ON THE GAP BETWEEN TRIVIAL AND NONTRIVIAL INITIAL SEGMENT PREFIX-FREE COMPLEXITY

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Abstract. An infinite sequence X is said to have trivial (prefix-free) initial segment complexity if the prefix-free Kolmogorov complexity of each initial segment of X is the same as the the complexity of the sequence of 0s of the same length, up to a constant. We study the gap between the minimum complexity $K(0^n)$ and the initial segment complexity of a nontrivial sequence, and in particular the nondecreasing unbounded functions f such that

$$(\star)\quad K(X\upharpoonright_n) \leq K(0^n) + f(n) + c \text{ for a constant } c \text{ and all } n$$

for a nontrivial sequence X, where K denotes the prefix-free complexity. Our first result is that there exists a Δ^0_3 unbounded nondecreasing function f which does not have this property. It is known that such functions cannot be Δ^0_2 hence this is an optimal bound on their arithmetical complexity. Moreover it improves the bound Δ^0_4 that was known from Csima and Montalbán [CM06].

Our second result is that if f is Δ^0_2 then there exists a non-empty Π^0_1 class of reals X with nontrivial prefix-free complexity which satisfy $(\star)$. This implies that in this case there uncountably many nontrivial reals X satisfying $(\star)$ in various well known classes from computability theory and algorithmic randomness; for example low for Ω, non-low for Ω and computably dominated reals. A special case of this result was independently obtained by Bienvenu, Merkle and Nies [BMN11].

1. Introduction

Kolmogorov complexity measures the absolute amount of information that is coded into a binary string (a program). The algorithmic randomness of an infinite binary sequence, a *real*, is reflected by the complexity of its initial segments. If we restrict the underlying machines to prefix-free domains, we get a refined measure which is known as prefix-free complexity and is a standard tool for studying the initial segment complexity of reals.

In the last 10 years or so, particular emphasis has been drawn to reals with low prefix-free complexity. A lower bound on the prefix-free complexity of n bits (up to a constant) is the complexity of n zeros. The reals that have trivial prefix-free initial segment complexity, namely below this lower bound up to a constant, are called K-trivial and have been the object of intense study in recent years. See Nies [Nie09, Chapter 5] or Downey and Hirschfeldt [DH10, Chapter 11].

Key words and phrases. Kolmogorov complexity, initial segment prefix-free complexity, K-triviality, low for Ω.

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In this paper we are interested in reals that have very low prefix-free initial segment complexity, but not necessarily trivial. Let $K(\sigma)$ denote the prefix-free complexity of the sequence σ and let $K(n)$ denote $K(0^n)$. We study the following question.

(1.1) **Question.** Which reals X satisfy $\exists c\ \forall n\ (K(X\restriction_n) \leq K(n) + f(n) + c)$, when f is an unbounded nondecreasing function?

In order to compare rates of growth of functions we use the arithmetical hierarchy of complexity. For example, there is a Δ^0_5 unbounded nondecreasing function that grows more slowly than any Δ^0_4 unbounded nondecreasing function. As far as question (1.1) is concerned, we found that the difference between Δ^0_2 and Δ^0_3 functions is of special significance.

Our first result is the existence of a Δ^0_3 unbounded nondecreasing function f such that all reals of (1.1) have trivial complexity. This provides an optimal bound on the arithmetical complexity of such functions and extends work by Csima and Montalbán [CM06] (see Section 1.1). Our second result, discussed in Section 1.2, is that if f if Δ^0_2 then the class of reals in (1.1) is large and includes nontrivial members of some well known classes from computability theory and algorithmic randomness. A special case of the latter was independently obtained by Bienvenu, Merkle and Nies [BMN11, Theorem 8]. In Section 1.3 we give some terminology that is adopted in the technical sections of this paper.

1.1. An optimal bound on the complexity of gap functions for K-triviality.

Suppose that we are given a nondecreasing unbounded function $f : \mathbb{N} \to \mathbb{N}$. It seems plausible that given f, one can construct a set X which is not K-trivial but $K(X\restriction_n) \leq K(n) + f(n) + c$ for some constant c and all $n \in \mathbb{N}$. Intuitively, we would try to construct X such that the difference between complexity of the first n bits of it and $K(n)$ increases when f is sufficiently large. Since $\lim_s f(s) = \infty$ one would hope that we can achieve $\limsup(K(X\restriction_n) - K(n)) = \infty$ so that X is not K-trivial.

Surprisingly, this is not the case. This was shown by Csima and Montalbán in [CM06], where an unbounded nondecreasing function f was constructed such that for each set X and any constant c

(1.2) if $K(X\restriction_n) \leq K(n) + f(n) + c$ for all $n \in \mathbb{N}$, then X is K-trivial.

Following Downey and Hirschfeldt [DH10, Section 10.12], an analysis of the proof shows that the function f is Δ^0_4. Barmpalias and Vlek [BV11, Section 5] called such functions 'gap functions for K-triviality' and they showed that they cannot be Δ^0_2. In fact, it was shown that if f is Δ^0_2, unbounded and nondecreasing, then there exists a c.e. set X which is not K-trivial but its initial segment complexity is bounded by $K(n) + f(n) + c$ for some constant c. In Section 3 we use a result from Barmpalias and Sterkenburg [BS11] in order to show that there is a Δ^0_3 unbounded nondecreasing function f satisfying (1.2).

Theorem 1.1. *There exists a Δ^0_3 unbounded nondecreasing function f such that all reals X which satisfy $\exists c\ \forall n\ (K(X\restriction_n) \leq K(n) + f(n) + c)$ are K-trivial.*

The diagonalization employed in the proof of Theorem 1.1 (originally from [CM06]) is particularly interesting since it deals with all possible sequences X via a compactness argument. A discussion for cases where the sequences X are restricted in a certain arithmetical class can be found in Barmpalias and Vlek [BV11, Section

5]. For example, it was shown that for some Δ^0_2 function f with $\lim_n f(n) = \infty$ the property (1.2) holds for all Σ^0_1 sets. On the other hand it was shown that there is no unbounded Δ^0_2 nondecreasing function f such that (1.2) holds for all Σ^0_1 sets. Finally, also in [BV11, Section 5], it was shown that with no restriction on the complexity of f, the property $\lim_n(f(n) - K(n)) = \infty$ implies that there are uncountably many infinite sequences X satisfying the condition occurring in (1.1). In general, one can ask if given a Δ^0_2 unbounded function f one can construct a set X which is not K-trivial but the prefix-free initial segment complexity of it is bounded by $K(n) + f(n) + c$ for some constant c. In this paper we focus on the special case where f is nondecreasing.

1.2. **A large collection of reals with very low initial segment complexity.** We are interested in the collection of reals X which have very low but nontrivial initial segment complexity, in the sense that $K(X \upharpoonright_n) - K(n)$ is bounded by a very slow-growing unbounded nondecreasing function f. In particular, we focus on the case where f is Δ^0_2 since this is the weakest assumption on the arithmetical complexity of it under which this collection of reals is nonempty (by Theorem 1.1).

Let us say that a function g is *right c.e.* if it can be uniformly approximated from above in a computable way. In other words, if there is a computable function of two arguments h such that $h(n, k+1) < h(n, k)$ for all $n, k \in \mathbb{N}$ and $g(n) = \lim_k h(n, k)$ for all $n \in \mathbb{N}$. Bienvenu, Merkle and Nies showed in [BMN11, Theorem 8] that given any right c.e. function g such that $\lim_n(g(n) - K(n)) = \infty$, there exist uncountably many reals X such that $K(X \upharpoonright_n) \leq g(n) + c$ for some constant c and all $n \in \mathbb{N}$. On the other hand it was observed by Barmpalias and Vlek [BV11, Section 5] that every Δ^0_2 unbounded nondecreasing function is bounded from below by a right c.e. unbounded nondecreasing function. If follows that if f is an unbounded Δ^0_2 nondecreasing function then there are uncountably many reals X satisfying (1.1). Independently, as reported by Barmpalias and Vlek [BV11, end of Section 5], we obtained the following version of this result.

