ALGORITHMIC RANDOMNESS AND MEASURES OF COMPLEXITY

GEORGE BARMPALIAS

Abstract. We survey recent advances on the interface between computability theory and algorithmic randomness, with special attention on measures of relative complexity. We focus on (weak) reducibilities that measure (a) the initial segment complexity of reals and (b) the power of reals to compress strings, when they are used as oracles. The results are put into context and several connections are made with various central issues in modern algorithmic randomness and computability.

Contents

Barmpalias was supported by the *Research fund for international young scientists* number 611501-10168 from the National Natural Science Foundation of China, and an *International Young Scientist Fellowship* number 2010-Y2GB03 from the Chinese Academy of Sciences; partial support was also received from the project *Network Algorithms and Digital Information* number ISCAS2010-01 from the Institute of Software, Chinese Academy of Sciences. The preparation of this article started when the author was a visiting fellow at the Isaac Newton Institute for the Mathematical Sciences, Cambridge U.K., in the programme 'Semantics & Syntax'. The author wishes to thank André Nies and the referees for helpful feedback on previous drafts of this article.

1. Introduction

The study of the continuum from an algorithmic perspective is largely based on *reductionism*, i.e. the idea that *a whole can be understood if we understand its parts, and the relationships between them.* In this respect, a considerable part of the research in this area concerns various notions of reductions amongst reals and the algebraic study of the associated degree structures.

Within this framework, algorithmic information theory (in the tradition of Kolmogorov, Solomonoff, Martin-Löf, Chaitin and Levin) has received a great deal of attention from researchers in computability theory. As a result, a considerable body of research has been produced on the interface of computability theory with algorithmic randomness. Much of this development is documented in the recent monographs [DH10, Nie09]. The introduction of reducibilities and the application of methodologies from computability theory to algorithmic randomness has been a considerable part of this movement; for example see [DHL04]. At the same time, many of the traditional techniques in computability theory proved inadequate to deal with certain problems (see below) and new methods and novel types of arguments were developed, establishing a new area that lies between classical computability theory and algorithmic randomness. Furthermore, certain concepts and results in this area turned out to be very useful for a number of problems in classical computability theory.

In recent years the author has begun a systematic program organising and delineating the various measures of relative randomness. We have found surprising and intricate interactions between the measures and between measures of relative complexity like classical Turing (and other) degrees. In the present article we report on our work and related contemporaneous work, putting the material in a historical framework.

The purpose of this survey is threefold. Firstly, we wish to present a number of recent developments on relative randomness, in the context of the existing work in the more general area of randomness and computability. This presentation aims to give easy access to and an overview of these developments. Second, in the light of these advances we wish to take a step back and reconsider the underlying measures of complexity that form the basis of this work. Are they faithful formalizations to the intuitive notions that they are supposed to represent? Our analysis and comparison of different measures provide a rigorous context in which this question can be formally addressed. For example, we give examples of measures that are supposed to represent the same intuitive notion, yet their theories differ on a very basic level. The critical layer of this survey focuses on the exposition of such anomalies. Third, we suggest research directions in the form of a number of open questions that stem from and are motivated by our discussion.

In order to make the text more readable, many of the results that we discuss are not given in full generality. Moreover the list of citations is not complete; our choice represents the topics and the issues that we wish to highlight. The focus is on measures of complexity in the form of (weak) reducibilities that measure

(a) the initial segment complexity of reals;
(b) the power of reals to compress strings, when they are used as oracles.

The overall goal is to provide a coherent and readily accessible picture of this topic and point to interesting research directions. The most recent accounts on the progress in this area are [MN06, DHNT06] and the above mentioned monographs, each of which featuring a considerable number of open problems. Although the focus of these accounts is broader, the present survey includes an overview of the solution to a number of the problems posed in these publications.

In Section 2 we give a brief overview of the theory of algorithmic randomness. Based on the fundamental notions of complexity and randomness, we can define various measures of relative complexity (these are preorders that partially order the continuum) and develop tools for the classification of the continuum in terms of relative initial segment complexity.

Already in this introductory section the reader can find recent results and research trends, as well as open questions. Section 3 focuses on aspect (a) above, namely the measures of complexity that concern the initial segment complexity of reals. The global and local structures of the degrees of randomness are discussed, as well as special topics like 'reals with very low nontrivial initial segment complexity' and reals that are 'bounded by a random real'. Section 4 focuses on aspect (b) above, namely the measures of complexity that concern the compression power of reals when they are used as oracles. Finally the last section contains a comparison of the various measures of complexity that are discussed in the previous sections, and reveals several crucial differences on measures that purport to formalise the same intuitive notions.

Throughout the text there are displays of statements in-between the main text. There are three types of these displays. Firstly, there are theorems which are written precisely, often with mathematical notation. Secondly, there are theorems which are written in a more informal manner, in plain English. Any ambiguity that may arise from this style of presentation is resolved in the sentence that follows it in the main text. The third type of these displays is the one where the text is enclosed in double quotation marks. These are informal sentences about the complexity of sets which admit more than one interpretation, in terms of the different definitions that we consider for the quantification of the complexity of reals. The precise interpretations of this type of displays are discussed in the main text that follows them.

Some results appear more than once throughout the text, in different contexts. This controlled repetition is desirable since the purpose of this survey is to provide a coherent picture of this research topic.

2. Measures of algorithmic complexity

There were several proposals for a mathematical definition of randomness in the 20th century. These are usually grouped under the three banners *typicalness, compressibility,* and *unpredictability.* We will primarily use the paradigm of *compressibility*, which is due to Kolmogorov. Equivalent approaches, such as the definition of Martin-Löf, will be occasionally discussed. Section 2.1 is a brief introduction to the ideas and language of Kolmogorov complexity that are used in this presentation.

Kolmogorov complexity provides a robust mathematical definition of the initial segment complexity of a string, as a function of the lengths of the initial segments. Given this basic definition, a fruitful way to study the initial segment complexity of a sequence is to compare it with the initial segment complexity of other sequences. Similarly, one may measure the power of an oracle X to compress (i.e. give short descriptions of) programs by comparing the distribution of complexities of programs relative to X with the corresponding distribution relative to other oracles Y.

