# $K$-trivial Closed Sets and Continuous Functions

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**Abstract.** We investigate the notion of $K$-triviality for closed sets and continuous functions. Every $K$-trivial closed set contains a $K$-trivial real. There exists a $K$-trivial $\Pi^0_1$ class with no computable elements. For any $K$-trivial degree $\mathbf{d}$, there is a $K$-trivial continuous function of degree $\mathbf{d}$.

## 1 Introduction

The study of algorithmic randomness has been an active area of research in recent years. The basic problem is to quantify the randomness of a single real number. Here we think of a real $r \in [0,1]$ as an infinite sequence of 0's and 1's, i.e as an element in $2^{\mathbb{N}}$. There are three basic approaches to algorithmic randomness: the measure theoretic, the compressibility and the betting approaches. All three approaches have been shown to yield the same notion of (algorithmic) randomness. Here we will only use notions from the compressibility approach, incorporating a number of non-trivial results in this area. For background and history of algorithmic randomness we refer to [11, 10, 13].

Prefix-free (Chaitin) complexity for reals is defined as follows. Let $M$ be a prefix-free function with domain $\subset \{0,1\}^*$. For any finite string $\tau$, let $K_M(\tau) =$

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$min\{|\sigma| : M(\sigma) = \tau\}$. There is a *universal* prefix-free function $U$ such that, for any prefix-free $M$, there is a constant $c$ such that for all $\tau$

$$K_U(\tau) \leq K_M(\tau) + c.$$

We let $K(\sigma) = K_U(\sigma)$. Then $x$ is said to be *random* if there is a constant $c$ such that $K(x\lceil n) \geq n - c$ for all $n$. This means a real $x$ is random exactly when its initial segments are not *compressible*.

In a series of recent papers [1–4], P. Brodhead, S. Dashti and the authors have defined the notion of (algorithmic) randomness for closed sets and continuous functions on $2^{\mathbb{N}}$. Some definitions are needed. For a finite string $\sigma \in \{0,1\}^n$, let $|\sigma| = n$. For two strings $\sigma, \tau$, say that $\tau$ *extends* $\sigma$ and write $\sigma \prec \tau$ if $|\sigma| \leq |\tau|$ and $\sigma(i) = \tau(i)$ for $i < |\sigma|$. For $x \in 2^{\mathbb{N}}$, $\sigma \prec x$ means that $\sigma(i) = x(i)$ for $i < |\sigma|$. Let $\sigma^\frown\tau$ denote the concatenation of $\sigma$ and $\tau$ and let $\sigma^\frown i$ denote $\sigma^\frown(i)$ for $i = 0, 1$. Let $x\lceil n = (x(0), \ldots, x(n-1))$. Two reals $x$ and $y$ may be coded together into $z = x \oplus y$, where $z(2n) = x(n)$ and $z(2n+1) = y(n)$ for all $n$. For a finite string $\sigma$, let $I(\sigma)$ denote $\{x \in 2^{\mathbb{N}} : \sigma \prec x\}$. We shall call $I(\sigma)$ the *interval* determined by $\sigma$. Each such interval is a clopen set and the clopen sets are just finite unions of intervals. Now a nonempty closed set $P$ may be identified with a tree $T_P \subseteq \{0,1\}^*$ where $T_P = \{\sigma : P \cap I(\sigma) \neq \emptyset\}$. Note that $T_P$ has no dead ends. That is, if $\sigma \in T_P$, then either $\sigma^\frown 0 \in T_P$ or $\sigma^\frown 1 \in T_P$. For an arbitrary tree $T \subseteq \{0,1\}^*$, let $[T]$ denote the set of infinite paths through $T$. For a detailed development of $\Pi^0_1$ classes, see [5].

