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Annals of Pure and Applied Logic 141 (2006) 51–60
ANNALS OF PURE AND APPLIED LOGIC
www.elsevier.com/locate/apal

The *ibT* degrees of computably enumerable sets are not dense

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Received 8 September 2005; received in revised form 7 October 2005; accepted 13 October 2005
Available online 8 November 2005
Communicated by R.I. Soare

Abstract

We show that the identity bounded Turing degrees of computably enumerable sets are not dense.

Keywords: *ibT* degrees; Computably enumerable sets; Density

1. Introduction

1.1. The ibT degrees

In the study of relative computations it is natural to consider computations of a set A from another set B in which to compute $A \upharpoonright n$ we are only allowed to ask membership questions of $B \upharpoonright n$.

Definition 1. We say that A is identity bounded Turing reducible to B ($A \leq_{ibT} B$ for short) if there is a Turing functional Γ such that $\Gamma^B = A$ and the use of the computations is bounded by the identity function i.e. on each argument n the B-queries are for numbers $\leq n$. The induced degrees are called *ibT* degrees.

This gives a reducibility which is complexity sensitive and which, in particular, preserves most notions of randomness for binary strings (see [2,7,8]). Moreover it is closely related to a 'domination' reducibility which was used by Nabutovsky, Soare and Weinberger (see [18,15,5]) in proving some results in differential geometry. In this paper we study the *ibT* degrees of computably enumerable (c.e.) sets, i.e. sets which can be effectively listed.

1.2. The Lipschitz connection

The *ibT* reducibility is also closely related with what we like to call the *computably Lipschitz* (or cl) reducibility. This first appeared as a measure of relative randomness in [7,8] under the name sw reducibility (from strong wtt).

Definition 2 (*Downey et al. [7,8]*). We say $A \leq_{\text{cl}} B$ if there is a Turing functional Γ and a constant c such that $\Gamma^B = A$ and the use of this computation on any argument n is bounded by $n + c$. The Turing functionals which have their use restricted in such a way are called cl functionals.

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0168-0072/$ - see front matter
doi:10.1016/j.apal.2005.10.001

This can be seen as a strong version of the wtt (also called *bounded Turing* or bT) reducibility—hence the name sw—but it is something more than that. In fact, if we see functionals as operators on strings, the cl functionals are a reasonable effectivization of the Lipschitz continuous operators—hence the name cl—as Proposition 1 shows. The relationship between Lipschitz conditions and sw-reducibility was first mentioned in [10].

Definition 3. A partial operator Γ from a (pseudo-) metric space (X, d) to itself is Lipschitz continuous if there is a constant C (often called Lipschitz constant) such that

$$d(\Gamma(x), \Gamma(y)) \leq C \cdot d(x, y)$$

for all x, y in the domain of Γ.

Proposition 1 (*Barmpalias and Lewis [2]*). *A cl functional is a partial computable and Lipschitz continuous operator from* $(2^{<\omega}, d)$ *to itself. Conversely, every partial computable and Lipschitz continuous operator*

$$\Gamma : (2^{<\omega}, d) \to (2^{<\omega}, d)$$

equals a cl functional on infinite strings.

Here d is the (pseudo-) metric on the space of finite/infinite binary strings defined as $d(\sigma, \tau) = 2^{-n}$ where n is the least position where σ, τ differ (and if they do not differ, $d(\sigma, \tau) = 0$). Now working in the same way, the *ibT* functionals can be seen as an effectivization of the Lipschitz continuous operators with Lipschitz constant 1 in the sense of the following proposition.

Proposition 2. *An ibT functional is a partial computable and Lipschitz continuous operator with Lipschitz constant* 1 *from* $(2^{<\omega}, d)$ *to itself. Conversely, every partial computable and Lipschitz continuous operator*

$$\Gamma : (2^{<\omega}, d) \to (2^{<\omega}, d)$$

with Lipschitz constant 1 *equals an ibT functional on infinite strings.*

1.3. The non-density result

The following theorem asserts that the identity bounded Turing degrees of c.e. sets are not dense.

Theorem 1. *There are c.e. sets* W, V *such that* $V <_{ibT} W$ *and for every c.e. set* A,

$$V \leq_{ibT} A \leq_{ibT} W \Rightarrow A \leq_{ibT} V \ \vee \ W \leq_{ibT} A.$$

In computability theory it is often the case that proofs are considered as a game with two players. One of the players is us, trying to prove the desired result, while our opponent is trying to prove its negation. The sets that are under our control are usually represented by the first letters of the Latin alphabet while the opponent gets the last few letters of the Latin alphabet. Sometimes it is useful to take a step back from the game, forget which side we support, see which side is going to win and adopt it. This is what we did after trying to show density and it turned out that the opponent had a winning strategy; so we adopted it and along with that, the names W, V of his sets.