Measures of relative complexity provide a formal way to do this. Formally, these are preorders (i.e. reflexive and transitive relations) that partially order the continuum. A preorder induces an equivalence relation on the continuum and we often refer to the equivalence classes as 'degrees'; the partially ordered structure of the degrees (according to the original preorder) is often called the 'degree structure' that is induced by the preorder. For example, the Turing degrees were introduced in [KP54] based on the concept of *oracle computation* from [Tur39] and the associated *Turing reducibility*. When the preorder represents a measure of complexity (formalizing the notion that a sequence is more complex than another sequence) we regard the sequences in a single degree as having the same 'amount of complexity'. In the above example of the Turing degrees, this means that two oracles in the same Turing degree contain the same information, coded in perhaps different ways.

When we set out to invent a measure of relative complexity we are confronted with the problem of choosing amongst many appealing alternatives. Different measures have distinctive qualities that may be advantageous in certain situations (e.g. restricted to certain classes of sequences) but not in others. In Section 2.2 (concerning initial segment complexity) and Section 2.3 (concerning oracle power to compress programs) we introduce the reader to a number of different measures that are appealing in some ways and are based on a clear intuitive idea. This is not simply a list of definitions that will be used in the following sections; rather, it is an exploration of the ways that one might proceed for the invention of an appropriate measure of relative complexity.

A large part of the current research in the interface between algorithmic randomness and computability theory today is devoted to the study of classes of reals with very low complexity. Such triviality notions are often obtained by considering the sequences which are 'below' all sequences with respect to some preorder that is related to initial segment complexity. Section 2.4 is devoted to this subtopic which is quite central to our study. In Section 2.5 we elaborate on our programme of comparing various measures of complexity, thus motivating the results presented in the main part of this survey which facilitate these comparisons.

2.1. Algorithmic randomness and complexity.

A standard measure of the complexity of a finite string was introduced by Kolmogorov in [Kol65] (an equivalent approach was due to Solomonoff [Sol64]). The basic idea behind this approach is that simple strings have short descriptions relative to their length while complex or random strings are hard to describe concisely. Kolmogorov (and Solomonoff) formalized this idea using the theory of computation. In this context, Turing machines play the role of our idealized computing devices, and we assume that there are Turing machines capable of simulating any mechanical process which proceeds in a precisely defined and algorithmic manner. Programs can be identified with binary strings.

A string τ is said to be a description of a string σ with respect to a Turing machine M if this machine halts when given program τ and outputs σ. Then the Kolmogorov complexity of σ with respect to M (denoted by $C_M(\sigma)$) is the length of its shortest description with respect to M. It can be shown that there exists an *optimal* machine V, i.e. a machine which gives optimal complexity for all strings, up to a certain constant number of bits. This means that for each Turing machine M there exists a constant c such that $C_V(\sigma) < C_M(\sigma) + c$ for all finite strings σ. Hence the choice of the underlying optimal machine does not change the complexity distribution significantly and the theory of Kolmogorov complexity can be developed without loss of generality, based on a fixed underlying optimal machine U. We let C denote the Kolmogorov complexity with respect to a fixed optimal machine.

When we consider randomness for infinite strings, it becomes important to consider machines whose domain satisfies a certain condition; the machine M is called *prefix-free* if it has a prefix-free domain (which means that no program for which the machine halts and gives output is an initial segment of another). The use of prefix-free machines means that we do not allow the length of a description of a string to be used in the computation that outputs the string. Hence the information of a string is measured solely in terms of the least number of bits which can reproduce it. In this sense we obtain a more refined measure of information. Similar to the case of ordinary Turing machines, there exists an *optimal* prefix-free machine U so that for each prefix-free machine M the complexity of any string with respect to U is up to a constant number of bits larger than the complexity of it with respect to M. We let K denote the prefix-free complexity with respect to a fixed optimal prefix-free machine.

In order to define randomness for infinite sequences, we consider the complexity of all finite initial segments. A finite string σ is said to be *c-incompressible* if $K(\sigma) \geq |\sigma| - c$. Levin [Lev73] and Chaitin [Cha75] defined an infinite binary sequence X to be random (also called 1-random) if there exists some constant c such that all of its initial segments are c-incompressible. By identifying subsets of $\mathbb{N}$ with their characteristic sequence we can also talk about randomness of sets of numbers. Moreover the above definitions and facts relativize to an arbitrary oracle X when the machines that we use have access to this external source of information. For example, in this case we write K^X for the corresponding function of prefix-free complexity. An infinite binary sequence that is random relative to the halting problem $\emptyset'$ is called 2-random (and similarly for the various iterations of the halting problem). Further variations (for example weak 2-randomness, see [Nie09, Section 3.6]) may be obtained by varying the definitions.

This definition of randomness (i.e. 1-randomness) of infinite sequences is independent of the choice of underlying optimal prefix-free machine, and coincides with other definitions of randomness like the definition given by Martin-Löf in [ML66]. The coincidence of the randomness notions resulting from various different approaches may be seen as evidence of a robust and natural theory.

2.2. **Initial segment complexity.** Comparing the complexity of a real with the complexity of other reals is a classical method of measuring complexity in the theory of computation. In the case of initial segment complexity, a number of such measures were introduced by Downey, Hirschfeldt and LaForte in [DHL01, DHL04] (the definitions and results in this section are from this work, unless otherwise stated). Perhaps one of the most straightforward choices for such a measure is $\leq_K$ which is defined as

$$X \leq_K Y \overset{\text{def}}{\iff} \exists c \forall n \, (K(X \restriction_n) \leq K(Y \restriction_n) + c). \tag{2.1}$$

We may express the fact that $X \leq_K Y$ simply by saying that the prefix-free initial segment complexity of X is less than (or equal to) the prefix-free initial segment complexity of Y. The plain complexity version $\leq_C$ of the above relation is defined analogously.

$$X \leq_C Y \overset{\text{def}}{\iff} \exists c \forall n\ (C(X\restriction_n) \leq C(Y\restriction_n) + c). \tag{2.2}$$

Although these relations are preorders (i.e. reflexive and transitive relations), many would argue that they do not constitute 'reducibilities'. Indeed, unlike most traditional relative measures of complexity such as Turing reducibility, they do not have an underlying effective procedure that connects (transforms) one member of the relation to the other. We use the term 'weak reducibilities' to refer to reducibilities that are weaker than Turing reducibility, and note that many of them are measures of complexity with no underlying reduction. The distinction between measures of relative complexity that are reducibilities and those that are not is discussed in more depth in Section 2.5.