We define a measure $\mu^*$ on the space $\mathcal{C}$ of closed subsets of $2^{\mathbb{N}}$ as follows. Given a closed set $Q \subseteq 2^{\mathbb{N}}$, let $T = T_Q$ be the tree without dead ends such that $Q = [T]$. Let $\sigma_0, \sigma_1, \ldots$ enumerate the elements of $T$ in order, first by length and then lexicographically. We then define the code $x = x_Q = x_T$ by recursion such that for each $n$, $x(n) = 2$ if both $\sigma_n{}^\frown 0$ and $\sigma_n{}^\frown 1$ are in $T$, $x(n) = 1$ if $\sigma_n{}^\frown 0 \notin T$ and $\sigma_n{}^\frown 1 \in T$, and $x(n) = 0$ if $\sigma_n{}^\frown 0 \in T$ and $\sigma_n{}^\frown 1 \notin T$. We then define $\mu^*$ by setting

$$\mu^*(\mathcal{X}) = \mu(\{x_Q : Q \in \mathcal{X}\}) \tag{1}$$

for any $\mathcal{X} \subseteq \mathcal{C}$ and $\mu$ is the standard measure on $\{0,1,2\}^{\mathbb{N}}$. Informally this means that given $\sigma \in T_Q$, there is probability $\frac{1}{3}$ that both $\sigma^\frown 0 \in T_Q$ and $\sigma^\frown 1 \in T_Q$ and, for $i = 0, 1$, there is probability $\frac{1}{3}$ that only $\sigma^\frown i \in T_Q$. In particular, this means that $Q \cap I(\sigma) \neq \emptyset$ implies that for $i = 0, 1$, $Q \cap I(\sigma^\frown i) \neq \emptyset$ with probability $\frac{2}{3}$. Brodhead, Cenzer, and Dashti [2] defined a a closed set $Q \subseteq 2^{2^{\mathbb{N}}}$ to be (Martin-Löf) random if $x_Q$ is (Martin-Löf) random. Note that the equal probability of $\frac{1}{3}$ for the three cases of branching allows the application of Schnorr's theorem that Martin-Löf randomness is equivalent to prefix-free Kolmogorov randomness. Then in [2, 3], the following results are proved. Every random closed set is perfect and contains no computable elements (in fact, it contains no $n$-c.e. elements). Every random closed set has measure 0 and has box dimension $\log_2 \frac{4}{3}$.

A continuous function $F : 2^{\mathbb{N}} \to 2^{\mathbb{N}}$ may be represented by a function $f : \{0,1\}^* \to \{0,1\}^*$ such that the following hold, for all $\sigma \in \{0,1\}^*$.

– $|f(\sigma)| \leq |\sigma|$.
– $\sigma_1 \prec \sigma_2$ implies $f(\sigma_1) \prec f(\sigma_2)$.
– For every $n$, there exists $m$ such that for all $\sigma \in \{0,1\}^m$, $|f(\sigma)| \geq n$.
– For all $x \in 2^{\mathbb{N}}$, $F(x) = \bigcup_n f(x\lceil n)$.

We define the space $\mathcal{F}$ of representing functions $f : \{0,1\}^* \to \{0,1\}^*$ to be those which satisfy clauses (1) and (2) above. There is a one-to-one correspondence between $\mathcal{F}$ and $\{0,1,2\}^{\mathbb{N}}$ defined as follows. Enumerate $\{0,1\}^*$ in order, first by length and then lexicographically, as $\sigma_0, \sigma_1, \ldots$. Thus $\sigma_0 = \emptyset$, $\sigma_1 = (0), \sigma_2 = (1), \sigma_3 = (00), \ldots$. Then $r \in \{0,1,2\}^{\mathbb{N}}$ corresponds to the function $f_r : \{0,1\}^* \to \{0,1\}^*$ defined by declaring that $f_r(\emptyset) = \emptyset$ and that, for any $\sigma_n$ with $|\sigma_n| \geq 1$,

$$f_r(\sigma_n) = \begin{cases} f_r(\sigma_k), & \text{if } r(n) = 2; \\ f_r(\sigma_k)^\frown i, & \text{if } r(n) = i < 2. \end{cases}$$

where $k$ is such that $\sigma_n = \sigma_k{}^\frown j$ for some $j$. Every continuous function $F$ has a representative $f$ as described above, and, in fact, it has infinitely many representatives. We define a measure $\mu^{**}$ on $\mathcal{F}$ induced by the standard probability measure on $\{0,1,2\}^{\mathbb{N}}$. Brodhead, Cenzer, and Remmel [4] defined an (Martin-Löf) random continuous function on $2^{\mathbb{N}}$ which has a representation in $\mathcal{F}$ which is Martin-Löf random. The following results are proved in [1, 4]. Random $\Delta^0_2$ continuous functions exist, but no computable function can be random and no random function can map a computable real to a computable real. The image of a random continuous function is always a perfect set and hence uncountable. For any $y \in 2^{\mathbb{N}}$, there exists a random continuous function $F$ with $y$ in the image of $F$. Thus the image of a random continuous function need not be a random closed set. The set of zeroes of a random continuous function is a random closed set (if nonempty).