In fact, there are reasons why one might initially be encouraged to believe that density holds. In the world of c.e. sets, the only non-density results we know about concern computations like those of the tt and the m reducibilities which ask membership queries of certain elements and based on that answer *alone* they decide on the membership of a number. Note that the weak truth table (i.e. bounded Turing) and the identity bounded Turing computations do not have this property.

The nature of truth table computations allows us to show the existence of minimal c.e. degrees in the corresponding structures by a construction closely related to that used in order to prove the existence of *maximal sets*. For example, *every maximal c.e. set has minimal m-degree* [19,14] and *every* η*-maximal semicomputable (i.e. semirecursive in Jockusch's sense) c.e. incomputable set has minimal tt-degree* [6,13,11]. Thus it is not surprising that we need different methods in order to prove Theorem 1. As a further illustration of the essential difference between the *ibT* and the truth-table reducibilities (and the induced degree structures) we note that *there are no minimal c.e. ibT-degrees* (this can be seen using a standard permitting argument). In the following all relative computations will be *ibT* computations and all sets will be c.e. For background in computability theory we refer the reader to [17,16].

2. The failure of the Sacks density method

The density method of Sacks can be used to show the density of the c.e. sets in the Turing and bounded Turing (i.e. wtt) degrees (in the second case only a finite version of Sacks's original argument is needed, see [12]). Moreover, it seems to be *the* way to prove these results in the sense that it is very likely that the ideas it employs are necessary (there are certainly no direct constructions known which prove these results in an essentially different way). But when we try to apply similar strategies in a natural way in order to show density in the case of the *ibT* computations, we are faced with the immediate problem of lack of space for coding. The density method requires double coding into the set A we construct in between V and W: there is the coding of V together with the partial coding of W into A. It is not hard to see that it is impossible to incorporate both codings within such tight use bounds. This, however, does not necessarily mean the failure of the method: we can try to make the proof non-uniform (in contrast to the original Sacks argument) with the hope of making enough space in A to afford the required coding. Actually, this idea worked when we showed the following theorem which, amongst other things implies that *the theories of the Solovay degrees of c.e. reals and the Solovay degrees of c.e. sets are not elementarily equivalent*.

Theorem 2 (*Barmpalias [1]*). *There are no maximal elements in the structure of the computably Lipschitz degrees of c.e. sets. That is, for every c.e. set W there is a c.e. set A such that $W <_{cl} A$.*

Indeed, the Solovay degrees of c.e. sets are identical with the cl degrees of c.e. sets (see [8]) and there is a Solovay complete c.e. real Ω. The proof of Theorem 2 also shows the following result.

Corollary 1. *There are no maximal elements in the structure of the identity bounded Turing degrees of c.e. sets. That is, for every c.e. set W there is a c.e. set A such that $W <_{ibT} A$.*

In this situation we had to code into A the content of W and some additional information coming from diagonalizations which refuted the computability of A from W. Again the problem was that there is not enough space in A for both codings. We dealt with this by making the construction non-uniform: we assumed we have a good approximation to the (sup of the) density of A (i.e. the proportion of 1s in segments of the characteristic sequence of A) and based on this we created extra space for the diagonalizations. This construction produced three sets, one of which is strictly above W. A feature of this construction is that in the coding of W into A the codes can drop arbitrarily lower than the elements that they code (and there seems no way to avoid this). This is exactly why this approach cannot work in proving density: in the density method making the codes of the lower set smaller affects the Sacks restraints argument which shows that the constructed set is not above the higher one. In the case of the *ibT* reducibility these restraints are damaged and so the argument cannot work.

3. Motivation and intuition toward non-density

The sensitivity and idiosyncrasies of *ibT* computations do not only give us problems but also useful information. For example, it is not difficult to show the following.

Proposition 3. *If W is non-computable then $W + 1 <_{ibT} W$ and $2W <_{cl} W$, where $W + 1 = \{n + 1 \mid n \in W\}$ and $2W = \{2n \mid n \in W\}$.*

The first clause of the proposition says that if we shift the characteristic sequence of W one place ahead (leaving a 0 at the first position) we get something strictly less informative (in *ibT* terms) than W. This is because *ibT* does not accept even a constant advantage (e.g. 1) of the oracle access relative to the argument. This does not hold if we consider $<_{\mathrm{cl}}$, in which case we need to *spread* W i.e. create some distance between the bits in its characteristic sequence. In this way any constant advantage of the oracle access will be transcended by the increasingly large displacement of the bits of W, making the computability of W from the other set (e.g. $2W$) impossible.

Now it natural to ask whether there can be W c.e. non-computable and some c.e. A such that $W+1 <_{ibT} A <_{ibT} W$. The answer is yes and it is quite easy to construct such sets.

