \sigma_{i+1}$ it follows by Lemma 2.3 that if $\alpha = \Psi_i^\beta$ then $\beta \leq_T \alpha$. We have that α is not random since it clearly does not satisfy (†).

3. Maximality

Having proved that quasi-maximality does not characterize the random reals it is natural to ask whether maximality might provide such a characterization. With the following theorem, however, we are able to answer this question in the negative.

Theorem 3.1. *No real is of maximal ℓ-degree.*

Proof. Let the ℓ-functionals Φ_0 and Φ_1 be defined inductively as follows. Suppose $d \in \{0, 1\}$.

(i) For both strings τ of length 1 we define $\Phi_d^\tau = d$.
(ii) If $|\tau|$ is of the form 2^n for some $n \geq 1$ then let τ_0 be the initial segment of τ of length $2^n - 1$. There exists a unique $\tau_1 \neq \tau_0$ of length $2^n - 1$ such that $\Phi_d^{\tau_1} = \Phi_d^{\tau_0}$. If τ_0 is the leftmost of τ_0, τ_1 then define $\Phi_d^\tau = \Phi_d^{\tau_0}0$ and otherwise define $\Phi_d^\tau = \Phi_d^{\tau_0}1$.
(iii) If $|\tau|$ is not of the form 2^n for any $n \geq 0$ then let τ_0 be the initial segment of τ of length $|\tau| - 1$. Let $c = \tau(|\tau| - 1)$ and define $\Phi_d^\tau = \Phi_d^{\tau_0}c$.

It is important to have an intuitive picture of the above inductive definition. Consider the range of Φ_0. We begin by branching the empty sequence with two 0s. From then on, at levels 2^n (for any n) we extend with either two 1s or two 0s according to whether there is another node of the identity tree which is on the left and which is Φ_0-mapped to the same string as the node we are on or not. At all other levels we extend the strings as we would the identity tree—that is, a 0 on the left branch and a 1 on the right branch. It can easily be seen that Φ_0 has the following properties.

- For every τ, $\Phi_0^\tau \downarrow$ and is a string of the same length.
- For every string σ which begins with 0 there exist exactly two incompatible τ_0, τ_1 such that
$$\Phi_0^{\tau_0} = \Phi_0^{\tau_1} = \sigma.$$
- If $|\sigma| = 2^k + c < 2^{k+1}$ consider the two τ_i such that $\Phi_0^{\tau_0} = \Phi_0^{\tau_1} = \sigma$. Then τ_0, τ_1 differ at their c-th bit from the end, i.e. their $|\sigma| - c - 1$ bit. In particular, if σ is of length 2^k they differ on their last bit.
- For every real α which begins with 0 there is a unique β such that $\Phi_0^\beta = \alpha$.

So now suppose given a real α and without loss of generality that $\alpha(0) = 0$. Then there exists a unique β such that $\Phi_0^\beta = \alpha$. If β is of ℓ-degree strictly above α then we are done. So suppose instead that we are given i such that $\Psi_i^\alpha = \beta$. We shall define Φ_2 for which there exists a total tree of reals β' such that $\Phi_2^{\beta'} = \alpha$. This suffices to give the result, since then we can pick β' on this tree which is not Turing below α. Pick n_0 large enough such that $2^{n_0} - c_i > 2^{n_0 - 1} + 1$.

(i) For all τ which are of length $< 2^{n_0}$, $\Phi_2^\tau = \Phi_0^\tau$.
(ii) If $|\tau| > 2^{n_0}$, but is not of the form 2^n for any $n \geq 0$ then let τ_0 be the initial segment of τ of length $|\tau| - 1$. Let $c = \tau(|\tau| - 1)$ and define $\Phi_2^\tau = \Phi_2^{\tau_0}c$.
(iii) If $|\tau|$ is of the form 2^n for some $n \geq n_0$, then let τ_0 be the initial segment of τ of length $2^n - 1$. Let $\sigma = \Phi_2^{\tau_0}$, $c = \Psi_i^\sigma(2^{n-1} - 1)$ and define $\Phi_2^\tau = \Phi_2^{\tau_0}c$.

Now if $n \geq n_0 - 1$, then for every string τ of length 2^n such that Φ_2^τ is compatible with α, there exist two strings $\tau' \supset \tau$ of length 2^{n+1} such that $\Phi_2^{\tau'}$ is compatible with α—the point being that $\beta(2^n - 1) = \alpha(2^{n+1} - 1)$. □

Acknowledgements

Both authors were partially supported by the NSFC Grand International Joint Project, No 60310213, New Directions in the Theory and Applications of Models of Computation.

References

[1] G. Barmpalias, Computably enumerable sets in the Solovay and the strong weak truth table degrees, in: Proceedings of the CiE 2005 Conference in Amsterdam, in: Lecture Notes in Computer Science.

[2] G. Barmpalias, A.E.M. Lewis, A c.e. real that cannot be sw-computed by any Ω-number, Notre Dame Journal of Formal Logic 47 (2) (2006) 197–209.

[3] G. Barmpalias, A.E.M. Lewis, Random reals and Lipschitz continuity, Mathematical Structures in Computer Science 16 (5) (2006).

[4] G. Chaitin, Algorithmic Information Theory, Cambridge University Press, 2004.

[5] R. Downey, D. Hirschfeldt, G. LaForte, Randomness and reducibility, Journal of Computer and System Sciences 68 (2004) 96–114.

[6] R. Downey, D. Hirschfeldt, G. LaForte, Randomness and reducibility, in: Mathematical Foundations of Computer Science, in: Lecture Notes in Comput. Sci., vol. 2136, Springer, Berlin, 2001, pp. 316–327.

[7] R. Downey, D. Hirschfeldt, Algorithmic Randomness and Complexity, Monograph (in preparation).

[8] A. Kucera, Measure, Π_0^1 Classes and Complete Extensions of PA, in: Springer Lecture Notes in Mathematics, vol. 1141, Springer-Verlag, 1985, pp. 245–259.

[9] C. Schnorr, A unified approach to the definition of a random sequence, Mathematical Systems Theory 5 (1971) 246–258.

[10] R. Soare, Computability theory and differential geometry, Bulletin of Symbolic Logic (in press).

[11] L. Yu, D. Ding, There is no SW-complete c.e. real, Journal of Symbolic Logic 69 (4) (2004) 1163–1170.

Further reading

[1] G. Barmpalias, A.E.M. Lewis, The ibT degrees of computably enumerable sets are not dense, Annals of Pure and Applied Logic (in press).

[2] C.S. Calude, A characterisation of c.e. random reals, Theoretical Computer Science 271 (1–2) (2002) 3–14.

[3] R. Downey, D. Hirschfeldt, A. Nies, Randomness, computability, and density, SIAM Journal of Computation 31 (4) (2002) 1169–1183.

[4] R. Downey, Some recent progress in algorithmic randomness, 2004 (preprint).

[5] P. Odifreddi, Classical Recursion Theory, North-Holland, Amsterdam Oxford, 1989.

[6] R. Soare, Recursively Enumerable Sets and Degrees, Springer-Verlag, Berlin, London, 1987.