There has been a considerable amount of work on studying reals whose complexity is "low" or trivial from the point of view of randomness. Chaitin defined a real $x$ to be $K$-trivial if $K(x\lceil n) \leq K(1^n) + O(1)$. We recall that there are noncomputable c.e. sets which are $K$-trivial and that the $K$-trivial reals are downward closed under Turing reducibility. The latter is a highly non-trivial result of Nies [15] who also showed that the $K$-trivial reals form a $\Sigma^0_3$-definable ideal in the Turing degrees. In particular, this means that if $\alpha$ and $\beta$ are $K$-trivial, then the join $\alpha \oplus \beta$ is also $K$-trivial.

The main goal of this paper is to study $K$-triviality for closed subsets of $2^{\mathbb{N}}$ and for continuous functions on $2^{\mathbb{N}}$. We define a closed set $Q$ to be $K$-trivial if the code $x_Q$ is $K$-trivial and we define a continuous function $F : 2^{\mathbb{N}} \to 2^{\mathbb{N}}$ to be $K$-trivial if it has a representing function $f \in \mathcal{F}$ which is $K$-trivial.

## 2 *K*-trivial closed sets

Since every $K$-trivial real is $\Delta^0_2$, we have that every $K$-trivial closed set is a strong $\Pi^0_2$ class. Note also that the canonical code of a $\Pi^0_1$ class has c.e. degree and that there are $K$-trivial reals with non-c.e. degree. Hence there are $K$-trivial closed sets which are not $\Pi^0_1$ classes.

Analogous to the existence of c.e. $K$-trivial reals, we will construct several examples of $K$-trivial $\Pi^0_1$ classes. Note that a $\Pi^0_1$ class $P$ is said to be *decidable* if the canonical tree $T_P$ is computable, which is if and only if the canonical code for $P$ is computable. Thus we want to construct $K$-trivial $\Pi^0_1$ classes which are not decidable. The degree of a closed set $Q$ is the degree of the tree $T_Q$ and also the degree of the canonical code for $T_Q$.

We begin with those non-decidable $\Pi^0_1$ classes with the simplest structure, that is, countable classes with a unique limit path. Our first construction relies on the following notion. If $A = \{a_0 < a_1 < \cdots\}$ is an infinite set, then $A$ is said to be *retraceable* if there is a partial computable function $\phi$ such that $\phi(a_{n+1}) = a_n$ for all $n$. The initial subsets of $A$ are $A$ together with the finite sets $\{a_0, \ldots, a_{n-1}\}$ for each $n$. Dekker and Myhill [9] showed that every c.e. degree contains a retraceable $\Pi^0_1$ set $A$. Cenzer, Downey, Jockusch and Shore [6] showed that a $\Pi^0_1$ set $A$ is retraceable if and only if the family $I(A)$ of initial subsets is a $\Pi^0_1$ class. Clearly $I(A)$ has unique limit element $A$.

**Theorem 1.** *For any noncomputable $K$-trivial c.e. degree* $\mathbf{d}$*, there exists a $K$-trivial $\Pi^0_1$ class $P$ of degree* $\mathbf{d}$ *such that $P$ has a unique, noncomputable limit element.*

*Proof.* Let $A$ be a retraceable $\Pi^0_1$ set of degree $\mathbf{d}$. Then $A$ is $K$-trivial and noncomputable and is the unique limit element of the $\Pi^0_1$ class $P = I(A)$ as shown above. It remains to show that the tree $T_P$ has the same degree as $A$. Certainly $T_P \leq_T A$, since

$$\sigma \in T_P \iff (\forall i < |\sigma|)[\sigma(i) = 1 \rightarrow (i \in A \;\&\; (\forall j < i)(j \in A \rightarrow \sigma(j) = 1))].$$

On the other hand, $A \leq_T T_P$ since

$$a \in A \iff (\exists \sigma \in \{0,1\}^{a+1})(\sigma \in T_P \;\&\; \sigma(a) = 1).$$

$\square$

We next construct a $K$-trivial class having only computable members.