Proposition 4. *There are c.e. non-computable sets W, A such that*

$$W + 1 <_{ibT} A <_{ibT} W.$$

Fig. 1. Straight and shifted registrations of W on A.

We have the non-computability requirements for W and also the requirement that every enumeration $n \searrow W$ must be followed by $n \searrow A$ or $n+1 \searrow A$. The last condition must hold because the enumeration $n+1 \searrow W+1$ must be coded into A by enumerating something $\leq n+1$, which is permitted by W i.e. is $\geq n$ (see Fig. 1). If $n \searrow A$ we say that the W-enumeration is registered *straight* (in A) and if $n+1 \searrow A$ we say that it is registered *shifted*.

Also, we have to make A *strictly* in between the two sets. We do this as follows: call an enumeration $n \searrow W$ *true* if $W \upharpoonright n$ does not change during the stages after the enumeration. The requirement $\Phi^A \neq W$ is satisfied by a true enumeration below the relevant length of agreement which is A-registered *shifted* and $\Psi^{W+1} \neq A$ is satisfied by a true enumeration below the length of agreement which is A-registered *straight*. So each of these requirements needs a single action for their satisfaction (each of them will have a personal witness) and all strategies can be put together satisfying all the requirements with only a finite injury effect.

The motivation for the other direction is that in satisfying the A diagonalization requirements above (making A *strictly* in between $W, W+1$) there is an implicit co-operation of W, A: for example we cannot satisfy $\Phi^A \neq W$ by choosing a witness n where $n+1$ is already in A (because n will be forced to register straight and not shifted). Similarly for $\Psi^{W+1} \neq A$ and as we see in the next section we can regularly block the choice of straight/shifted A-registration if we make W work *against* A. This kind of *blocking* can be seen in Fig. 1.

4. Proof of Theorem 1

If we choose $V = W+1$ the satisfaction of the following requirements is enough to imply the theorem.

$$\mathcal{P}: \quad W \text{ incomputable}$$
$$\mathcal{Q}_A: \ V \leq_{ibT} A \leq_{ibT} W \Rightarrow A \leq_{ibT} V \ \vee \ W \leq_{ibT} A$$

where A runs over all c.e. sets. In a more detailed fashion they can be written as

$$\mathcal{P}_\Gamma: \qquad \Gamma \neq W$$
$$\mathcal{Q}_{\Gamma,\Delta,A}: \ V = \Gamma^A \wedge A = \Delta^W \Rightarrow A = \Phi^V \ \vee \ W = \Psi^A$$

where Γ, Δ run over all partial computable *ibT* functionals and A runs over all c.e. sets. These parameters belong to the opponent while we control W and the functionals Φ, Ψ which we build separately for each requirement. In the following we describe the strategies considering characteristic parts of the construction and as we go on we generalize, arriving finally at a full description of the construction. The proof will be an infinite injury and in particular a $\mathbf{0}''$ tree argument. We often use the words *below* and *above* about the relative position of strategies/requirements, meaning their position on a priority list or even on the final tree (which we imagine as growing downwards). So when a strategy is *above* another one it is of higher priority.

4.1. One $\mathcal{Q}$ above all $\mathcal{P}$

The priority list now looks like

$$\mathcal{Q} > \mathcal{P}_0 > \mathcal{P}_1 > \cdots.$$

To estimate whether the set A of $\mathcal{Q}$ is computable from W and computes V (i.e. the first clause of $\mathcal{Q}$ holds) we use the parameter

$$\ell_{\mathcal{Q}} = \min(\ell(\Gamma^A, V), \ell(\Delta^W, A))$$

where $\ell(\Gamma^A, V), \ell(\Delta^W, A)$ are the lengths of agreement of the relevant reductions. Of course at every stage we only have an approximation of these according to the state of A, V, Γ etc. and this holds for all the parameters involved in the description of the construction.

Finitary outcome. The set A of the opponent should wait for a W-permission before any enumeration. In other words, it should not enumerate numbers below the current value of $\ell_{\mathcal{Q}}$ but instead it should wait until this drops below the number it wishes to enumerate. If it does not follow this safe strategy and enumerates $n \searrow A$ where $n < \ell_{\mathcal{Q}}$, $W \upharpoonright (n+1)$ will be protected by restraints and the disagreement $A(n) \neq \Delta^W(n)$ will be preserved. This gives us a more precise picture of how some A which seriously attempts to refute $\mathcal{Q}$ behaves. In this or any other case where $\ell_{\mathcal{Q}}$ reaches a finite limit (e.g. due to insufficient enumeration of Γ axioms) the outcome will be $\boxed{f}$.