(1.3) If f is Δ^0_2, nondecreasing and unbounded then there is a constant c and a perfect non-empty Π^0_1 class P such that (1.1) holds for all $X \in P$.

We note that (1.3) can also be obtained from the proof of [BMN11, Theorem 8] by an easy modification.

The use of Π^0_1 classes in computability theory is connected with the various existence theorems that apply to these classes, the so-called basis theorems. For example, every Π^0_1 class contains a low set X (i.e. such that $\Sigma^0_1(X) \subseteq \Delta^0_2$) and a computably dominated set Y (i.e. such that every function that can be computed by Y is dominated by a computable function). However these are all 'lowness' properties, in the sense that any computable set satisfies them. For this reason, basis theorems are applied to Π^0_1 classes that do not have computable members—the so-called *special* Π^0_1 classes. In this way, the various basis theorems yield nontrivial examples of reals with special properties.

In our study and more general in the theory of initial segment prefix-free complexity, *trivial* means K-trivial. Hence Π^0_1 classes without K-trivial members play the role that special Π^0_1 classes play in classical computability theory. In other words we can use the various basis theorems in order to exhibit reals with certain computational and initial segment complexity lowness properties as long as the Π^0_1 classes that we use do not have K-trivial members (otherwise the reals that we exhibit may satisfy the properties in a trivial and uninteresting way). For example, if we apply the low basis theorem to the class in (1.3), even if we managed to make the class special we would possibly get a K-trivial set satisfying (1.1) which is clearly uninteresting. For this reason we would like to show that the class in (1.3) may be chosen without any K-trivial members.

Theorem 1.2. *If f is Δ^0_2, nondecreasing and unbounded then there is a constant c and a perfect non-empty Π^0_1 class P with no K-trivial members such that the condition in (1.1) holds for all $X \in P$.*

If every perfect nonempty Π^0_1 class contained a nonempty Π^0_1 with no computable members, we would be able to obtain a weaker version of Theorem 1.2 simply from (1.3). However it can be shown using standard methods in computability theory that this is not the case.

(1.4) There is a perfect Π^0_1 class such that every nonempty Π^0_1 subclass of it has computable members.

This can be seen as evidence that the proof of Theorem 1.2 requires a more involved argument.

The difficulty of constructing a Π^0_1 class with no K-trivial members depends heavily on the additional properties that we require. For example in Kučera and Slaman [KS07, Section 2] (perhaps the first paper which showed the importance of such classes) the construction was not much harder than the construction of a special Π^0_1 class. However in Barmpalias, Lewis and Stephan [BLS08, Section 2] the construction of such a class was considerably more challenging in the presence of certain additional requirements. Unfortunately the methods developed in this paper are not sufficiently general to deal with certain constructions of Π^0_1 classes, including the one of Theorem 1.2 and the main construction in Barmpalias [Bar10c]. In Section 4 we give a proof of Theorem 1.2 which uses a more flexible method for ensuring that a Π^0_1 class has no K-trivial members.

In Section 2 we give a number of applications of Theorem 1.2 and its proof. First, we use a number of basis theorems to obtain large collections of reals X in certain known classes that have very low initial segment complexity. Recall that Ω is the halting probability of a universal prefix-free machine. Also, given a Martin-Löf random sequence Y we say that X is low for Y if Y is Martin-Löf random relative to X. We show that given an unbounded Δ^0_2 nondecreasing function f there exist uncountably many low for Ω reals (of nontrivial complexity) which satisfy the property in (1.1). The same holds for non-low for Ω, and computably dominated reals. Second, we observe that our method for ensuring that a Π^0_1 class does not have K-trivial paths is also applicable in Barmpalias [Bar10c] which deals with the structure of the LK degrees. We say that $A \leq_{LK} B$ if there is some constant c such that $K^B(\sigma) \leq K^A(\sigma) + c$ for all strings σ. In other words, if oracle B can compress at least as efficiently as A. In [Bar10c] it was shown that for every Δ^0_2 set B which is not K-trivial, there exist uncountably many sets X such that $X \leq_{LK} B$; in fact, a perfect Π^0_1 class of such sequences. The strategy that was elaborated in Section 4.3 for ensuring that the constructed Π^0_1 class does not have K-trivial paths is applicable in the proof in [Bar10c] without additional complications. Hence

(1.5) If B is Δ^0_2 and not K-trivial then there exists a Π^0_1 class such that all of the members of it X are not K-trivial and satisfy $X \leq_{LK} B$.

The importance of (1.5) lies again on the use of basis theorems. For example, it implies that for every Δ^0_2 set B which is not K-trivial there exists a c.e. set A such that $\emptyset <_{LK} A <_{LK} B$. This was one of the results in Barmpalias [Bar10b]. Moreover it implies (1.6).

(1.6) In the LK degrees, every non-zero LK degree which contains a Δ^0_2 set has uncountably many predecessors, each of them having countably many predecessors.

As we elaborate in Section 2, this corollary is obtained via the use of a low for Ω basis theorem.

1.3. **Terminology.** In Sections 3 and 4 we use the notion of a tree in the Cantor space. This can be defined in the following two different ways:

(i) As a downward $\subseteq$-closed set of strings.

(ii) As a partial map from strings to strings, which preserves compatibility and incompatibility relations.

Perfect trees correspond to total maps in clause (ii). For convenience, in Section 3 we refer to the first formulation while in Section 4 we refer to the latter one. Level n in a tree under (i) is the collection of strings of length n which belong to the tree. On the other hand, if $T : 2^{<\omega} \to 2^{<\omega}$, $\sigma \mapsto T_\sigma$ is a tree under (ii), level n of T refers to the collection of the strings T_τ such that T_τ is defined and τ has length n. If in any level of T the strings have the same length (as will be the case in Section 4), this length is said to be *the height* of this level. Concatenation of strings is denoted by '$*$'.

The *weight* of a prefix-free set S of strings is defined to be the sum $\sum_{\sigma\in S} 2^{-|\sigma|}$. The *weight* of a prefix-free machine M is defined to be the weight of its domain. Prefix-free machines are most often built in terms of *request sets*. A request set L is a set of tuples $\langle \rho, \ell\rangle$ where ρ is a string and ℓ is a positive integer. A 'request' $\langle \rho, \ell\rangle$ represents the intention of describing ρ with a string of length ℓ. The weight of a request $\langle \rho, \ell\rangle$ is $2^{-\ell}$. The weight of a request set is the sum of the weights of the requests that it contains. A request set is a *bounded request set* if the weight of it is bounded by 1. The Kraft-Chaitin theorem (see e.g. Downey and Hirschfeldt [DH10, Section 2.6]) says that for every bounded request set L which is c.e., there exists a prefix-free machine M such that for each $\langle \rho, \ell\rangle \in L$ there exists a string τ of length ℓ with $M(\tau) = \rho$.