**Theorem 2.** *For any $K$-trivial c.e. degree* $\mathbf{d}$*, there exists a $K$-trivial $\Pi^0_1$ class of degree* $\mathbf{d}$ *with unique limit path $0^\omega$ and all elements computable.*

*Proof.* Let $B$ be a co-c.e. set of degree $\mathbf{d}$ and let $Q = \{0^\omega\} \cup \{\{n\} : n \in B\}$. Clearly $Q$ has all elements computable and unique limit element $0^\omega$. It is easy to check that $T_Q \equiv_T B$. $\square$

Next we wish to obtain a $\Pi^0_1$ class with no computable members (a *special* $\Pi^0_1$ class) such that the code for the class is $K$-trivial. To do so we rely heavily on the fact that $K$-triviality is closed under Turing equivalence. Note first that since the $K$-trivials form an ideal in the Turing degrees, the separating class for two $K$-trivial sets $A, B$ will be $K$-trivial, as the set of its extendible nodes (and hence its code) is Turing-equivalent to $A \oplus B$. It remains to show there are recursively inseparable $K$-trivial sets. The following proof due to Steve Simpson.

**Theorem 3.** *There is a $K$-trivial $\Pi^0_1$ class with no computable members.*

*Proof.* Let $B$ be a noncomputable c.e. $K$-trivial set. Split $B$ into disjoint noncomputable c.e. $A_1, A_2$ as in the Friedberg splitting theorem. Ohashi [17] observed that the proof of the Friedberg splitting theorem in fact gives that $A_1, A_2$ are recursively inseparable. By the downward closure of $K$-triviality, they are also $K$-trivial. Let $S$ be their separating class. Then by the discussion above, $S$ is a special $K$-trivial $\Pi^0_1$ class. □

Now a separating class always has measure zero. Next we construct $K$-trivial classes of arbitrarily large positive measure yet still containing no computable members. The proof makes use of the well-established *cost function* method from the area of algorithmic randomness, first used in Kucera-Terwijn [14] and later made explicit, e.g. in Downey-Hirschfeldt-Nies-Stephan [12].

**Theorem 4.** *There is a $K$-trivial $\Pi^0_1$ class (of arbitrarily large measure) with no computable paths (thus perfect).*

*Proof.* There is a well established framework for constructing $K$-trivial reals in the Cantor space $2^\omega$ in terms of *cost functions.* A good presentation of this can be found in Nies [16]. It is clear that the same method applies to the space $3^\omega$. Let $K$ be the prefix-free complexity and

$$cost(x,t) = \sum_{x<w\leq t} 2^{-K_t(w)}.$$

It is well known that $\lim_x \sup_t cost(x,t) = 0$. In order to construct a $K$-trivial $\Pi^0_1$ class $P$ it suffices to give a monotone approximation $(P_t)$ to $P$ (in the sense that $P_t \supseteq P_{t+1}$) such that if $c_t$ is the code for $P_t$ and $x_s$ is the least number such that $c_{s-1}(x) \neq c_s(x)$ then

$$\sum_{s>0} cost(x_s, s) \leq 1. \tag{2}$$

Indeed in [16] it is shown that $c$ is $K$-trivial iff it has a $\Delta^0_2$ approximation $(c_t)$ which satisfies (2). To make sure that there are no computable paths through $P$ it suffices to satisfy the following requirements:

$$R_e : \Phi_e \text{ is total } \Rightarrow \Phi_e \notin P$$

where $(\Phi_e)$ is an effective enumeration of all Turing functionals with binary values. The strategy for $R_e$ is to modify the code $c$ at some stage so that the tree represented by $c$ no longer extends some initial segment of $\Phi_e$. This is done by switching a 2 in $c$ to a 0 or 1 according to which has the desired effect. First note that each $c_t$ will consist of all 2's except for a finite initial segment, so we will find a suitable digit to switch. Second note that when we change a position in $c$ from 2 to something else (0 or 1), we can effectively adjust the tail of $c$ (the digits after the modified digit) so that the code describes the tree that we get if we cut that branch from the branching node corresponding to the 2 above. This means that if we let $R_e$ act on $c$ in the way described above, we get an approximation to $P$ which is co-c.e. (so $P$ is a $\Pi^0_1$ class).