The relations defined in (2.1) and (2.2) were already implicit in the widely circulated manuscript of Solovay [Sol75], where the following reducibility (now known as Solovay reducibility) was introduced for the study of Chaitin's halting probabilities of universal prefix-free machines. A real number α is called c.e. (for 'computably enumerable') if the set of rational numbers that are smaller than α is computably enumerable. Given two left c.e. reals α, β we say that $\alpha \leq_S \beta$ (in words, α is Solovay reducible to β) if there is a constant c and a partial computable functions $f : \mathbb{Q} \to \mathbb{Q}$ such that for each rational number $q < \beta$ we have $f(q)\downarrow$, $f(q) < \alpha$ and $\alpha - f(q) < c \cdot (\beta - q)$. Informally, this means that any increasing computable sequence that converges to β can be effectively transformed to an increasing computable sequence that converges to α at least as fast. Solovay reducibility measures the hardness of monotone approximation for c.e. reals (from below) and, as demonstrated in [Sol75] it also provides a way to quantify the randomness of a c.e. real. For example, Solovay showed that $\alpha \leq_S \beta$ implies $\alpha \leq_K \beta$ and $\alpha \leq_C \beta$; moreover the random c.e. reals are greatest elements in the partial order of c.e. reals under $\leq_S$. The associated degree structure is known as the Solovay degrees and has a maximum, consisting of the halting probabilities of prefix-free machines (see [CHKW01, KS01]). This structure has been extensively studied in the literature (for example it is an upper semi-lattice with the usual addition of reals as a join operator, it is dense and has an undecidable first order theory, see [DHN02, DHL07]) and we will not focus on it in this survey. However Solovay reducibility will be discussed in relation with the other measures of complexity that are the main subject of discussion.

Since Solovay reducibility is defined via monotone effective approximations, it cannot serve as a measure of relative complexity for reals that do not have a computable approximation. A proposal for a measure which extends $\leq_S$ and is defined uniformly on all sequences, yet (contrary to $\leq_K$ and $\leq_C$) resembles a reducibility, is the *relative K reducibility* (in symbols, rK). This is defined in terms of relative Kolmogorov complexity. In particular, let $K(\sigma|\tau)$ denote the length of the shortest description of σ, when the underlying universal prefix-free machine uses finite oracle τ (relative plain Kolmogorov complexity is defined similarly).

$$X \leq_{rK} Y \overset{\text{def}}{\iff} \exists c \forall n\ (K(X\restriction_n \mid Y\restriction_n) \leq c).$$

Note that $X \leq_{rK} Y$ can be defined equivalently using plain complexity, by the relation $\exists c \forall n\ (C(X\restriction_n \mid Y\restriction_n) \leq c)$. This follows from the basic relations between plain and prefix-free complexity, namely the fact that there exists a constant d such that $C(\sigma|\tau) \leq K(\sigma|\tau) + d$ and $K(\sigma|\tau) \leq 2C(\sigma|\tau) + d$ for all strings σ, τ. It is often convenient to use the following characterization.

> $X \leq_{rK} Y$ iff there exists a partial computable function $f : 2^{<\omega} \times \mathbb{N} \rightarrow 2^{<\omega}$ and a constant k such that $\forall n \exists j < k\ (f(B\restriction_n, j)\downarrow = A\restriction_n)$.

The precursor of $\leq_{rK}$ was a severe restriction of oracle computation that is now known as computably Lipschitz reduction (in symbols $\leq_{\mathrm{cl}}$). Formally, $A \leq_{\mathrm{cl}} B$ if the first n bits of A can be computed by a machine from the first $n + c$ bits of B (uniformly in n) for some constant c. Although $\leq_{\mathrm{cl}}$ is sensitive to initial segment considerations, it is too restricted for the role of a measure of randomness. However it has been thoroughly studied in [BL06a, BL06c, Day10, ASDFM11] and can be seen as a notion of efficient oracle computation. Moreover it is occasionally relevant in discussions about randomness. For example, a formal counterpart of the following rather surprising fact was obtained in [BL07].

> *If a typical sequence is computed efficiently by another sequence, then the two sequences have the same information.*

The formal version of this statement may be obtained by replacing 'typical' with 1-random, 'computed efficiently by' with '$\leq_{\mathrm{cl}}$-reducible to' and letting 'same information' mean 'in the same Turing degree'. Another example of the relevance of $\leq_{\mathrm{cl}}$ to algorithmic randomness is the fact that, in the class of computably enumerable sets it coincides with $\leq_S$ (see [DHL01, DHL04]). Stephan (see [BDG10]) has shown that there are two random c.e. reals which are not equivalent with respect to $\leq_{\mathrm{cl}}$.

2.3. **Computational strength.** Turing reducibility is the archetypical example of a measure of computational strength. It formalizes the notion that (ignoring the various limitations of computational resources) all the information encoded in one sequence can be recovered from another sequence, in an algorithmic manner. In algorithmic randomness we often need to formalize notions like 'a real A can compress finite programs at least as well as a real B'. If B can compute A then it can simulate any algorithmic procedure that uses A as an oracle. However, as we see in the next sections, the converse does not hold. More generally, an oracle B may be able to perform a class of algorithmic tasks (such as compression of programs) as efficiently as A can, although it is incapable of computing A. These considerations lead to the introduction of weaker measures of computational strength.

In order to formalize the above example we may use K^X, which is the prefix-free complexity function relative to oracle X. A natural way to compare the compression power of oracles was introduced in [Nie05] in the form of the reducibility $\leq_{LK}$.

$$X \leq_{LK} Y \overset{\text{def}}{\iff} \exists c \forall \sigma\ (K^Y(\sigma) \leq K^X(\sigma) + c).$$

In other words $X \leq_{LK} Y$ formalizes the notion that Y can achieve an overall compression of the strings that is at least as good as the compression achieved by X. A related measure is $\leq_{LR}$ which formalizes the notion that every real whose initial segments can be compressed by one of the oracles, can also be compressed by the other oracle. More precisely (and in a contrapositive form), $X \leq_{LR} Y$ if every random (i.e. 'incompressible') sequence relative to Y is also random relative to X. Clearly $X \leq_{LK} Y$ implies $X \leq_{LR} Y$. Surprisingly, the converse also holds, so that the two relations coincide [KHMS12]. Based on this coincidence (for reasons of uniformity) in this presentation we refer to various results that were originally proved for $\leq_{LR}$ in terms of $\leq_{LK}$. In principle, a proof about $\leq_{LR}$ can be routinely 'translated' in terms of $\leq_{LK}$ and vice-versa. One may define analogues of $\leq_{LR}$ based on different notions of randomness. For the case of 'weak 2-randomness' we refer to [BMN12, Section 4].