2. Applications of Theorem 1.2

First, we give a simple application which can also be derived from Bienvenu, Merkle and Nies [BMN11, Theorem 8]. We are interested in the cardinality of the sequences with initial segment complexity bounded by $c \cdot K(n) + d$, where $c \geq 1$ is a real number and $d \in \mathbb{N}$. By the coding theorem (see e.g. Nies [Nie09, Theorem 2.2.26]) if $c = 1$ and d is any integer there are finitely many such sequences. We show how to use (1.3) in order to show that for any $c > 1$ there exists $d \in \mathbb{N}$ such that there are continuum many sequences with initial segment complexity bounded by $c \cdot K(n) + d$. First we need the following.

Lemma 2.1. *Let $c > 1$ be a real number. There exists an unbounded Δ^0_2 nondecreasing function f such that $K(n) + f(n) \leq c \cdot K(n)$ for all $n \in \mathbb{N}$.*

Proof. Let $q > 0$ be a rational number such that $q+1 < c$. Then $K(n)+q\cdot K(n) \leq c\cdot K(n)$ for each $n \in \mathbb{N}$. Let $f(n)$ be the largest $t \in \mathbb{N}$ such that $t \leq q \cdot K(i)[s]$ for all $i \geq n$ and all $s \in \mathbb{N}$. Clearly f is computable from $\emptyset'$, hence Δ^0_2. Moreover since $K(i)[s]$ is non-increasing in s, for each $n \in \mathbb{N}$ the number $f(n)$ is the largest t such that $t \leq q \cdot K(i)$ for all $i \geq n$. Hence f is nondecreasing. Since $\lim_i K(i) = \infty$ it is also unbounded. By the definition of f we have $K(n) + f(n) \leq K(n) + q \cdot K(n) \leq c \cdot K(n)$ for all $n \in \mathbb{N}$. □

By combining Lemma 2.1 and (1.3) we have the following.

Corollary 2.2. *Let $c > 1$ be a real number. For some $d \in \mathbb{N}$ there exist uncountably many sequences X with initial segment complexity bounded by $c \cdot K(n) + d$.*

Second, we show how the combination of Theorem 1.2 with various basis theorems for Π^0_1 classes yield examples of reals with low nontrivial initial segment complexity with various properties. For example, since every Π^0_1 class with no computable members contains uncountably many computably dominated sets (e.g. by Nies [Nie09, Theorem 1.8.44]), Theorem 1.2 implies the following.

Corollary 2.3. *Given an unbounded nondecreasing Δ^0_2 function f there exists a constant c and uncountably many computably dominated reals X which are not K-trivial but $K(X\restriction_n) \leq K(n) + f(n) + c$ for all n.*

Note that Corollary 2.3 already follows from a weaker version of Theorem 1.2, namely the one where we merely require the class to avoid computable paths. Indeed, this is because if a computably dominated set is not computable, then it is not Δ^0_2 hence it is not K-trivial. This weaker version of Theorem 1.2 is much easier to prove. In view of Lemma 2.1 the following is a special case of Corollary 2.3.

Corollary 2.4. *Let $c > 1$ be a real number. For some $d \in \mathbb{N}$ there exist uncountably many computably dominated reals X with initial segment complexity bounded by $c \cdot K(n) + d$.*

The low basis theorem can be applied in a similar way. Perhaps the following application is worth mentioning.

Corollary 2.5. *Let $c > 1$ be a real number. For some $d \in \mathbb{N}$ there exists a low real X with nontrivial initial segment complexity bounded by $c \cdot K(n) + d$.*

Next, we discuss the information that Theorem 1.2 can give us about the initial segment complexity of the low for Ω sequences. The low for Ω basis theorem says that every non-empty Π^0_1 class has a low for Ω member. It is an easy consequence of compactness and was shown in Reimann and Slaman [RS10] and independently in Downey, Hirschfeldt, Miller and Nies [DHMN05]. We will need the following generalized version of the low for Ω basis theorem.

Theorem 2.6. *Let Z be a set and X be Z-random. Every nonempty $\Pi^0_1(Z)$ class contains a nonempty $\Pi^0_1(Z \oplus X)$ subclass class which consists of low for X sets.*

Proof. Let P be a $\Pi^0_1(Z)$ class and let (U_i) be a universal oracle Martin-Löf test. For each $i \in \mathbb{N}$ let V_i be the set of reals A which are in U_i^Y for all $Y \in P$. Clearly $\mu(V_i) < 2^{-i}$ and by compactness V_i is a $\Sigma^0_1(Z)$ class (uniformly in i). Therefore (V_i) is a Martin-Löf test relative to Z. Since X is random relative to Z, there is some $i_0 \in \mathbb{N}$ such that $X \notin V_{i_0}$. This means that there are paths Y in P such that

$X \notin U_{i_0}^Y$. Let us denote the collection of these sets by Q. Clearly Q is a nonempty $\Pi^0_1(X \oplus Z)$ subclass of P. Since (U_i) was chosen universal, for any path $Y \in Q$ the set X is random relative to Y. □

We are particular interested in the following special case of Theorem 2.6.

Corollary 2.7. *Every nonempty Π^0_1 class contains a nonempty $\Pi^0_1[\emptyset']$ subclass which consists entirely of low for Ω sets.*

Note that if the given Π^0_1 class does not have any K-trivial members then the $\Pi^0_1[\emptyset']$ subclass given by Corollary 2.7 has no Δ^0_2 members. This follows from the fact that a Δ^0_2 low for Ω real is necessarily K-trivial. The latter follows from a number of results in Hirschfeldt, Nies and Stephan [HNS07] and is also presented in Nies [Nie09, Theorem 8.1.18].

Corollary 2.8. *Let g be an unbounded nondecreasing Δ^0_2 function. There exists a constant c and uncountably many low for Ω sequences X such that $K(X\restriction_n) \leq K(n) + g(n) + c$ for all n.*

Proof. Consider the Π^0_1 class of Theorem 1.2 for the given g. Then use Corollary 2.7 to obtain a non-empty subclass P of it, which is $\Pi^0_1[\emptyset']$. Since every Δ^0_2 low for Ω sequence is K-trivial, P does not have Δ^0_2 members. Since it is a $\Pi^0_1[\emptyset']$ class it follows that it is perfect, hence uncountable. □

It is worth stating the following special case of Corollary 2.8.

Corollary 2.9. *Let $c > 1$ be a real number. For some $d \in \mathbb{N}$ there exists a low for Ω real X with nontrivial initial segment complexity bounded by $c \cdot K(n) + d$.*

In the following we show analogous results for sequences that are not low for Ω. We need a basis theorem for Π^0_1 classes that establishes the existence of sequences that are not low for Ω. The following is the first step towards this basis theorem. We say that a countable class $\mathcal{C} \subseteq 2^\omega$ is uniformly $\emptyset'$-computable if it can be presented as $\{\Phi^{\emptyset'}_{f(e)} \mid e \in \mathbb{N}\}$, where f is a computable sequence of indices of total $\emptyset'$-computable functions.

Lemma 2.10. *Let $T : 2^{<\omega} \to 2^{<\omega}$ be a perfect Δ^0_2 tree and let $\mathcal{C} \subseteq 2^\omega$ be a uniformly $\emptyset'$-computable class. Then there is a Δ^0_2 path of T which is not in $\mathcal{C}$.*

Proof. Let $\mathcal{C} = \{\Phi^{\emptyset'}_{f(e)} \mid e \in \mathbb{N}\}$, where f is a computable function and $\Phi^{\emptyset'}_{f(e)}$ is total for every $e \in \mathbb{N}$. Define a path A inductively as follows. If $A\restriction_n = \sigma$ is defined, let $A\restriction_{n+1}$ be $\sigma * i$, where i is chosen such that $T_{\sigma * i} \not\subseteq \Phi^{\emptyset'}_{f(n)}$. Clearly A and $T_A := \cup_{\sigma \subset A} T_\sigma$ are Δ^0_2 and $T_A \neq \Phi^{\emptyset'}_{f(e)}$ for all $e \in \mathbb{N}$. □

Now we can state a strong version of the promised basis theorem. Observe the contrast with the low for Ω basis theorem. The following proof uses the fact that the K-trivial sets form a uniformly $\emptyset'$-computable class. This can be derived from Chaitin [Cha76] but appears more explicitly in Nies [Nie09, Theorem 5.3.28].