Infinitary outcomes. We imagine $\ell_{\mathcal{Q}}$ as a black area which covers an initial segment of the natural numbers at each stage and expands or shrinks at various stages while $\ell_{\mathcal{Q}} \to \infty$. If we assume that the finitary outcome above does not hold then every time $\ell_{\mathcal{Q}}$ shrinks, this must have happened due to a W-enumeration $\leq \ell_{\mathcal{Q}}$. Conversely, *every change of* W *on argument* $\leq \ell_{\mathcal{Q}}$ *implies a change of* A *on some argument* $\leq \ell_{\mathcal{Q}}$. Indeed, in the construction every W-enumeration will be either strictly below $\ell_{\mathcal{Q}}$ or strictly above $\ell_{\mathcal{Q}}$ (*we* choose the witnesses) and a change $\leq \ell_{\mathcal{Q}}$ will be a change $< \ell_{\mathcal{Q}}$. So there will be a V-change on some argument $\leq \ell_{\mathcal{Q}}$ and hence $\leq \ell(\Gamma^A, V)$. This will bring an A-enumeration $\leq \ell_{\mathcal{Q}}$ since we assumed that $\ell_{\mathcal{Q}} \to \infty$.

So every time $\ell_{\mathcal{Q}}$ drops down, a W-enumeration $\leq \ell_{\mathcal{Q}}$ occurs followed by an A-enumeration $\leq \ell_{\mathcal{Q}}$ before $\ell_{\mathcal{Q}}$ grows larger than ever before (i.e. arrives in an *expansionary stage*). We consider that A-enumeration as a *registration* of the W-enumeration that caused it.

In fact, if $n \searrow W$ is the enumeration, the A-registration cannot be other than $n \searrow A$ or $n+1 \searrow A$ (or both). Indeed for the rectification of Γ we need a number $\leq n+1$ and W has only permitted numbers $\geq n$. If both $n, n+1$ are enumerated into A we consider as A-registration the smallest one. By the usual restraints that we impose below finitary outcomes in tree constructions it follows that every time $\ell_{\mathcal{Q}}$ drops down there will not be another W enumeration below it until it reaches an expansionary stage. So *every* backward movement of $\ell_{\mathcal{Q}}$ is associated with a single A-registration which can be *straight* or *shifted* according to whether $n \searrow A$ or not (exactly as in the sketch of the proof of Proposition 4 above).

We will be keeping the reduction $A = \Phi^V$ up to $\ell_{\mathcal{Q}}$ with the hope that (almost) all registrations are shifted. If this holds it is easy to see that $A = \Phi^V$ will be total and correct. This will be the first infinitary outcome $\boxed{i_1}$. But this assumption may not hold and each straight registration will create an error (permanent, if the related enumeration $n \searrow W$ is true i.e. $W \upharpoonright n$ does not change anymore) on Φ which will have to be initialized. This happening infinitely often, the strategy based on shifted registrations will be injured infinitely many times, ending up unsuccessful. Under this outcome, which we write $\boxed{i_2}$, we use the fact that there are infinitely many straight registrations and pass control to a successful strategy.

Under $\boxed{i_2}$ the strategy is based on the observation that if n has been registered straight $n-1$ has to register straight as well (since n will already be in A). Since $\boxed{i_2}$ guarantees infinitely many straight registrations we can forget the work we did in satisfying the $\mathcal{P}$ requirements under other outcomes (thus cancelling any related witnesses) and start dealing with them anew, exclusively with witnesses $n-1$ such that n is an enumerated witness which has been registered straight (so it is in A and W). We call such numbers $\mathcal{Q}$-*blocked*. This way we can refresh $\Psi^A = W$ up to $\ell_{\mathcal{Q}}$ every time we pass to $\boxed{i_2}$ and we can be sure that no number below the Ψ-length of agreement will register shifted. Indeed the only W-witnesses in that segment will be the ones of the strategies below $\boxed{i_2}$ (we have cancelled the rest) which, if enumerated, are forced to make straight registrations. So $\Psi^A = W$ will be total and correct. What we do here is to create a situation where from a certain point of view (that of Ψ) *all* registrations are straight. Note that we pass to $\boxed{i_2}$ when a witness from below $\boxed{i_1}$ is registered straight.

Of course we will need to take some care as to the way we choose witnesses under outcome $\boxed{i_1}$ but this is not a big deal. For the atomic case described in this section it is enough to choose witnesses (under $\boxed{i_1}$) such that any two are not successive numbers. For the general strategy we will need a stronger condition but this will not be difficult to achieve. We order the outcomes naturally as

$$\boxed{i_2} < \boxed{i_1} < \boxed{f}$$

where $<$ is interpreted as 'to the left' if we think of outcomes as branches on a tree (see Fig. 2). For the $\mathcal{P}$ strategy the outcomes are naturally $\boxed{d} < \boxed{w}$ (i.e. it has diagonalized or it is waiting).