A remarkable connection between $\leq_{LK}$ and $\leq_K$ was obtained in [MY08].

$$\text{If } X, Y \text{ are 1-random then } X \leq_K Y \Rightarrow Y \leq_{LK} X. \tag{2.3}$$

This result is based on van Lambalgen's theorem, which says that $X \oplus Y$ is 1-random if and only if X is 1-random and Y is 1-random relative to X. In fact, much of the study of $\leq_K$ in [MY08] is based on the use of a related measure of complexity that is called 'van Lambalgen's reducibility' and serves as a connection between randomness and relative computational power. An informal interpretation of (2.3) is the following.

> *Amongst 1-random reals, more randomness implies less ability to derandomize and in some sense, less information.*

Such interactions between randomness and information is one of the main themes in this area.

2.4. **Triviality notions.** As much as we are interested in random sequences, the other end of the complexity spectrum has turned out to be equally interesting and an integral part of the study of high complexity. Computational weakness refers to the least 'degree' of complexity and a number of precise expressions of it may be obtained by considering the sequences that are 'reducible' (or 'weakly reducible') to every other sequence with respect to the measures that we discussed in Sections 2.2 and 2.3. These classes of computationally weak sequences are supersets of the computable sequences, and an interesting feature is that in many cases they are proper supersets. For example, consider the sequences in the least K-degree. These are the sequences X which have the least possible initial segment prefix-free complexity, i.e. $\exists c \forall n \; K(X \upharpoonright_n) \leq K(n) + c$. Here $K(n)$ denotes the prefix-free complexity of the string 0^n. In other words, these sequences are as simple as the infinite sequence of 0s; they are called K-trivial sequences and Solovay [Sol75] showed that they can be noncomputable. On the other hand in [DHNS03] there where shown to be Turing-incomplete, hence providing an arguably natural solution to Post's problem (which asks for incomplete non-zero c.e. Turing degrees). In contrast, for the case of plain Kolmogorov complexity Chaitin [Cha76] showed that any set with trivial initial segment complexity must be computable.

Another class of computationally weak sequences consists of the sequences in the lowest LK degree. These are the sequences Y such that K^Y and K are equal, modulo a constant number of bits. In other words, if they are used as an oracle they do not improve the compression of strings. In [Nie05] it was shown that a sequence is low for K if and only if it is K-trivial, i.e.

(2.4) *The class of oracles that do not improve the compression of strings is equal to the class of oracles with trivial initial segment prefix-free complexity.*

This is a surprising coincidence between easily describable and computationally weak sequences. This seminal result provided the first alternative characterization of the K-trivial sequences and motivated a considerable volume of research on 'lowness classes'. A number of different characterizations of the K-trivial sequences have been obtained since. This can be seen as evidence that this class is robust and plays an important role in computability and randomness. The following characterization is from [BL11b]. If A is a c.e. set then $\{X \mid X \leq_K A\} \subseteq \Delta^0_2$ (see [BV11]). It turns out that A is K-trivial if and only if the above class is *uniformly* Δ^0_2.

Theorem 2.1. *A c.e. set B is K-trivial if and only if the family of sets of lesser prefix-free complexity is uniformly computable from $\mathbf{0}'$.*

One direction in this equivalence is the fact that the class of K-trivial sets is uniformly Δ^0_2. This a highly nontrivial consequence of the main result from [Nie05] that we discussed above. The other direction is a dual argument that shows that for each c.e. set of nontrivial initial segment complexity, the class of sets of lesser complexity is 'effectively large' (but countable).

An analogous characterization was obtained in [Bar10c] with respect to a measure of computational strength from Section 2.2.

(2.5) *A Δ^0_2 set A is K-trivial if and only if $\{X \mid X \leq_{LK} A\}$ is countable.*

Moreover, if $A \in \Delta^0_2$ is not K-trivial, then the class $\{X \mid X \leq_{LK} A\}$ is 'effectively uncountable' in the sense that it contains the paths through a perfect computable tree. With some additional effort, this tree can be chosen so that it does not have K-trivial paths. Such a stronger version of (2.5) can be used to show that $\{X \mid X \leq_{LK} A\}$ contains reals from many well known classes from computability theory, provided that A is not K-trivial. This is done through the application of basis theorems from computability theory (see [BB12] for more details).

More triviality notions may be obtained by considering various 'lowness notions'. For example, a set X is *low for random* if every random real is also random relative to X (in symbols, if $X \leq_{LR} \emptyset$). Moreover Y is 'low for K' if the function $\sigma \mapsto K(\sigma)$ is equal to $\sigma \mapsto K^Y(\sigma)$ within a constant (in symbols, $Y \leq_{LK} \emptyset$). By [Nie05] both of these notions coincide with K-triviality. We may relax these conditions in order to obtain larger classes. Chaitin's Ω is a certain 1-random real. A set X is 'low for Ω' if Chaitin's Ω is random relative to X. This relaxed version of 'low for random' was introduced in [NST05], where it was shown that it is different from the original notion.

A relaxation of 'low for K' is obtained by requiring that $\liminf_n(K(\sigma) - K^Y(\sigma))$ is finite, instead of $\limsup_n(K(\sigma) - K^Y(\sigma)) < \infty$. These sets are called 'weakly low for K'. Miller [Mil10] showed that a set is low for Ω if and only if it is weakly low for K. He also showed that this class has measure 1. The characterization (2.5) can be generalized to all sets if we consider low for K reals, which form a superclass of the K-trivial reals.

Theorem 2.2. *Given a set A, the class $\{X \mid X \leq_{LK} A\}$ is countable if and only if $\liminf_n \big(K(\sigma) - K^Y(\sigma)\big) < \infty$.*

This result is from [BL11a] and provided an answer to a question from [Mil10] (which also appeared in [Nie09, Problem 8.1.13]).

We may also attempt a relaxation of K-triviality by requiring (in analogy with the 'weakly low for K') that $\liminf_n(K(X \restriction_n) - K(n)) < \infty$. The reals X which satisfy this condition are called 'infinitely often K-trivial' and have proved very useful in the study of the K-degrees. In [BV11] it was shown that all (weakly) 1-generic and all c.e. sets are infinitely often K-trivial. Moreover such reals exist in every truth-table degree, although they are contained in a null class. Similar considerations with respect to plain complexity define the 'infinitely often C-trivial' reals and the above observations hold in this case too. As we discuss in Section 3.5, these notions are likely to characterise the degrees with the countable predecessor property in the degrees of randomness. Curiously enough, the only classes of reals that we know are infinitely often K-trivial are also infinitely often C-trivial. This observation motivates the following open question.