Corollary 2.11. *Every perfect Δ^0_2 tree contains a path which is not low for Ω. In particular, there is no perfect Π^0_1 class containing only low for Ω paths.*

Proof. Let $T \subseteq 2^{<\omega}$ be a perfect Δ^0_2 tree. Since the K-trivial sets form a uniformly $\emptyset'$-computable class, we may apply Lemma 2.10 to get a Δ^0_2 path X of T which is not K-trivial. As we discussed above, every Δ^0_2 low for Ω set is K-trivial. Therefore the path X of T is not low for Ω. □

An application of the above result to the class of Theorem 1.2 gives the following.

Corollary 2.12. *Let f be an unbounded nondecreasing Δ^0_2 function. Then there exists a constant c and a sequence X that is not low for Ω and $K(X\upharpoonright_n) \le K(n) + f(n) + c$ for all n.*

Again, we may isolate the following special case, in view of Lemma 2.1.

Corollary 2.13. *Let $c > 1$ be a real number. For some $d \in \mathbb{N}$ there exists a real X which is not low for Ω and has initial segment complexity bounded by $c \cdot K(n) + d$.*

Finally we mention an application to the theory of LK degrees of the method for avoiding K-trivial sequences in a Π^0_1 class that is elaborated in Section 4. The basic framework that we use in the proof of Theorem 1.2 is the same as the one used in Barmpalias [Bar10c] for the proof that every Δ^0_2 set with nontrivial complexity has uncountably many LK predecessors. The atomic strategies in [Bar10c] where considerably more complex than in the argument of Section 4 but the overall convergence of the Π^0_1 classes obey the same rules in both arguments. However in [Bar10c] the class was merely perfect and a question was raised if it can be made without any K-trivial paths.

A straightforward incorporation of the strategies elaborated in Section 4 to the main argument in [Bar10c] gives the desired construction and proves (1.5). Since there are no additional obstacles in this argument, other than the ones that were elaborated in [Bar10c] and in Section 4, we do not present this construction. The value of (1.5) as opposed to the result in [Bar10c] (and the reason why this question was asked) lies again in the application of basis theorems. The additional results that we get is that every Δ^0_2 nontrivial LK-degree bounds a Σ^0_1 nontrivial LK degree (which was proved in Barmpalias [Bar10b] using different methods) and the new result (1.6). The latter is a consequence of the application of Corollary 2.7 to (1.5), in the same way that Corollary 2.8 was obtained from the class of Theorem 1.2. Indeed, by Miller [Mil10] the low for Ω sets have countably many LK predecessors.

3. Proof of Theorem 1.1

3.1. **Preliminary facts.**

In this section we give a basic fact about the K-trivial sets, which is largely a consequence of the work done in Barmpalias and Sterkenburg [BS11]. First, we need the following 'uniformity' lemma.

Lemma 3.1. *Given a $\emptyset'$-computable sequence (T_i) of trees with finitely many paths such that $T_i'' \le_T \emptyset''$ uniformly in i, there is a $\emptyset''$-computable function f such that for each i the number $f(i)$ is a code for a finite set of indices t_j, $j < k_i$ such that there are exactly k_i paths through T_i and these are $\Phi^{\emptyset'}_{t_j}$ for $j < k_i$.*

Proof. Given i we show how to define $f(i)$ computably in $\emptyset''$. First we ask the cardinality k_i of $[T_i]$. This can be decided in $\emptyset''$, see [BS11, Corollary 2.10]. Then we can search for k_i incomparable strings $\sigma_j, j < k_i$ of the same length, such that for each $j < k_i$ the subtree of T_i below σ_j has a unique infinite path. By the definition of k_i such strings exist. Moreover the check amounts to asking for a given string σ if for all levels ℓ above $|\sigma|$ there exists a level $n > \ell$ such that there exists exactly one string of level ℓ which extends σ and has an extension at level n. This is a Π^0_2 question. Hence the condition can be checked computably in T_i''. Since $T_i'' \le_T \emptyset''$ (uniformly in i) the strings can be found computably in $\emptyset''$. Once we determine

σ_j, $j < k_i$ we can effectively obtain the indices $t_j, j < k_j$ as follows. Given $j < k_i$ we let t_j be the program that defines the unique path of T_i extending σ_j. This definition is sound since given a Δ^0_2 tree with a unique path we can effectively get a Δ^0_2 definition of this path from the tree. □

For each $e \in \mathbb{N}$ fix T_e to be the set of strings σ such that $K(\sigma\restriction_i) \leq K(i) + e$ for each $i \leq |\sigma|$. Clearly the trees T_e, $e \in \mathbb{N}$ are uniformly Δ^0_2. Moreover for each $e \in \mathbb{N}$ the set $[T_e]$ consists of the finitely many K-trivial infinite sequences with constant e (i.e. the infinite sequences X such that $K(X\restriction_j) \leq K(j) + e$ for all j). By [BS11, Corollary 3.4] there exists a uniformly c.e. sequence (Q_e) of trees and a constant c such that $[Q_e] = [T_e]$ and Q_e (as a set of strings) is K-trivial with constant $2e + c$ for each $e \in \mathbb{N}$. Moreover, given a constant via which a set Q is K-trivial one can $\emptyset''$-effectively obtain a reduction $Q'' \leq_T \emptyset''$ (see [BS11, Proposition 3.6]). If we combine these facts with Lemma 3.1 we obtain the following.

Proposition 3.2. *There is a $\emptyset''$-computable function f such that for each i the number $f(i)$ is a code for a finite set of indices t_j, $j < k_i$ such that the K-trivial sequences with constant e are exactly the ones given by $\Phi^{\emptyset'}_{t_j}$ for $j < k_i$.*

Using Proposition 3.2 one can revisit the argument of Csima and Montalbán in [CM06] and explicitly make sure that the function f of (1.2) is Δ^0_3. Instead we give a different, more direct presentation of this argument in the following section.

3.2. **Construction of f of Theorem 1.1.** Let us denote by $\mathcal{K}_e$ the class of K-trivial sequences with constant e. In the argument below we freely use the fact that:

(3.1) Given $X \leq_T \emptyset'$ and $e \in \mathbb{N}$ we can $\emptyset''$-computably decide if $X \in \mathcal{K}_e$.

Here the set X is given in the sense of a reduction of it to $\emptyset'$. We define an increasing sequence (n_k) and let $f(t)$ be the least k such that $n_k \geq t$. Given k define $n_k > n_{k-1}$ to be the least number such that for each $e \leq k$:

- For all $X \in \mathcal{K}_{e+k+2} - \mathcal{K}_e$ there exists $i < n_k$ such that $K(X\restriction_i) > K(i) + e$.
- If $k > e + 1$ and for some set X the least number i such that $K(X\restriction_i) > K(i) + e$ is in $[n_{k-2}, n_{k-1})$ then there exists $j < n_k$ such that $K(X\restriction_j) > K(j) + e + k$.