The last consideration is that $R_e$ cannot change digit $n$ at stage $s$ unless $cost(n,s) < 2^{-(e+1)}$. This will make $c$ $K$-trivial. Let $\mathbb{N}^{[e]}$ be the $e$-th column of $\mathbb{N}$, i.e. the set of numbers of the form $\langle e,t\rangle$ for some $t \in \mathbb{N}$ where $\langle .,.\rangle$ is a computable bijection from $\mathbb{N}\times\mathbb{N}$ to $\mathbb{N}$. The symbol $\restriction$ denotes restriction of the object that precedes it to the numbers $< x$. For example $\Phi_e \restriction x \downarrow$ means that $\Phi_e$ is defined at all arguments $< x$. All parameters in the construction are in formation and only have current values which correspond to the current stage.

**Construction.** At stage $s$ look for the least $e < s$ such that $R_e$ has not acted and there is a positive $x \in \mathbb{N}^{[e]}$ with the property that

- $\Phi_e \restriction x \downarrow$ and is on $P_s$
- $cost(k(x,s),s) < 2^{-(e+1)}$, where $k(x,s)$ is the position of node $\Phi_e \restriction (x-1)$ in the code $c_s$ of $P_s$.

If there is no such $e$ go to the next stage. Otherwise note that since $R_e$ has not acted and $x \in \mathbb{N}^{[e]}$, no strategy has chopped any branch from node $\Phi_e \restriction (x-1)$ and so the latter is branching. Now switch $k(x,s)$ from 2 to $1-\Phi_e(x-1)$ (so that $\Phi_e \restriction x \notin P$) and let larger positions describe the tree that we get by chopping that branch. Go to the next stage.

For the verification, the comments before the description of $R_e$ explain why the approximation $(c_t)$ defined in the construction corresponds to a co-c.e. approximation of $P$, so that $P$ is a $\Pi^0_1$ class. Each $R_e$ is satisfied by the standard cost-function argument: there is some $x_0$ such that for all $x > x_0$ and all $s$, $cost(x,s) < 2^{-(e+1)}$ (by the properties of *cost*). Finally $c$ is $K$-trivial since the approximation $(c_t)$ given in the construction satisfies (2) (that each $R_e$ acts at most once and contributes cost at most $2^{-(e+1)}$). Finally note that by choosing the witnesses $x$ sufficiently large we can make sure that $P$ has measure arbitrarily close to 1. □

**Theorem 5.** *If $P$ is a $K$-trivial $\Pi^0_1$ class then the leftmost path is a $K$-trivial real.*

*Proof.* The leftmost path is computable from the (code of the) $\Pi^0_1$ class $P$ and since $K$-triviality is downward closed under Turing reductions it must be $K$-trivial. □

By Nies' top low$_2$ theorem (see [11]), there is a low$_2$ c.e. degree above all $K$-trivial degrees. By Theorem 5, this means that the sets computed by it form a basis for the $K$-trivial $\Pi^0_1$ classes (while no incomplete c.e. degree has this property with respect to all $\Pi^0_1$ classes). The following theorem shows that such a c.e. degree cannot be low. Note however that there are low $PA$ degrees, i.e. low degrees such that the sets computed by them form a basis for all $\Pi^0_1$ classes. The corresponding problem for $K$-trivial reals—whether there is a low degree bounding all $K$-trivials—is a major open problem.