Question 1. *Are the classes of infinitely often K-trivial reals and infinitely often C-trivial reals different?*

Another related result from [BV11] is the following disjunction. A set X is called 'complex' if its initial segment complexity is bounded from below by a computable order (i.e. a nondecreasing and unbounded function). It is not hard to see that this notion is invariant under the version of initial segment complexity that is used (plain or prefix-free).

Every real is either complex or infinitely often K-trivial, or both.

Moreover the same statement holds in terms of plain complexity.

The class of K-trivial sequences is far from trivial and, in fact, has very rich structure. There are several more ways one can reveal the complexities of this class. From the point of view of classical computability theory, the study of the ideal of the K-trivial sequences in the Turing degrees has attracted considerable attention. A number of results about the upper bounds of this ideal where established in [KS09, BN11, BD12], in response to [MN06, Questions 4.2 and 4.3]. The study of the quotient structure of the c.e. Turing degrees modulo the K-trivial degrees is also of interest. Intuitively, it gives information about the degrees of unsolvability of c.e. sets when K-trivial information is available 'for free'. The following is a direct consequence of [BD12].

Theorem 2.3. *The quotient upper semi-lattice of the c.e. Turing degrees modulo the K-trivial degrees has no minimal pairs.*

We do not know much more about this structure; for example, the following basic question is open.

Question 2. *Is the quotient upper semi-lattice of the c.e. Turing degrees modulo the K-trivial degrees dense?*

We may also explore the complexity of it as a Σ^0_3 class. Given a constant c, by the coding theorem (see [Nie09, Theorem 2.2.26] for a modern presentation) there are only finitely many infinite binary sequences X that are K-trivial with constant c (i.e. $\forall n\ K(X\upharpoonright_n) \leq K(n) + c$). If we denote the latter finite class by $\mathcal{K}_c$, the class of the K-trivial sequences is stratified in the cumulative hierarchy of the finite classes $\mathcal{K}_i$, $i \in \mathbb{N}$. The function $c \mapsto |\mathcal{K}_c|$ giving the sizes of the classes in the hierarchy is clearly Δ^0_4 and (it is not hard to show that it is) not Δ^0_2. In computability theory it is common for sets to have the maximum complexity not explicitly ruled out by their definition or their construction (for example a Σ^0_1 set is likely to be Σ^0_1-complete unless it is obviously computable).[1] Rather surprisingly, the function $c \mapsto |\mathcal{K}_c|$ is actually considerably simpler than it looks: it is Δ^0_3.

(2.6) *Given input c, the number of reals with prefix-free complexity bounded by $K(n) + c$ can be computed by $\mathbf{0}''$.*

---

[1] A specific version of this for constructions of c.e. sets is discussed in [JS72] (also see [Soa87, Exercise V.5.6]) under the name 'maximum degree principle'.

This result from [BS11] provided an answer to a question from [Nie09, Problem 5.2.16] and [DH10, Section 10.1.4]. The reasons for this unexpected complexity reduction are rather deep and highly related to the fact that K-trivial sequences cannot be Turing complete. The incompleteness of the K-trivial sequences provides an arguably natural solution to Post's problem (see [DH10, Section 11.1.2]) and a further contrast to the 'maximum degree principle' of [JS72].

Although (2.6) gives the exact arithmetical complexity of the function $c \mapsto |\mathcal{K}_c|$, we do not know how powerful this function is when it is used as an oracle.

Question 3. *Does the function $c \mapsto |\mathcal{K}_c|$ compute $\mathbf{0}'$ or even $\mathbf{0}''$?*

We note that the answer to this question may depend on the choice of the underlying universal machine.

2.5. **Comparison of different measures of complexity.** We have introduced a number of different measures of complexity. In the following sections we seek to provide an in-depth critical review of these measures, and assess their effectiveness in faithfully representing the intuitive notions upon which they were defined. We do this by studying their properties, and by making comparisons amongst them.

A very basic task that authors often perform upon the introduction of a new measure of complexity is to separate it from previously known measures (or, in some cases, show that it coincides with a known measure). This is similar to separating complexity classes or randomness notions and it often amounts to using a technical argument (in our case, a diagonalization or a priority argument) for the construction of special purpose sequences that demonstrate the separation (e.g. see [DHL01, DHL04] and [MS07] for such separations concerning a number of the measures that we introduced).

In our analysis we will not be concerned with such 'artificial' separations. Instead, we seek to expose essential differences between the various measures, like simple order-theoretic properties that one may satisfy while others may not. Such differences are especially interesting in the case where two measures purport to be formalizations of the same notions. For example, in comparing the initial segment complexity of sequences one may choose to use the plain complexity $\leq_C$ measure or the prefix-free complexity measure $\leq_K$. It is rather easy to produce artificial (i.e. special-purpose) examples X, Y such that $X \leq_K Y$ but $X \not\leq_C Y$. However such local differences do not reveal any intrinsic difference between the two measures. In contrast, consider the following statement.

> "In the c.e. degrees of randomness, every pair of non-trivial
> degrees has a non-trivial lower bound."

This is known to hold for $\leq_K$ and is known not to hold for $\leq_C$ (where non-trivial means 'not in the lowest degree'). Not only is this a definable 2-quantifier statement in the $\leq_K$ and $\leq_C$ degree structures of the c.e. sets but it is a natural algebraic property that is often considered in the study of partial orders. Yet the two models based on $\leq_K$ and $\leq_C$ give a different answer (despite the fact that they both purport to model the structure of relative initial segment complexity amongst reals). The following is a property of a different kind, but serves the same purpose in the comparison of $\leq_K$ and $\leq_C$. A splitting of a c.e. set A is a pair of disjoint c.e. sets B, C such that $B \cup C = A$.

> "Every c.e. set can be split into two disjoint c.e. sets of the same degree."

This property is known to hold for $\leq_K$ but fails for $\leq_C$ (and, in fact, for all other measures considered in this survey). Our analysis of measures of complexity focuses on such intrinsic properties that provide information about their nature. Sections 3 and 4 discuss various results about relative initial segment complexity and compressing power respectively. Section 5 provides detailed comparisons of different measures of the type we indicated above, based on the results that are presented in the previous sections.