By Proposition 3.2 and (3.1) using $\emptyset''$ we can determine a large enough n_k satisfying the first condition. For the second condition, note that by the previous step (the definition of n_{k-2}) if the least number i such that $K(X\restriction_i) > K(i) + e$ is in $[n_{k-2}, n_{k-1})$ then we have that $X \notin \mathcal{K}_{e+k}$. Hence for each such set, the string $X\restriction_{n_{k-1}}$ is not extendible in (i.e. is not a prefix of an infinite path through) the tree T_{e+k}. Hence by König's lemma there exists a level ℓ in T_{e+k} at which no extendible in T_{e+k} string has $K(\sigma\restriction_i) > K(i) + e$ for $i \in [n_{k-2}, n_{k-1})$. This level ℓ can be calculated using $\emptyset'$ and is lower bound for n_k satisfying the second condition. This concludes the definition of (n_k) and shows that $f \leq_T \emptyset''$.

Now suppose that some set X satisfies $K(X\restriction_n) \leq K(n) + f(n) + e$ for some $e > 1$ and all $n \in \mathbb{N}$. For a contradiction, suppose that X is not K-trivial. So let t be the least number $> n_e$ such that $K(X\restriction_t) > K(t) + e$. Let k be such that $t \in [n_{k-2}, n_{k-1})$. Then by the second condition of the definition of n_k there exists some $j < n_k$ such that $K(X\restriction_j) > K(j) + e + k$. But this contradicts the fact that $K(X\restriction_j) \leq K(j) + f(j) + e$ since $f(j) < k$. This concludes the proof that f satisfies (1.2) and is Δ^0_3.

4. Proof of Theorem 1.2

It was observed in Barmpalias and Vlek [BV11, Section 5] that every Δ_2^0 unbounded nondecreasing function is bounded from below by a right-c.e. unbounded nondecreasing function. Therefore we can assume that the function f of Theorem 1.2 has a computable approximation $f[s]$ such that $n \mapsto f(n)[0]$ is the identity, $f(i)[s] \leq f(j)[s]$ for all $i < j$, and for each s there exists a unique n such that $f(n)[s] \neq f(n)[s+1]$, in which case $f(n)[s+1] = f(n)[s] - 1$.

The parameters $f(n)[s]$ can be viewed as movable markers that can only move from right to left and their initial position is n. Note that at most one marker can move at each stage, and each marker can only move by one position (i.e. decrease its value by 1). We wish to define a perfect Π_1^0 tree T and a prefix-free machine M such that $K_M(X\restriction_n) \leq K(n) + f(n) + c$ for all $n \in \mathbb{N}$, some constant c and all $X \in [T]$. Equivalently, its suffices to ensure that at each stage s

$$K_M(\sigma) \leq K(|\sigma|)[s] + f(|\sigma|)[s] \text{ for all } \sigma \text{ on } T[s] \text{ with } |\sigma| \leq s. \tag{4.1}$$

In Section 4.1 we describe the basic framework of a dynamic construction of a perfect non-empty Π_1^0 class with paths X satisfying (1.1). Although there are more concise arguments that demonstrate this fact, the proof of Theorem 1.2 requires a dynamic approach and Section 4.1 provides an introduction to it. Sections 4.2, 4.3 and 4.4 deal with various aspects of the proof of Theorem 1.2. The formal construction and verification for this theorem occurs in Sections 4.5 and 4.6 respectively.

4.1. **Dynamic construction of the class in (1.3).** We define an effective sequence of 1–1 maps $T[s] : 2^{<\omega} \to 2^{<\omega}$ which preserve the prefix relations of the strings. We denote the image of σ under $T[s]$ by $T_\sigma[s]$. These are essentially uniformly computable perfect trees, and we can consider the set of infinite paths through them:

$$[T[s]] = \{X \mid \forall n \exists \sigma\ (|\sigma| = n\ \wedge\ T_\sigma[s] \supseteq X \restriction n)\}$$

which is a Π_1^0 class. We will also ensure that $[T[s+1]] \subseteq [T[s]]$ for each $s \in \mathbb{N}$ and that $T_\sigma = \lim_s T_\sigma[s]$ exists for each $\sigma \in 2^{<\omega}$. Then the downward closure of the range of the map T is a Π_1^0 tree T and $[T] = \cap_s [T[s]]$ is a perfect Π_1^0 class, where T is the limit map $\sigma \mapsto T_\sigma$. Level n of tree $T[s]$ consists of the nodes (markers) $T_\sigma[s]$ for $\sigma \in 2^n$.

Intuitively, this formal description amounts to starting from a certain map $\sigma \mapsto T_\sigma[0]$ and at each stage $s > 0$ moving the markers $T_\sigma[s-1]$ to possibly new positions (i.e strings) $T_\sigma[s]$. The position of a marker at a given stage is its current value. The movement of the markers T_σ will satisfy the following conditions at each stage.

(i) The map $\sigma \mapsto T_\sigma[0]$ is the identity.

(ii) $\sigma \subset \tau \iff T_\sigma[s] \subset T_\tau[s]$

(iii) Each new position is the position of some marker at the previous stage.

(iv) All nodes of a level of T have the same length.

(v) If some node T_σ moves, then all nodes of the same or larger levels move.

(vi) If a level of T moves at stage s, it moves to a number $\geq s$.

In item (vi), a level of T moves to number n when all nodes on that level move to strings of length n. In order to meet (4.1) it suffices to control the enumeration of

M-descriptions at stage s of the construction by the following clause.

$$\text{For all strings } \sigma \text{ on } T[s] \text{ of length} \leq s \text{ such that } K_M(\sigma)[s-1] > K(|\sigma|)[s] + f(|\sigma|)[s], \text{ request an } M\text{-description of } \sigma \text{ of length } K(|\sigma|)[s] + f(|\sigma|)[s]. \tag{4.2}$$

Let $n_{-1}[s] = 0$ and for each $k \in \mathbb{N}$ let $n_k[s]$ be the least number such that $f(n_k)[s] > 2k$ and $n_k[s] > n_{k-1}[s]$. By the approximation properties of f we have $n_k[0] = 2k+1$. In the non-realistic case that the approximation $f[s]$ to f was constant (hence f was computable), each $n_k[s]$ would also be constant in s and $k \mapsto n_k$ would be computable. In that case it would suffice to let T be any computable tree such that T_σ has length $n_{|\sigma|}$ (in particular, the markers T_σ do not move). Indeed, in that case the weight of the domain of M would be at most

$$\sum_k \sum_{|\sigma|=k} \Big(\sum_{i=|T_{\sigma^-}|}^{|T_\sigma|-1} u_i \cdot 2^{-f(i)} \Big) \leq \sum_k 2^k \Big(\sum_{i=n_{k-1}}^{n_k-1} u_i \cdot 2^{-2k} \Big) \leq \sum_i u_i < 1 \tag{4.3}$$

where σ^- is the predecessor of σ and u_i is the weight of the descriptions of i of the universal machine. Also if σ is the empty sequence then (by convention) T_{σ^-} is the empty sequence. In this ideal scenario, each node T_σ is responsible for the weight of the M-descriptions that are issued for strings between T_{σ^-} and T_σ. By the choice of $k \mapsto n_k$ and since $|T_\sigma| = n_{|\sigma|}$ we have that the weight for which T_σ is responsible is at most $2^{-|\sigma|} q_\sigma$, where q_σ is the weight of the U-descriptions of numbers between $|T_{\sigma^-}|$ and $|T_\sigma|$. Since for each k there are 2^k nodes T_σ with $|\sigma| = k$ it follows that the total weight of all M-descriptions is bounded.