**Theorem 6.** *If $A$ is c.e. and low then there is a $K$-trivial $\Pi^0_1$ class which contains no $A$-computable paths. In other words, there is no c.e. low set $A$ such that the sets computed by $A$ form a basis for the $K$-trivial $\Pi^0_1$ classes.*

*Proof.* This is similar to the proof that for every c.e. low $A$ there is a $K$-trivial $B$ such that $B \not\leq_T A$ (in the same way that the proof of Theorem 4 is similar to the construction of a non-computable $K$-trivial set). If the reader is not familiar with that construction, (s)he might like to have a look at it [16]. We wish to follow the construction of Theorem 4 only now we need to satisfy the following more demanding requirements:

$$R_e : \Phi_e^A \text{ is total } \Rightarrow \Phi_e^A \notin P.$$

In general it is impossible to satisfy these requirements but if we know that $A$ is low we can use the following trick (due to Robinson) to succeed. During the construction we will ask $\emptyset'$ a $\Sigma^0_1(A)$ question (for the sake of $R_e$). Note that since $A$ is low, $\emptyset'$ can answer such questions. At each stage we will only have an approximation to $\emptyset'$ and so we will get a correct answer possibly after a finite number of false answers. Requirement $R_e$ will use *witnesses* (in the sense of the proof of Theorem 4) from $\mathbb{N}^{[e]}$. We will ask the following:

Is there a stage $s$ and a witness $x$ such that
– $\Phi_e^A \restriction x[s] \downarrow$ with correct $A$-use and $\Phi_e^A \restriction x[s] \in P_s$
– $cost(k(x,s),s) < 2^{-(n_e+e+3)}$
where $n_e$ is the number of times that some branch of $P$ has been pruned (i.e. some digit of $c$ has been changed) for the sake of $R_e$?

First notice that the above question refers to the partial computable sequences $(P_s)$, $(n_e[s])$ which are defined during the very construction. By the recursion theorem we can ask such questions and approximate the right answers: given any partial computable sequence $(P'_s)$ of $\Pi^0_1$ classes and uniformly partial computable sequences $(n'_e[s])$, we will effectively define a construction in which the questions refer to the given parameters. All of these constructions will define a sequence $(P_s)$ of $\Pi^0_1$ classes which monotonically converges to a $K$-trivial $\Pi^0_1$ class $P$ which however does not necessarily satisfy the other requirements; also each will define a uniformly partial computable sequence $(n_e[s])$. The (double) recursion theorem will give a construction in which the questions asked actually refer to $(P_s)$ and $(n_e[s])$. Such a construction will succeed in satisfying all requirements. Let $g(e,s)$ be a computable function approximating the true answer to the questions above, when these are set to refer to the given parameters $(P'_s)$, $(n'_e[s])$.

**Construction.** For stage $s$ and each $e < s$ such that there is an *unused* $x \in \mathbb{N}^{[e]}$ satisfying $\Phi_e^A \restriction x[s] \in P_s$ and $cost(k(x,s),s) < 2^{-(n_e[s]+e+3)}$ (where $n_e[s]$ is as above) do the following. Wait for a stage $t \geq s$ such that $g(e,t) = 1$ or the computation $\Phi_e^A \restriction x[s]$ has been spoiled. In the first case switch $k(x,s)$ from 2 to $1 - \Phi_e(x-1)$ (so that $\Phi_e \restriction x \notin P$) and let larger positions describe the tree that we get by chopping that branch (say that $x$ has been *used*); proceed to stage $s+1$. In the latter case do nothing and test the next value of $e$. If the above has run over all $e < s$ and we are still at stage $s$, go to stage $s+1$.

For the verification, note that if $x$ is unused at some stage, then currently all nodes of the $x$th level of $P$ are branching. So each construction defines a (possibly finite) monotone sequence of clopen sets $P_s$ (and so a $\Pi^0_1$ class $P$ as a limit). Also, for every values of the input $(P'_s)$, $(n'_e[s])$ the resulting class $P$ is $K$-trivial as the condition (2) from the proof of Theorem 4 holds (at any stage at most one requirement acts and the cost of that action is small by construction). By the double recursion theorem there is a construction such that