Turing reducibility (and its variations) is the archetypical reducibility in computability theory. The theory of the Turing degrees is more developed than any theory of degrees related to algorithmic complexity. Hence when we are confronted with a problem regarding one of the newer measures of complexity that we introduced, we often attempt to adapt a method that works in the Turing degrees to the new preorder. After all, the preorders of Sections 2.2 and 2.3 are all Σ^0_3, as $\leq_T$ is. Sometimes this approach succeeds. The one theorem (and associated method of proof) about $\leq_T$ that applies successfully and uniformly to $\leq_r$ for $r \in \{ibT, \mathrm{cl}, S, rK, K, C\}$ and other related measures of complexity is the following result from [Sac63], which is known in the literature as the Sacks splitting theorem.

> "Every c.e. set of nonzero degree **a** can be split into two disjoint c.e. sets of strictly lesser incomparable degrees which have least upper bound **a**."

The proof for the various reducibilities is a direct adaptation of the original argument. See [Ste11, Chapter 2], [Bar11a, Section 5] (and [BHLM13] for a generalized version).[2] A similar splitting theorem also holds for $\leq_{LK}$ (see Section 4.2).

In some cases (e.g. c.e. splitting inside a degree) Turing degree methods and the associated results cannot be transferred to the above measures of complexity. As it is discussed in [BV11], this can often be explained by the fact that Turing reducibility can be characterized in terms of arithmetical definability, and a good number of Turing degree techniques are based on this special property. On the other hand, in some cases such a transfer is possible but requires additional effort. In such cases that concern weak reducibilities, an often fruitful methodology is to exhibit parts of the structures where the link with definability survives to some extend. This was demonstrated in [BV11] (where the lower cones below infinitely often K-trivial sets were used for $\leq_K$) and [Bar10a] (where the lower cones below the weakly low for K sets were used for $\leq_{LK}$).

3. INITIAL SEGMENT COMPLEXITY OF INFINITE SEQUENCES

The oscillations of the initial segment complexity of a real are rather unpredictable and often hard to control. As an illustration, consider the following example from [CM06]. There exists an order (i.e. unbounded nondecreasing function) g such that no real can be constructed with $K(X\upharpoonright_n)$ restricted in the interval $(K(n), K(n)+g(n))$ unless $K(X\upharpoonright_n)$ is $K(n)$ (modulo a constant). In other words,

---

[2] Some care is needed in the details of this adaptation since for example, with respect to these reducibilities the degree of $B \oplus C$ is not always the least upper bound of the degrees of B and C. However it is not hard to verify that if B, C is a splitting of a c.e. set A then the degree of A is the least upper bound of the degrees of B, C.

although we are allowed an eventually unbounded number of extra bits of complexity (namely $g(n)$ at length n) it is not possible to use them in increasing the complexity of $X\upharpoonright_n$.[3]

Another case of interest is the oscillations of $K(A\upharpoonright_n)$ in $(K(n), 4\log n)$ when A is a c.e. set. In this case $K(A\upharpoonright_n)$ has to drop to $K(n)$ infinitely often. If A is not K-trivial then these 'dips' of complexity happen on lengths n of high Kolmogorov complexity (hence, at unpredictable lengths); moreover this property is shared by a considerable number of other classes of reals from computability theory like the generics (see Section 3.3).

3.1. **Oscillations of initial segment complexity of random reals.**

In the case of random reals X, the complexity $K(X\upharpoonright_n)$ oscillates between n and $n+K(n)$. Van Lambalgen envisioned these oscillations as a way to quantify randomness.

> *"Although this oscillatory behaviour is usually considered to be a nasty feature, we believe that it illustrates one of the great advantages of complexity: the possibility to study degrees of randomness."* [vL87]

This suggestion was followed up by a number of authors, giving concrete results which show that the properties of initial segment complexity oscillations of random reals often indicate how random the real is. For example, the following characterisation was established in [Mil10],

Theorem 3.1. *A set X is random relative to $\emptyset'$ if and only if there is a constant c such that $K(X\upharpoonright_n)$ is larger than $n+K(n)-c$ for infinitely many n.*

Moreover a corresponding statement was obtained for plain complexity in [NST05, Mil04]: X is random relative to $\emptyset'$ if and only if $C(X\upharpoonright_n)$ is larger than $n-c$ for some constant c and infinitely many n.

The study of oscillations of the initial segment complexity of reals also gives results about the degrees of randomness; this was demonstrated in [MY10]. We give some examples of the oscillation properties which can be used in order to derive various basic structural properties of the K-degrees of random reals (see Section 3.5). The ample excess lemma from [MY08] says that if X is 1-random then $K(X\upharpoonright_n)-n$ grows fast, in the sense that $\sum_n 2^{n-K(X\upharpoonright_n)}$ is finite (the converse is obvious). In particular $K(X\upharpoonright_n)-n$ tends to infinity as $n\to\infty$ (an older result by Chaitin). Given these facts, the following open question comes into focus.

Question 4. *Are there 1-random X, Y such that $\liminf_n \big(K(Y\upharpoonright_n)-K(X\upharpoonright_n)\big)$ is finite but $X<_K Y$?*

By [LV97, Exercise 3.6.3(a)], if $\sum_n 2^{-f(n)}=\infty$ for a computable function f then for each real X we have $K(X\upharpoonright_n)<n+K(n)-f(n)$ for infinitely many n. In the same fashion but somewhat more generally, the upward oscillations in the complexities of almost all reals are described in the following result from [MY10].

> *If $\sum_n 2^{-g(n)}<\infty$ for some function g, then for almost all reals X there exist infinitely many n such that $K(X\upharpoonright_n)<n+g(n)$.*

Some reals have rather high initial segment complexity without being 'random'. The following result from [BD09] illustrates an instance of this phenomenon.

---

[3] This is because these extra bits are given very 'slowly'. Such orders g are not definable in arithmetic by formulas with less than two quantifiers; see Section 3.3.

Theorem 3.2. *If h is any function that tends to infinity, then there exists a set X which is not 1-random and $\exists c \forall n\ K(X \upharpoonright_n) \geq n - h(n) - c$.*

On the other hand some 1-random reals may have certain ‘dips’ in their initial segment complexity, as the following result from [MY10] illustrates.