We will modify the above argument to deal with the real possibility that the 'markers' $f(k)[s]$ may move to smaller numbers during the stages s. We will allow the revision of the positions of the markers T_σ which define the tree T as discussed above. The movement of the markers T_σ and the enumeration of M will satisfy the following additional condition.

$$\text{At each stage } s \text{ the height of the } k\text{th level of } T[s] \text{ is} \geq n_k[s] \text{ for } k < s. \tag{4.4}$$

Now the main challenge is to bound the weight of the domain of M. The weight of the requested M-descriptions will be distributed to the markers T_σ as follows. Each T_σ is responsible for

(a) The M-descriptions of strings between the final position of T_{σ^-} and the final position of T_σ.

(b) The M-descriptions of strings ρ which were on $T[s]$ at some stage s, $\rho \supset T_\sigma[s]$ but at $s+1$ level $|\sigma|$ was the least to move, and moved so that ρ is no longer on $T[s+1]$.

The following fact is crucial for the verification of the construction.

Lemma 4.1. *If a computable sequence of maps $\sigma \mapsto T_\sigma[s]$ meets conditions* (i)-(vi) *and condition* (4.4)*, then a prefix-free machine M that is enumerated according to* (4.2) *has bounded domain.*

Proof. Let A_σ contain the M-descriptions under clause (a) above. Note that A_σ is empty unless T_{σ^-} reaches a limit. Also note that the sets A_σ may not be uniformly (in σ) computably enumerable, although $T_\sigma[s]$ is uniformly computable in σ and s. Let B_σ be the c.e. set of M-descriptions that are attributed to T_σ under clause (b). Note that any M-description that is issued must either fall under clause (a), or under clause (b). Hence it suffices to show that the sets $\cup_\sigma A_\sigma$ and $\cup_\sigma B_\sigma$ (where σ ranges over all strings) have bounded weight.

By (4.4), the definition of $k \mapsto n_k[s]$ and the definition of M by (4.2), the weight of $\cup_\sigma A_\sigma$ is bounded. For $\cup_\sigma B_\sigma$ fix a string σ. Let u_i be the weight of the descriptions of i of the universal machine. Let $s_1, s_2, \dots$ be the stages (finitely many or infinitely many) where enumerations into B_σ occurred and $s_0 = 0$. These are typically the stages where level $|\sigma|$ was the least to move. Fix $i > 0$. Following the calculation (4.3) adapted to the snapshot at stage s of the restriction of T on extensions of σ, the weight of the descriptions that where enumerated in B_σ at s_i, $i > 0$ is at most $2^{-2|\sigma|} \cdot \sum_{j=s_{i-1}}^{s_i-1} u_j$. In this calculation we use (4.4), property (vi) and the fact that at stage s only strings of length $\leq s$ may have M-descriptions. Hence

$$\texttt{wgt}(B_\sigma) \leq \sum_i \Big(2^{-2|\sigma|} \cdot \sum_{j=s_{i-1}}^{s_i-1} u_j\Big) \leq 2^{-2|\sigma|} \cdot \sum_i u_i < 2^{-2|\sigma|}.$$

Since there are 2^i strings of length i we have

$$\texttt{wgt}(\cup_\sigma B_\sigma) \leq \sum_i \Big(\sum_{\sigma \in 2^i} \texttt{wgt}(B_\sigma)\Big) \leq \sum_i \Big(\sum_{\sigma \in 2^i} 2^{-2|\sigma|}\Big) \leq \sum_i 2^{-i} \leq 2.$$

Since both $\cup_\sigma A_\sigma$ and $\cup_\sigma B_\sigma$ have bounded weight, so does the domain of M. □

Given the above framework, the proof of (1.3) is straightforward. At stage $s+1$:

(I) if there is some $k \leq s$ such that $n_k[s+1] > n_k[s]$, pick the least one and let t be the least number $> n_k[s+1]$ such that the t-th level of $T[s]$ is $> s$ (i.e. $|T_\rho[s]| > s$ if $|\rho| = t$). Then move level k of T to the current level t as follows. For each σ of length k let $T_\sigma[s+1]$ equal to $T_{\sigma^*}[s]$ where $\sigma^* = \sigma * 0^{t-k}$. Also, let $T_{\sigma*\eta}[s+1] = T_{\sigma^* * \eta}[s]$ for all strings η.

(II) for all strings σ on $T[s+1]$ of length $< s$, if $K_M(\sigma)[s] > K(|\sigma|)[s+1] + f(|\sigma|)[s+1]$ request an M-description of σ of length $K(|\sigma|)[s+1] + f(|\sigma|)[s+1]$.

It is clear that the sequence of maps $\sigma \mapsto T_\sigma[s]$ that we define in the construction meets conditions (i)-(vi). Moreover it satisfies (4.4) and the machine M is enumerated according to (4.2). By Lemma 4.1 the requested M-descriptions have bounded weight. Hence by the Kraft-Chaitin theorem there is a prefix-free machine M that gives descriptions as requested in the construction. Clause (II) of the construction explicitly ensures (4.1). It remains to show that the markers $T_\sigma[s]$ reach a limit. i.e. they are eventually permanently defined. We do this by induction on the levels of the trees $T[s]$. By the hypothesis on f and the definition of $n_k[s]$, for each k the sequence $(n_k[s])_{s\in\mathbb{N}}$ converges. Let t_k be the modulus of convergence of $(n_k[s])_{s\in\mathbb{N}}$. We show that for each $n \in \mathbb{N}$, level n of $T[s]$ reaches a limit with respect to s. By construction level 0 of T will reach a limit by stage t_0. Assume that all levels $< k$ have reached a limit by stage m and $m_\star = \max\{m, t_k\}$. By construction level k of T will reach a limit by stage $m_\star$. This concludes the induction step and the proof of (1.3).

4.2. **Dynamic construction of a Π^0_1 class with no K-trivial members.** A perfect Π^0_1 class is constructed by introducing 'splits' along every path in the class. That is, we make sure that each path splits into two paths at infinitely many lengths. In order to construct a Π^0_1 class which does not contain any K-trivial paths one has to introduce 'clumps' in the tree instead of mere splits. By choosing the 'clumps' large enough, we can be sure that they contain strings of appropriately high Kolmogorov complexity. This is based on the following well known and widely used fact (e.g. see Barmpalias [Bar10a, Theorem 2.6] or Downey and Greenberg [DG08]).

(4.5) There exists a computable function $f(e,n)$ such that for all $e,n \in \mathbb{N}$ and any string σ of length n there exists an extension τ of σ of length $f(e,n)$ such that $K(\tau\restriction_i) > K(i) + e$ for some $i < |\tau|$.

Hence removing the branches of a certain length that have low complexity (a computably enumerable event) leaves us with a non-empty class with the desired property. However this rather crude method is not compatible with the framework of Section 4.1. Indeed, in that argument the nodes T_σ of the constructed Π^0_1 tree T were given as limits of their computable approximations $T_\sigma[s]$. If we tried to implement a strategy based on (4.5) we would have to ask for larger and larger 'clumps' above T_σ, each time this marker moves. Hence the sums in the calculations of the weight of the domain of the machine would no longer be bounded, even if one considers modifications of the function $k \mapsto n_k$.

The solution is a more dynamic approach within the framework of Section 4.1 which is compatible with keeping the size of the 'clumps' above each movable node *constant*. Since we wish to obtain a Π^0_1 class with no K-trivial paths, we will enumerate a c.e. set Q of strings such that

(4.6) $[T]-[Q]$ is non-empty and does not contain any K-trivial sequence

where for a set of strings S we let $[S] = \{X \mid \exists\sigma \in S,\ \sigma \subset X\}$. Since the set of infinite paths $[T]$ of T is a Π^0_1 class, the class $[T]-[Q]$ is also Π^0_1 . Note that one construction will build Q, T simultaneously.