$$n_e[s] = n'_e[s] \ \wedge \ P_s = P'_s$$

for all $s, e$; i.e. the input and output as (double) partial computable sequences are the same. This construction must be total (in the sense that it passes through all stages) since every search halts (for example if $\Phi^A_e \upharpoonright x[s] \in P_s$, $cost(k(x,s),s) < 2^{-(n_e[s]+e+3)}$ and the computation is true then $g(e)$ has to settle at 1 as it guesses correctly). Finally suppose that $R_e$ is not satisfied. This means that the answer to the $e$-question is a negative one. So $g(e)$ would settle to 0 (since it approximates the correct answer to the $e$-question) and $R_e$ would act finitely often. But then the cost requirement (in particular $n_e$) would remain constant and (by the properties of *cost*) for some large enough $x, s$ the computation $\Phi^A_e[s] \upharpoonright x$ will be correct and $\Phi^A_e \upharpoonright x[s] \in P_s$, $cost(k(x,s),s) < 2^{-(n_e[s]+e+3)}$ which is a contradiction. $\square$

## 3 *K*-trivial continuous functions

In [4], the notion of randomness was extended to continuous functions on $2^{\mathbb{N}}$. Thus it will be natural to consider $K$-trivial continuous functions. It was shown in [4] that a random continuous function maps any computable real to a random real. It follows immediately from the closure under join of $K$-trivial degrees that a $K$-trivial continuous function maps any computable real to a $K$-trivial real. It was shown in [4] that the set of zeroes of a random continuous function is either empty or random. It follows by downward closure of the $K$-trivial degrees that the set of zeroes of a $K$-trivial continuous function is either empty or $K$-trivial.

We consider a continuous functions $F : 2^{\mathbb{N}} \to 2^{\mathbb{N}}$ always in terms of one its representing functions $f : 2^{<\mathbb{N}} \to 2^{<\mathbb{N}}$, or, equivalently, in terms of the code of one of its representing functions. Note that by slowing the convergence of the function on finite strings, we may code information into the code of the function. Hence the codes of a given function on Cantor space are always closed upwards in the Turing degrees, so the $K$-degree of a function should be the $K$-degree of the canonical code, that which converges as rapidly as possible. However, the canonical code of a function $F$ may be computed from *any* code, so it follows from the downward closure of $K$-triviality that $F$ is $K$-trivial if and only if the canonical code is $K$-trivial.

**Theorem 7.** *For any $K$-trivial degree* $\mathbf{d}$*, there is a continuous function $F : 2^{\mathbb{N}} \to 2^{\mathbb{N}}$ with canonical code of degree* $\mathbf{d}$*. Moreover, if* $\mathbf{d}$ *is c.e., $F$ may be chosen to have left-c.e. canonical code.*

*Proof.* Let $A = \{a_1, a_2, \ldots\}$ be a set of degree $\mathbf{d}$. We define $F$ monotonically increasing such that $F(0^\omega) = 0^\omega$ and $F(1^\omega) = \chi_A$, the characteristic function of $A$. We work via $f : 2^{<\mathbb{N}} \to 2^{<\mathbb{N}}$. To begin, let $f(0) = 0^{(a_1+1)}$ and $f(1) = 0^{a_1}1$. Now suppose we have defined $f(\sigma) = \tau$ for $|\sigma| = n-1$, and that $a_n - a_{n-1} = m$. Then let $f(\sigma 0) = \tau 0^m$ and $f(\sigma 1) = \tau 0^{m-1}1$. It is clear that $f \equiv_T A$, so $f$ is of degree $\mathbf{d}$. Furthermore, if $\mathbf{d}$ is a c.e. degree and $A$ is chosen c.e., the code given by $f$ will be left-c.e., as shown by an analysis of the construction.

The code for $f$ may be thought of as composed of blocks of length $2^n$ for $n \geq 1$, in order of increasing size, corresponding to different levels of the tree. At level $n$, if $n-1 \notin A$, the block will be all zeros. If $n-1 \in A$ and $|A \upharpoonright n| = k$, the block will consist of $2^k$ subblocks of $2^{n-k}$ bits each, beginning with a subblock of all zeros and alternating to end with a subblock of all 1s. Thus the structure of the $n^{th}$ block is determined entirely by whether $n-1$ is in $A$, and if so, how many values $< n-1$ are also in $A$.