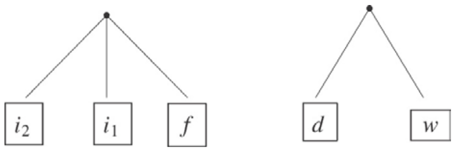

Fig. 2. Outcomes of $\mathcal{Q}$ and $\mathcal{P}$ nodes in the tree.

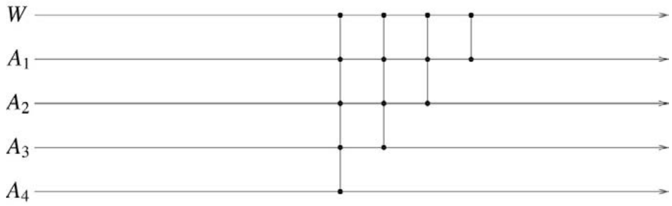

Fig. 3. The suffix of a witness of a node below $\mathcal{Q}_1^\frown\boxed{i_2}, \dots, \mathcal{Q}_n^\frown\boxed{i_2}$.

4.2. *Nesting $\mathcal{Q}$-strategies*

By now the reader should have already started thinking in terms of a tree where the nodes are the requirements/strategies and the branches their outcomes. In the following we will occasionally identify requirements and strategies with the nodes they occupy on the tree. Let us first look at the situation where $\mathcal{Q}_1^\frown\boxed{i_2}$ is above $\mathcal{Q}_2$ which has the various $\mathcal{P}$ requirements below its outcomes. We will require witnesses below $\mathcal{Q}_2^\frown\boxed{i_2}$ to be given $\mathcal{Q}_1$, $\mathcal{Q}_2$-blocked positions i.e. positions n such that $n+2$ is an enumerated witness below $\mathcal{Q}_1^\frown\boxed{i_1}$, $n+1$ is an enumerated witness below $\mathcal{Q}_2^\frown\boxed{i_1}$, and such that

- $n+1$ has been registered straight w.r.t. $\mathcal{Q}_1$, $\mathcal{Q}_2$;
- $n+2$ has been registered straight w.r.t. $\mathcal{Q}_1$.

In particular, $n+1, n+2$ are in W, A_1 and $n+1 \in A_2$. The existence of such positions follows easily: every time $\mathcal{Q}_2$ turns to $\boxed{i_2}$ there is a new straight registration of a witness from below $\mathcal{Q}_2^\frown\boxed{i_1}$. But these witnesses occupied $\mathcal{Q}_1$-blocked positions and so they have been registered straight w.r.t. $\mathcal{Q}_1$ also.

The reason that nested $\mathcal{Q}$ strategies work well is that if a $\mathcal{Q}$-blocked number is enumerated into A then its predecessor, if not in A, will also be $\mathcal{Q}$-blocked. The same argument holds for the case of many $\mathcal{Q}$-requirements $\mathcal{Q}_1 > \mathcal{Q}_2 > \cdots > \mathcal{Q}_n$ with $\boxed{i_2}$ outcomes on the same branch (see Fig. 3), where the existence of suitable witnesses below $\mathcal{Q}_n^\frown\boxed{i_2}$ can be shown inductively. Such a witness will be the first bit of a sequence $m, m+1, \dots, m+n$ of consecutive numbers such that $m+n-i$ is in W and $A_1, \dots, A_{i+1}$ for $0 \le i \le n-1$. Of course such positions m with $m \notin W$ may not exist if we choose witnesses too close to each other. Section 4.3 deals with the *depth* (distance of the witness to the closest witness on its left) that the witnesses should have when they are chosen, so that this pathology does not occur.

4.3. *The distance between witnesses*

Consider $\mathcal{P}$ (possibly on the leftmost path) whose branch contains

$$\mathcal{Q}_1^\frown\boxed{i_2}, \dots, \mathcal{Q}_k^\frown\boxed{i_2}$$

in order of priority. According to the above, this strategy can only pick witnesses n such that for $1 \le i \le k$, $n+i \in W$ and $n+i \in A_1, \dots, A_{k+1-i}$ (see Fig. 4). Since n must be outside W we need to put an additional condition on the *depth* of the witnesses in order that we will be able to show inductively the existence of suitable positions.