4.3. **Additional requirements and strategy.** To make sure that $[T]-[Q]$ does not contain any K-trivial sequences we add a set of parameters ℓ_e to the framework of Section 4.1 and satisfy the following additional requirements for all $e \in \mathbb{N}$.

(4.7) If a string σ on level ℓ_{e+1} of T has an (infinite) extension in $[T]-[Q]$ then $K(\sigma\restriction_i) > K(i)+e$ for some $i < |\sigma|$.

Intuitively, (4.7) says that by level ℓ_{e+1} all paths of T have been 'revealed' to be not K-trivial with constant e. In order to satisfy these conditions we will need to build an additional prefix-free machine N in order to gain partial control of $K(t)$, $t \in \mathbb{N}$ (i.e. establish certain inequalities of the type $K(t) < m$). By the recursion theorem we may use the index $c > 0$ of N in our construction. Then

(4.8) $K(t) \leq K_N(t) + c$ for all $t \in \mathbb{N}$.

For each e we have a strategy such that all these strategies together will make sure that the requirements of (4.7) are fulfilled. These strategies enumerate strings into Q and requests into N as follows. Let $\ell_0 = 0$ and $\ell_{e+1} = \ell_e + 2e + c + 3$. Thus every string at level ℓ_e of the tree T has 2^{2e+c+3} extensions at level ℓ_{e+1}. Moreover this holds at every stage s. A basic feature of the construction is that only levels ℓ_e can cause changes in the approximation of the tree.

Each strategy for (4.7) works in cycles. A cycle of the strategy corresponding to e will be interrupted upon a movement of a marker-node T_η for $|\eta| \leq \ell_e$. Note that this is an interaction of the strategy with the simpler argument of Section 4.1. Such events may be considered as *injuries* of the strategy. We will make sure that for each strategy for (4.7) they occur finitely often. The strategy corresponding to e is committed to keep the weight of the N-requests that it issues to at most 2^{-e-2}. In order to keep track of the weight that it adds in the domain of N with its requests, the strategy has a parameter $b_e[s]$. We let $b_e[0] = e + 4$. Each time the strategy is injured, the value of b_e increases by 1. Let $t_e[s]$ be the height of level ℓ_e on $T[s]$ plus $2e + c + 3$. Note that the height of level ℓ_{e+1} is $\geq t_e[s]$.

A cycle of the strategy corresponding to e starts at a stage $s + 1$ by ensuring that $K(t_e[s]) < b_e[s] + c$. It does this by enumerating an N-description of $t_e[s]$ of length $b_e[s]$. It continues as follows, as long as the strategy is not injured. If $K(\eta)[t] \leq b_e[s] + c + e$ at some later stage t for some η of length $t_e[s]$ on $T[s]$, it enumerates η into Q unless it is the last extension on $T[s]$ with that length of a node of $T[s]$ of level ℓ_e such that $[\eta] \cap [T[s]] - [Q[s]] \neq \emptyset$. In the latter event the cycle finishes. When the cycle finishes at some stage k, the strategy starts a new cycle by moving the nodes T_σ with $|\sigma| = \ell_e$ to strings on $T[k]$ of length $> k$ that do not have an initial segment in $Q[k]$ (unless $T_\sigma[s]$ already had a prefix in $Q[s]$). This is possible since at each stage s during the cycle, each string of level ℓ_e that does not have a prefix in $Q[s]$ has an infinite extension in $[T[s]] - [Q[s]]$.

(a) The cycle may be interrupted by injury.

(b) The cycle may finish.

(c) The cycle may never finish or be injured.

These are the three possible outcomes for a cycle of a sub-strategy. The weight that we spent in a cycle that started at stage s is $2^{-b_e[s]}$.

4.4. **Calculating the N-weight that is produced by a strategy.** We wish to obtain an upper bound on the weight of the N-requests that a strategy issues in the course of the construction. Every such request is issued during a cycle of the strategy. Moreover exactly one request is issued within a cycle of the strategy. First we consider the requests that were issued in a cycle that was either injured or never finished. In the latter case, note that no more cycles will occur. Since the parameter b_e increases by 1 upon each injury, we have the following bound on the N-weight that is attributed to the cycles that were either injured or never finished.

$$\sum_j 2^{-b_e[0]-j} = \sum_j 2^{-e-4-j} = 2^{-e-3}.$$

For the calculation of the weight of the requests that were issued during a cycle which finished we have to argue in a different way. If such a cycle begins at stage $s+1$, it adds weight $2^{-b_e[s]}$ to the weight of the domain of N and by the end of it at least 2^{2e+c+3} strings of length $t_e[s]$ have U-descriptions of length $\leq b_e[s]+c+e$. In other words, for each such increase on the weight of N we can count an increase in the domain of U which is 2^{e+3} times larger. Since $\mathtt{wgt}(U) < 1$ the total weight of the requests that are issued during such cycles is bounded by 2^{-e-3}. Adding the two kinds of N-weight increases, we get the following.

(4.9) The weight of the N-requests of the strategy corresponding to e is at most 2^{-e-2}.

Hence $\sum_e 2^{-e-2} = 2^{-1}$ is a bound for the total weight of the requests of N that are produced in the construction.

4.5. **Construction.** We use the definitions and conventions of Section 4.1. In order to make a precise application of the recursion theorem, at this point we may view c as an arbitrary parameter of the construction (not necessarily an index of N). Let $b_e[0] = e + 4$ and $t_e[0]$ be the height of level ℓ_e in $T[0]$ plus $2e + c + 3$. At stage $s + 1$, by "move level ℓ_e to level t" we mean the following.

(4.10) For each σ of length ℓ_e, if $T_\sigma[s]$ does not have a prefix in $Q[s+1]$ let ρ_σ be the leftmost extension of σ of length t such that $T_{\rho_\sigma}[s]$ does not have a prefix in $Q[s+1]$; otherwise let it be the leftmost extension of σ of length t. Let $T_\sigma[s+1] = T_{\rho_\sigma}[s]$. Also, let $T_{\sigma*\eta}[s+1] = T_{\rho_\sigma*\eta}[s]$ for all strings η. For all $i > e$ let $b_i[s+1] = b_i[s] + 1$ and for all $i \geq e$ let $t_i[s+1]$ be the height of level ℓ_e in $T[s+1]$ plus $2e + c + 3$.

The strategy corresponding to e *requires attention* at some stage $s + 1$ when there is an extension η of length $t_e[s]$ of $T_\sigma[s]$ for some σ of length ℓ_e with the property $K(\eta)[s] \leq b_e[s] + c + e$ and $\eta \notin Q[s]$.

At even stages $s + 1$ do the following:

(EI) *Ensure* (4.4). If there is some $k \leq s$ such that $n_k[s+1] > n_k[s]$, pick the least one and let t be the least number $\geq n_k[s+1]$ such that $|T_\rho[s]| > s$ for all ρ of length t. Let e be the largest number such that $\ell_e \leq k$. Then move level ℓ_e to level t and set $b_e[s+1] = b_e[s] + 1$.

(EII) *Enumerate requests into M.* For all strings σ on $T[s+1]$ of length $\leq s$, if $K_M(\sigma)[s] > K(|\sigma|)[s+1] + f(|\sigma|)[s+1]$ request an M-description of σ of length $K(|\sigma|)[s+1] + f(|\sigma|)[s+1]$.