Given an enumeration of $A$ as $A_s$, $s \in \omega$, we may define an approximation to the function $F$ with corresponding canonical code $C_s$. We show that as $s$ increases, a bit of $C_s$ holding a one may only change to zero if a preceding bit changes from zero to one; this shows that $C_s$ is an increasing approximation. As the enumeration $A_s$ is computable by assumption, the canonical code of $F$ is then left-c.e. Without loss of generality we consider a single level of the tree, $n$, and a single stage, $s$. If the corresponding block of $C_{s-1}$ is all zeros, this level causes no trouble at stage $s$: either it remains all zeros or half of its zeros change to ones. If the $n^{th}$ block of $C_{s-1}$ is half zeros and half ones, then enumeration into $A$ at stage $s$ may cause the subblocks to multiply and rearrange. However, this only occurs when some $k < n-1$ enters $A_s$, causing the corresponding earlier level to change from all zeros to half zeros and half ones. □

## 4 Medvedev degrees of $K$-trivial classes

The degrees of difficulty of $K$-trivial closed sets should be of interest. Simpson [18], Cenzer and Hinman [7] and others have developed the subject of the Medvedev (or strong) degrees of $\Pi^0_1$ classes. Here $P \leq_M Q$ means that there is a computable function mapping $Q$ into $P$. The Medvedev degrees form a lattice where the meet operation is the disjoint union and the join is the product, $P \otimes Q = \{\alpha \oplus \beta \mid \alpha \in P \text{ and } \beta \in Q\}$. There is a least degree $0_M$ consisting of the classes with a computable member and a highest degree $1_M$ which can be viewed as a universal $\Pi^0_1$ class. A related structure are the *Muchnik (or weak) degrees*, where $P$ is weakly reducible to $Q$ if for every $\beta \in Q$ there exists $\alpha \in P$ such that $\alpha \leq_T \beta$.

One general problem is where the $K$-trivial $\Pi^0_1$ classes fit into the Medvedev (or Muchnik) degrees of the $\Pi^0_1$ classes. We have only a few results so far.

Since the $K$-trivial reals form an ideal in the Turing degrees, it follows that the family of $\Pi^0_1$ classes which contain a $K$-trivial real form an ideal in the lattice of Medvedev degrees (and also in the lattice of Muchnik degrees). The following proposition says that the $K$-trivial $\Pi^0_1$ classes are closed under the meet and the join operation in the Medvedev degrees.

**Proposition 1.** *The $K$-trivial $\Pi^0_1$ classes are closed under disjoint unions and under products.*

*Proof.* The degree of the code of the disjoint union of two $\Pi^0_1$ classes is the join of the degrees of the codes of these $\Pi^0_1$ classes. The same holds for products and since $K$-triviality is invariant in the Turing degrees and closed under join (in the Turing degrees) the proposition follows. □

Note however that $K$-triviality (for $\Pi^0_1$ classes) is not closed under Medvedev equivalence. For example the least Medvedev degree contains $\Pi^0_1$ classes with computable leftmost path but with a canonical code which computes the halting problem. Hence we could call a Medvedev degree $K$-trivial if it contains a $K$-trivial class. Since there is no c.e. complete $K$-trivial real and any Medvedev complete $\Pi^0_1$ class is also c.e. complete, it follows that no $K$-trivial $\Pi^0_1$ class is Medvedev complete. A relevant question is whether there a top Medvedev degree among the $K$-trivials, or even a maximal one.

**Theorem 8.** *There is no maximal $K$-trivial Medvedev or Muchnik degree.*

*Proof.* Given any $K$-trivial $\Pi^0_1$ class $Q$ it suffices to construct a $K$-trivial $\Pi^0_1$ class $P$ which is not weakly reducible to $Q$. Indeed, in that case $P \otimes Q$ would be $K$-trivial strongly above $Q$ and not weakly below $Q$. We argue as follows. By the Low Basis Theorem, $Q$ contains a member $\alpha$ of low Turing degree. Now by Theorem 6, there is a $K$-trivial $\Pi^0_1$ class $P$ with no path computed by $\alpha$. This means that $P$ is not weakly reducible to $Q$. □

The above proof also shows that there is no $\Pi^0_1$ class $P$ which has low canonical code and is weakly above all $K$-trivial $\Pi^0_1$ classes.

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