> *Given any function h which tends to infinity, there exists a 1-random real X such that $K(X \upharpoonright_n) < n + h(n)$ for infinitely many n.*

For more results of this kind on downward and upward prefix-free complexity oscillations of reals we refer the reader to the citations of this section.

3.2. **Initial segment complexity of c.e. and Δ^0_2 sets.** If A is a c.e. set then $C(A \upharpoonright_n)$ oscillates between $C(n)$ and $2\log n$, ‘hitting’ the lower bound $C(n)$ infinitely often (the latter is an observation from [HKM09]). In particular, since $C(n)$ is bounded by $\log n$, there is no c.e. set A such that $C(A \upharpoonright_n)$ is always above $2\log n$ (this was originally observed in [Sol75]). Kummer [Kum96] showed that the initial segment complexity of certain c.e. sets achieves the upper bound $2\log n$ infinitely often. In fact, he illustrated the following ‘gap phenomenon’. Given any c.e. degree $\mathbf{a}$, then either there is $A \in \mathbf{a}$ such that $C(A \upharpoonright_n) \geq 2\log n - c$ for some c and infinitely many n, or for all $A \in \mathbf{a}$ and all orders f we have $C(A \upharpoonright_n) \leq \log n + f(n) + d$ for some d and all n. Informally,

> *either all sets in* $\mathbf{a}$ *have initial segment complexity asymptotically below* $\log n$
> *or some set in* $\mathbf{a}$ *has maximal complexity (i.e.* $2\log n$*) infinitely often.*

According to this analysis, the ‘complicated’ c.e. sets are the ones whose initial segment complexity ‘hits’ the upper bound $2\log n$ infinitely often. A stronger hardness property, which can be realized in the class of the c.e. sets, was introduced and studied in [KHMS06, KHMS11]. They called a set A *complex* if $C(A \upharpoonright_n) \geq f(n)$ for some computable order f.[4] Moreover they showed that a c.e. set A is complex if and only if $\emptyset' \leq_{\mathrm{wtt}} A$.

However none of these complexity properties indicate a completeness phenomenon regarding the initial segment complexity of c.e. sets. For example, some c.e. sets may achieve the upper bound $2\log n$ infinitely often, but they may do so at different lengths. There is no indication as to whether there are c.e. sets whose initial segment complexity bounds the complexity of any other c.e. set. Quite surprisingly (in view of the previous discussion) such complete c.e. sets were discovered in [BHLM13].

Theorem 3.3. *There exists a c.e. set A such that for every c.e. set W, there exists a constant c such that $\forall n\ C(W \upharpoonright_n | A \upharpoonright_n) \leq c$.*

According to the discussion in Section 2.2, this fact also holds with respect to prefix-free complexity. Moreover it implies that the plain or prefix-free complexity of the set A dominates (modulo a constant) the plain or prefix-free complexity of any c.e. set, respectively. This latter property suggests an analogy with the Chaitin Ω numbers. Indeed, the halting probabilities of universal prefix-free machines can be characterized as the c.e. reals with maximum initial segment complexity amongst the c.e. reals (and with respect to a variety of measures such as $\leq_S$, $\leq_C$, $\leq_K$); this

---

[4] It is not hard to see that this is equivalent to the property that $K(A \upharpoonright_n) \geq f(n)$ for some computable order f.

result was obtained cumulatively in [Sol75, KS01, CHKW01] (see [DH10, Section 9.2] for an integrated and simplified presentation).

Clearly, the c.e. sets with maximum initial segment complexity (amongst the c.e. sets) are complex (in the sense of [KHMS06, KHMS11]) but the converse does not hold (see [BHLM13]). It appears that this class of maximally complicated c.e. sets is new in computability theory. A natural example of a c.e. set with this property was recently discovered by Barmpalias and Zhenhao Li. This is the well-known set of nonrandom strings (i.e. set of strings σ such that $C(\sigma) < |\sigma|$) which was first introduced and studied by Kolmogorov.

The above results already indicate that translating results about complexity oscillations into structural properties in related degree structures is a fruitful approach also in the case of the c.e. and the Δ^0_2 sets. Another example supporting this claim is the use of the fact that c.e. sets have infinitely often trivial initial segment complexity in [BV11] in order to produce minimal pairs in the K-degrees of rather low arithmetical complexity (improving on results from [CM06, MS07] and using a somewhat simpler argument). This is also an example of the use of arithmetical definability in order to transfer methods from the Turing degrees to structures based on weak reducibilities. We elaborate on this method in Sections 3.6 and 4.2.

It is also interesting to compare the initial complexity oscillations of reals in different arithmetical complexity classes. For example, c.e. reals can have extremely low initial segment complexity which remains nevertheless nontrivial. There are several facts that illustrate this claim. For instance, given any Δ^0_2 order g, there exists a c.e. set A which is not K-trivial but $K(A\upharpoonright_n) \leq K(n) + g(n)$ holds for almost all n [BV11, Theorem 5.2]. Moreover given any two c.e. sets B_0, B_1 which are not K-trivial, there exists a c.e. set A which is not K-trivial such that $A \leq_K B_i$ for $i = 0, 1$ [Bar11b]. Despite these results, there is a Δ^0_2 real which is not K-trivial and the oscillations in its initial segment complexity do not permit a c.e. set of lesser and nontrivial initial segment complexity [BV11, Theorem 3.5].

(3.1) *There exists a nontrivial Δ^0_2 set X whose initial segment prefix-free complexity does not bound the initial segment prefix-free complexity of any nontrivial c.e. set.*

In view of the above facts about the initial segment complexity of the c.e. sets, (3.1) is rather surprising. Moreover its proof requires considerable effort (an infinite injury argument) compared to the corresponding statements in other degree structures like the C-degrees or the Turing degrees (where it is a rather simple finite injury argument). A basic study of the initial segment complexity of reals in all levels of the arithmetical hierarchy may be found in [BV11].

There are several other aspects that one can investigate concerning the oscillations of the initial segment complexity of reals that are possible. For example, the following question is open.

Question 5. *Is there a pair of sets X, Y which are not K-trivial and a constant c such that $\forall n \min\{K(X\upharpoonright_n), K(Y\upharpoonright_n)\} \leq K(n) + c$?*

Clearly, a pair of sets X, Y that meet the condition in Question 5 is a minimal pair in the K-degrees. On the other hand, minimal pairs in the K-degrees were constructed in [CM06, MS07, BV11] without requiring this strong property. In

[Bar11b] it was shown that the sets X, Y that are required in Question 5 cannot be c.e. (or even c.e. reals).