At odd stages $s + 1$ do the following

(OI) *Enumerate requests into N.* For all e such that $t_e[s] \leq s$ and no N-description of $t_e[s]$ of length $b_e[s]$ has been enumerated yet, enumerate an N-description of $t_e[s]$ of length $b_e[s]$.

(OII) *Enumerate into $Q[s+1]$.* Let e be the least number such that $t_e[s] \leq s$ for which the strategy corresponding to e requires attention. If no such e exists, end this stage. Otherwise let $Q_\star$ be the union of $Q[s]$ and the strings η of length $t_e[s]$ in $T[s]$ with the property $K(\eta)[s] \leq b_e[s] + c + e$. If for each σ of length ℓ_e

$$\text{either } T_\sigma[s] \text{ has a prefix in } Q_\star \text{ or } [T_\sigma[s]] \cap [T[s]] - [Q_\star] \neq \emptyset$$

let $Q[s+1] = Q_\star$. If not, let P be the set of strings ν of level ℓ_{e+1} that are the leftmost extension of $T_\sigma[s]$ for some σ with $|\sigma| = \ell_e$ such that either $T_\sigma[s]$ has a prefix in $Q_\star$ or $[T[s]] \cap [\nu] - [Q_\star] \neq \emptyset$. Let $Q[s+1] = Q_\star - P$, let t be the least number such that the height of level t is $> s$ and move level ℓ_e to level t.

4.6. **Verification.** A basic feature of the construction is that for all e, all σ of length ℓ_e and all stages s, either $T_\sigma[s]$ has a prefix in $Q[s]$ or $[T_\sigma[s]] \cap [T[s]] - [Q[s]] \neq \emptyset$. This property holds at stage 0 since $Q[0] = \emptyset$ and is preserved inductively throughout the construction via Step OII, Step EI and (4.10). In particular, (4.10) is justified as a way of moving the nodes of the tree (i.e. the conditions that it asks for the new positions of the nodes can be satisfied). As a consequence, since we never enumerate any prefix of $T_\emptyset$ into Q, we have $[T] - [Q] \neq \emptyset$.

Now one may view the construction as a computable function which takes c and returns a program for N (or rather the request set associated with N). By the recursion theorem we may choose c to be an index of N. After these necessary justifications we verify the desired properties of the set nonempty $[T] - [Q]$ in a series of lemmas.

Lemma 4.2. *There is a prefix-free machine M with the specification given in the construction.*

Proof. According to the justification above, the construction defines a computable sequence of maps $\sigma \mapsto T_\sigma[s]$. A simple inspection of the construction shows that this sequence meets conditions (i)-(vi). The same holds for condition (4.4) restricted to the even stages (since only at even stages descriptions are enumerated into M this restriction is allowed). The request set for the prefix-free machine M that is enumerated in the construction follows (4.2). By Lemma 4.1 the requests for M have bounded weight. This shows that the specification of M given in the construction corresponds to an actual prefix-free machine. □

The argument of Section 4.4 applies to the construction and shows that the weight of the requests for N is finite. Hence the following is a consequence of the Kraft-Chaitin theorem.

Lemma 4.3. *There is a prefix-free machine N with the specification given in the construction.*

The markers T_σ may move for two reasons. One is Clause (EI) and the other is Clause (OII) of the construction. The reason that the first type of movement stops is that the approximation to f converges. The second type of movement stops because the additional strategy corresponding to a level can only conclude a certain number of cycles, as we argued in Section 4.4. The proof of the following fact requires the combination of these arguments, in an induction.

Lemma 4.4. *Each movable marker T_σ reaches a limit.*

Proof. If levels move then the least level to move is level ℓ_e for some e, so it suffices to show that all levels ℓ_e reach a limit. Inductively assume that by stage s_0 all markers T_η with $|\eta| < \ell_e$ have reached a final value. We show that all markers T_ρ for $|\rho| = \ell_e$ reach a limit by some later stage. Let $s_1 > s_0$ be a stage after which n_{ℓ_e} remains constant.

After stage s_1 the strategy corresponding to e will not be 'injured'. In terms of the analysis of Section 4.3, each time it moves after stage s_1 it completes a cycle, while b_e remains constant. According to the same analysis, each time it completes a cycle after stage s_1, at least $2^{e+3-b_e[s_1]}$ additional weight can be counted in the domain of U. Alternatively, at least $2^{-b_e[s_1]}$ additional weight can be counted in the domain of N. Since the weight of the domain of a prefix-free machine is < 1, after stage s_1 the marker T_σ can only move at most $2^{b_e[s_1]}$ times. This concludes the induction step. □

Property (4.1) holds by the explicit action of step (EII) of the construction. Hence it remains to show that the additional strategies succeed in eliminating the K-trivial paths from $[T] - [Q]$.

Lemma 4.5. *There are no K-trivial paths in $[T]-[Q]$.*

Proof. It suffices to show that for each $X \in [T]-[Q]$ and for each $e \in \mathbb{N}$ there exists some $i \in \mathbb{N}$ such that $K(X \restriction_i) > K(i)+e$. Let σ be a string of length ℓ_e such that $T_\sigma \subset X$ (where T_σ refers to the final value of $T_\sigma[s]$). Also using Lemma 4.4, let s_0 be a stage after which level ℓ_e does not move. This means that $t_e[s]$ reaches a limit t_e after s_0. If $K(X \restriction_{t_e}) \leq K(t_e)+e$, by the choice of c (as a code for machine N, see (4.8)) we would also have $K(X \restriction_{t_e})[s] \leq b_e[s]+c+e$ at some even stage $s \geq s_0$. At that stage the strategy corresponding to e would require attention. Since $X \restriction_{t_e}$ was not enumerated into Q, according to Step (EII) of the construction $X \restriction_{t_e}$ was the last extension of $T_\sigma[s]$ that was not in Q and thus T_σ would move. This contradicts the choice of stage s_0. This contradiction shows that $K(X \restriction_{t_e}) > K(t_e)+e$. □

This concludes the proof of Theorem 1.2. We conclude this paper with a brief discussion of various constructions of Π^0_1 classes which do not contain K-trivial paths. The most direct way of avoiding the K-trivial members in a Π^0_1 class is to ensure that all members have very high initial segment complexity (e.g. they are random). In some cases, additional requirements on the members of the class require that the initial segment complexity of the members drops significantly at infinitely many segments. In certain situations, e.g. in [KS07, Lemma 2.3] and [BV11, Theorem 2.10], such a conflict can be handled in a rather simple way, which is not much different than the task of avoiding the computable embers in the class (such a strategy is based on (4.5)). However when certain conditions require that the class is 'thin' in some sense (i.e. it does not have 'many' extendible 'clumps') the task of avoiding the K-trivial members becomes less trivial. This is because a simple strategy is to introduce a large 'clump' and (knowing by counting that at least one path through the clump will have sufficiently large complexity) remove each path in the clump which turns out to have low complexity. But such a simple strategy contrasts the requirement that the class is 'thin' in some formal sense.

This was the situation in [BLS08, Theorem 7] as well as the proof of Theorem 1.2 in this paper. In this case the strategy is to work gradually, using clumps of fixed (relatively small) size. As we discussed in the proof of Theorem 1.2, each time we 'lose a clump' the opponent spends a certain fixed amount of weight, so he is forced to stop hence leaving us with a successful clump of paths of sufficiently high complexity. Another application of this method is on the theory of the LK degrees and was discussed in the end of Section 1.2 and the end of Section 2.

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