Definition 4. For any n and at any stage of the construction, consider the largest $m < n$ which has been a witness of some strategy during some previous stage (if such does not exist let $m = -1$). We say that the depth of n is $n - m$.

$$\underbrace{n}_{\substack{\text{Position for a}\\ \text{witness below}\\ \mathcal{Q}_k^\frown \boxed{i_2}}} < \underbrace{n+1}_{\substack{\text{enumerated}\\ \text{witness of}\\ \mathcal{Q}_k^\frown \boxed{i_1}}} < \cdots < \underbrace{n+i}_{\substack{\text{enumerated}\\ \text{witness of}\\ \mathcal{Q}_{k+1-i}{}^\frown \boxed{i_1}}} < \cdots < \underbrace{n+k}_{\substack{\text{enumerated}\\ \text{witness of}\\ \mathcal{Q}_1^\frown \boxed{i_1}}}$$

Fig. 4. A position for a witness of a node below $\mathcal{Q}_k^\frown \boxed{i_2}, \dots, \mathcal{Q}_1^\frown \boxed{i_2}$ which are written in increasing order of priority and belong to the same branch.

Every $\mathcal{Q}$ on the tree will have a parameter d which denotes the minimum depth requirement for the witnesses chosen below $\mathcal{Q}^\frown \boxed{i_1}$. This parameter relates particularly with the outcome $\boxed{i_1}$ of $\mathcal{Q}$. Every $\mathcal{Q}$ starts with $d = 2$ but every time it reaches $\boxed{i_2}$ it increases d by 1. Positive requirements will be expected to pick witnesses of depth larger than the depth levels d of higher $\mathcal{Q}^\frown \boxed{i_1}$ while they do not have to take into account the depth levels of $\boxed{i_2}$ strategies above. In this way, arguing inductively, an $\boxed{i_2}$ outcome now means the continuous production of suitably blocked positions of arbitrarily large depth.

4.4. $\mathcal{P}$ *in any place on the tree*

Consider some $\mathcal{P}$ sitting on a node α of the tree. If it has no $\boxed{i_2}$ predecessor it can freely choose a witness larger than the restraint and with depth larger than all d levels associated with $\boxed{i_1}$ on the branch above it. Otherwise it will have to pick a large enough witness as before but with the additional condition that it is forced to register straight for the $\mathcal{Q}$ strategies which have an $\boxed{i_2}$ edge above α. According to the previous discussions we ask it to be an *α-blocked position* in the sense of the following definition.

Definition 5. Consider a node α during the construction which has exactly $k > 0$ $\boxed{i_2}$ edges above it, corresponding to $\mathcal{Q}_1, \dots, \mathcal{Q}_k$ in order of increasing height on the tree. We say that a number (position) n is α-blocked if for all $1 \le i \le k$, $n + i$ is an enumerated witness of a node below $\mathcal{Q}_{k+1-i}$ which has been registered straight for $\mathcal{Q}_1, \dots, \mathcal{Q}_{k+1-i}$. In particular, for all $1 \le i \le k$, $n + i \in A_1, \dots, A_{k+1-i}$ and $n + i \in W$. If $k = 0$ every number is α-blocked.

Later we will give a rigorous definition of *registration* (Definition 6) so that everything mentioned in the actual construction is well defined. Note that the notion of α-blocked positions could have been defined independently of 'registration' by the condition 'for all $1 \le i \le k$, $n + i \in A_1, \dots, A_{k+1-i}$ and $n + i \in W$'. But Definition 5 is more intuitive since in order to show the existence of α-blocked positions during the construction we use facts about registrations of numbers; if α is on the leftmost (true) path, α-blocked positions will exist because of its position on the tree and in particular the $\boxed{i_2}$ edges above it (see Lemma 1 of the verification).

4.5. *The tree*

The tree is defined in a standard way based on a priority list (according to an effective numbering of the c.e. sets and the partial computable functionals)

$$\mathcal{Q}_0 > \mathcal{P}_0 > \mathcal{Q}_1 > \cdots \tag{1}$$

and the outcomes/branches of the strategies as described above. Proceeding inductively, if we are on a node which has been assigned a strategy and we want to assign a strategy/requirement to the end α of one of its branches we choose the highest priority requirement (w.r.t. (1)) which has not been assigned to a node above α. In this way in every infinite path of the tree all requirements are represented. Moreover there is the usual relation $<_L$ on nodes which means 'to the left of' and is the lexicographical ordering of nodes when they are seen as sequences of outcomes (it is induced from the ordering we defined on the outcomes). We can also talk of an outcome (i.e. *edge*) $\boxed{\cdot}$ being on the left of a node α if the node that $\boxed{\cdot}$ leads to is on the left of α. When we say that something is *below* a node α we mean that it belongs to the subtree with root α while *above* means that it belongs to the branch between the main root and α.

In the following we will state the actual strategies which will be part of the construction. So they can be thought of as strategies of particular nodes α on the tree with their own α-related parameters which change in α-time i.e. only in stages where α is accessed (*the α-stages*). Every node α has the standard restraint r which it has to respect and which is the largest stage in which some node or edge on the left of α was accessed. This will take care of all the numbers we want to protect once we adopt the usual convention:

$$\Gamma(n)^A[s]\downarrow \Rightarrow n \text{ and the } A\text{-use are } < s$$

for every functional Γ of the opponent (and this will imply the same condition on our functionals).

4.6. $\mathcal{Q}$-module

Before stating the module we make the following definition.

Definition 6. Suppose that $\mathcal{Q}$ sits on node α and $s_0 < s_1$ are two consecutive $\ell_{\mathcal{Q}}$-expansionary α-stages. If some n has entered $W \upharpoonright \ell_{\mathcal{Q}}[s_0]$ in the interval $[s_0, s_1)$ we say that n was registered at s_1. If $n \in A[s_1]$ we say that the registration is *straight*, otherwise it is *shifted*.

The module is as follows.

(1) If $\ell_{\mathcal{Q}}$ is larger than ever before go to the next step. Otherwise access $\boxed{f}$.
(2) If there have been any straight registrations of witnesses below $\boxed{i_1}$ since the last $\mathcal{Q}$-stage go to (3), otherwise go to (4).
(3) Access $\boxed{i_2}$, increase d by one and empty (i.e. initialize) Φ and enumerate axioms for $\Psi^A = W$ up to $\ell_{\mathcal{Q}}$;
(4) Access $\boxed{i_1}$ and enumerate axioms for $\Phi^V = A$ up to $\ell_{\mathcal{Q}}$.

4.7. $\mathcal{P}$-module on α

If $\mathcal{P}$ has been satisfied in previous stages access $\boxed{d}$. Otherwise proceed as follows.

(1) If it has a witness n: check if $\Gamma(n) = 0$, $n < \ell_{\mathcal{Q}}$ for all $\mathcal{Q}$ with $\boxed{i_1}$ or $\boxed{i_2}$ above $\mathcal{P}$. If so then enumerate $n \searrow W$, access $\boxed{d}$ and end the current stage. If not then access $\boxed{w}$.
(2) If it has no witness act as follows: look for a number which
   - is greater than r and any witnesses located in the nodes above;
   - has depth larger than all d associated with the $\boxed{i_1}$ edges above α;
   - is α-blocked.

   If such a number exists, pick it as a witness and access $\boxed{w}$. Otherwise access $\boxed{w}$ and end the stage.

4.8. Construction

At stage s successively access the nodes and edges of a branch of the tree of maximum length s, starting from the root and moving according to the instructions of the corresponding strategies. If we meet the instruction 'end the current stage' before the s-th node we stop developing the branch. If δ_s is the last node we access, initialize all strategies which lie below or on the right of δ_s, cancel any witnesses and any requests they have on $\boxed{i_2}$ edges and move on to stage $s + 1$.

4.9. Verification

Since we only allow certain positions to serve as witnesses, we need to show that such numbers always exist when we need them.

Lemma 1. *Let α be a node on the leftmost path. For every n, at the n-th time α is visited after the last stage at which it is initialized (i.e. at the particular substage) there is an α-blocked position with depth $\geq n$.*

Proof. Suppose $\mathcal{R}$ is on a node α on the leftmost path, Lemma 1 holds for all the predecessors of α and our argument takes place after the last stage s that $\mathcal{R}$ was initialized (i.e. $\delta_s <_L \alpha$). Note that if α has no $\boxed{i_2}$ predecessors then the result holds (any number big enough will do). Otherwise consider the lowest $\boxed{i_2}$ above α (which is an edge of, say, β). If α is visited for the n-th time, that $\boxed{i_2}$ edge has been visited at least n times. So the parameter d of β was at least $n+1$ after the last time $\boxed{i_2}$ was accessed previously. Since that stage there must have been a β-registration of a witness below $\beta^\frown\boxed{i_1}$ with depth at least $n+1$. This witness t must have been β-blocked and β-registered straight. Now $t-1$ has depth at least n and we show that it is α-blocked.

Indeed, $t \to W$ happened when the lengths of agreement ℓ associated with the $\boxed{i_2}$ above α were larger than t. So, since $t+1 \searrow V$ each of the corresponding sets A must enumerate something $\leq t+1$. But it should also be $\geq t$, otherwise it would not be permitted by W and α would not be on the leftmost path. For those strategies strictly above β it must be $t \searrow A$ since t is β-blocked (which implies that $t+1$ was already in A). For β it is also $t \searrow A$ since t is β-registered straight. Hence, $t-1$ is α-blocked. $\square$

Finally, we are going to show by induction that the strategies on the leftmost path are successful, i.e. satisfy the requirements they are working on.

Lemma 2. (1) *If $\mathcal{Q}$ with $\boxed{f}$ is on the leftmost path then $\ell_{\mathcal{Q}}$ reaches a finite limit and so $\mathcal{Q}$ is satisfied.*

(2) *If $\mathcal{Q}$ with $\boxed{i_1}$ is on the leftmost path then its parameter d reaches a finite limit, $\ell_{\mathcal{Q}} \to \infty$ and $\Phi^V = A$.*

(3) *For every $\mathcal{Q}$ with $\boxed{i_2}$ on the leftmost path, $\ell_{\mathcal{Q}} \to \infty$, $d \to \infty$ and $\Psi^A = W$.*

(4) *Every $\mathcal{P}$ on the leftmost path receives a final witness and is satisfied.*

Proof. By induction on the leftmost path: assume that Lemma 2 holds for the nodes above α which is on the leftmost path.

Case α is a $\mathcal{Q}$-node. Consider the following possibilities:

- If $\mathcal{Q}^\frown\boxed{f}$ is on the leftmost path it follows from the $\mathcal{Q}$-module that $\ell_{\mathcal{Q}}$ reaches a limit. So A is not between V, W via Γ, Δ and $\mathcal{Q}$ is satisfied.
- If $\mathcal{Q}^\frown\boxed{i_1}$ is true it is clear that d obtains a final value (since it increases only when $\boxed{i_2}$ is reached) and $\ell_{\mathcal{Q}} \to \infty$. We show that $\Phi^V = A$: since $\ell_{\mathcal{Q}} \to \infty$ there is continuous enumeration of axioms for all arguments and so, if this equality does not hold there must be a least permanent disagreement at some argument n. Let s_1 be the least stage where $\mathcal{Q}^\frown\boxed{i_1}$ was accessed and the disagreement was present and $s_0 < s_1$ the last stage before s_1 where $\mathcal{Q}^\frown\boxed{i_1}$ was accessed.

  Then n must have entered A during (s_0, s_1). But up to the point when we accessed $\mathcal{Q}^\frown\boxed{i_1}$ at s_0, W did not permit $n \searrow A$ since at all stages where $\mathcal{Q}^\frown\boxed{i_1}$ is accessed the axioms of Φ are for arguments less than $\ell_{\mathcal{Q}}$. Also, during the interval (s_0, s_1) no W enumeration which permits n (i.e. $t \searrow W$ with $t \leq n$) will happen since only nodes on the right of $\mathcal{Q}^\frown\boxed{i_1}$ are accessed. So there must have been some $t \searrow W$, $t \leq n$ during s_0 by a node below $\mathcal{Q}^\frown\boxed{i_1}$. So $t+1 \searrow V$ and in order for $\ell_{\mathcal{Q}}$ to be restored, n is either t or $t+1$. If it was t it would count as a straight registration and $\boxed{i_2}$ would be accessed, a contradiction. So $n = t+1$ and Φ^V would be rectified on n, also a contradiction.
- If $\mathcal{Q}^\frown\boxed{i_2}$ is true it is clear that $d \to \infty$ and $\ell_{\mathcal{Q}} \to \infty$. Because of the continuous enumeration of axioms for all arguments, if $\Delta^A = W$ was not true there would be a least n such that $\Delta^A(n) \neq W(n)$. As with the $\boxed{i_1}$ case above $n \searrow W$ at some $\alpha^\frown\boxed{i_2}$-stage by a node β below $\alpha^\frown\boxed{i_2}$. At that time n will be below $\ell_{\mathcal{Q}}$ and in a β-blocked position. So $n+1$ was already in A. When $n \searrow W$, $n+1 \searrow V$ and A has to change in $[n, n+1]$ (there is W permission for change $\geq n$ and the V coding requires change $\leq n+1$). Since $n+1$ is already in A, $n \searrow A$ by the next $\ell_{\mathcal{Q}}$ expansionary stage. So Δ is rectified, a contradiction.

Case α is a $\mathcal{P}$-node. According to the construction, when α is accessed all other positive requirements above it must have witnesses. Note that if a number n has depth $> d$ then $n \geq d$. So, by Lemma 1 there will be a stage where α is accessed and there is a witness larger than the restraint r and with depth larger than all d parameters associated with the $\boxed{i_1}$ edges above α. At this stage $\mathcal{P}$ will pick a witness n and it will keep it for ever (since by hypothesis it is not initialized later). If $\Gamma(n) \downarrow = 0$ at some point, $n \searrow W$ and $\mathcal{P}$ is satisfied. Otherwise it is also satisfied. □

Acknowledgements

The first author was supported by EPSRC grant No. GR /S28730/01. The second author was supported by EPSRC grant No. EP/C001389/1. Both authors were partially supported by the NSFC Grand International Joint Project, No. 60310213, New Directions in the Theory and Applications of Models of Computation.

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