Theorem 3.4. *Let $A_i, i < 2$ be c.e. sets (or c.e. reals) and not K-trivial. Then $\forall c \exists n\ \forall i < 2,\ K(A_i \restriction_n) > K(n) + c$.*

In plain words, this result says that in the world of c.e. reals (or c.e. sets) if the initial segment prefix-free complexity of each of two sequences raises above the trivial complexity $K(n)$ by an unbounded number of bits, then this must happen at certain lengths n *simultaneously* for the two sequences. In some sense, this statement may be interpreted informally as follows.

(3.2) "Left c.e. reals with nontrivial initial segment complexity have some sort of common information, or at least complexity."

Intuitively, this contrasts the existence of minimal pairs of c.e. reals in various degree structures that calibrate the complexity of sequences, like the Turing degrees, the Solovay degrees and the C-degrees. Indeed, the existence of minimal pairs with respect to a measure of complexity expresses formally the fact that the reals in the pair have no common information with respect to the given measure. Hence it is not surprising that Theorem 4.2 can be used in order to establish that there are no minimal pairs in the K-degrees of c.e. reals (or c.e. sets); this was demonstrated in [Bar11b]. Another formal expression of (3.2) (also derived from Theorem 4.2) is the lack of minimal pairs in the quotient structure of the c.e. Turing degrees modulo the K-trivial sets (see the brief discussion in Section 2.4). A third formal expression of (3.2) (which, however, was established without the use of Theorem 4.2) is the lack of minimal pairs in the LK-degrees of c.e. reals (or c.e. sets, or even Δ^0_2 sets); see Section 4.2 and [Bar10b].

3.3. **Sequences of very low but nontrivial initial segment complexity.** Some sequences have very low but non-trivial initial segment complexity. For example, the prefix-free complexity of X may be bounded by $K(n) + f(n)$ for all computable *orders* (i.e. nondecreasing unbounded functions) f and almost all n, but not bounded by $K(n)$. The sequences with the former property were called *ultracompressible* in [LL99] (where it was shown that they can be somewhat random, namely 'computably random'). The following fact from [BMN11, BB12] refers to an even more stringent upper bound on the initial segment prefix-free complexity of a sequence.

(3.3) *If g is a Δ^0_2 order then there exist uncountably many reals of complexity upper bounded by $K(n) + g(n)$.*

Note that since the K-trivial sequences form a countable class, the class in (3.3) contains non-trivial sequences. In fact, it is not hard to show that there are Turing-complete c.e. sets in this class (see [BV11, Section 5]). Furthermore, this class may be chosen to be effectively closed and without K-trivial members. This stronger result can be combined with basis theorems in order to establish the existence of reals with this property in many well known classes from computability theory (see [BB12] for details).

Can we require an even more stringent upper bound on the complexity without collapsing the class of reals satisfying this bound to the K-trivial reals? There is more than one answer to this question. One way to impose a lower complexity bound is to increase the complexity of the orders that we allow in (3.3). Indeed, for example, there are Δ^0_3 orders that grow more slowly than any Δ^0_2 order. Surprisingly, this route leads to a collapse to the class of the reals with trivial complexity.

> *There exists a* Δ^0_3 *order* g *such that any real with prefix-free complexity bounded by* $K(n) + g(n)$ *is in fact* K*-trivial.*

This is result from [CM06, BB12].

On the other hand, requiring g to be an order in (3.3) does not produce a realistic notion of what it means to be of 'low but nontrivial initial segment complexity'. Indeed, initial segment complexity oscillates in a non-monotonic manner and in fact many reals X happen to be *infinitely often* K*-trivial* in the sense that for some constant c, $\forall k \exists n > k, K(X \restriction_n) \leq K(n) + c$. Such reals are ubiquitous in computability theory. For example, it is not hard to see that retraceable sets (e.g. see [Odi89, Chapter II.6]) are infinitely often K-trivial. The following observations are from [BV11, Section 2]).

> *The infinitely often* K*-trivials include the sets that are computably enumerable or (weakly) 1-generic or do not compute a diagonally non-computable function.*

In particular, the class of infinitely often K-trivial sets is co-meager. A curious fact that follows from the above observations (see [BV11, Section 2] for details) is the following (briefly discussed in Section 2.4):

(3.4) *For any set* X *(at least) one of the following holds:*
(i) $\liminf_n (K(X \restriction_n) - K(n)) < \infty$*;*
(ii) there exists a computable order f *such that* $\forall n\ K(X \restriction_n) \geq f(n)$*.*

We remark that there are sets (e.g. the halting problem) for which both conditions of (3.4) hold.

With the above discussion it becomes clear that the bound $K(n) + g(n)$ in (3.3) is rather crude for a deeper exploration of sequences with low but nontrivial initial segment complexity. A more fruitful approach is to use the complexities of other c.e. sets as a measure of how low the initial segment complexity of a set is. Since we are interested in nontrivial initial segment complexities, we only consider the c.e. sets which are not K-trivial. For example we can ask about the initial segment complexity of the Turing complete c.e. sets. We know from [DHNS03] that this is nontrivial; but how low can it be? The following result from [Bar11b] gives a definitive answer.

(3.5) *There are Turing-complete c.e. sets of arbitrarily low (amongst the complexities of the c.e. sets) nontrivial initial segment prefix-free complexity.*

More precisely, given any c.e. set W which is not K-trivial, there exists a Turing complete c.e. set A such that $\exists c \forall n\ K(A \restriction_n) \leq K(W \restriction_n) + c$. Furthermore this holds uniformly for any finite collection of c.e. sets which are not K-trivial. For example, given any pair W, V of c.e. sets of nontrivial complexity there is a Turing complete c.e. set A such that $A \leq_K W$ and $A \leq_K V$. This stronger version of (3.5) gave an answer to a question in [DH10, Section 11.12] and [MS07] about minimal pairs in the structure of the K-degrees of c.e. reals.

We end this section with a methodological remark concerning K-triviality. Given the variety of characterizations of K-trivial sets (see Section 2.4) there are several ways to construct sets in this class or its complement. Since some of these characterizations are highly nontrivial, it is not surprising that the choice of which expression of K-triviality we deal with in a particular argument can have considerable implications to the complexity of the argument. As a concrete example, consider the task of showing that for every computable order g there exists a set A which is not K-trivial and $K(A\upharpoonright_n)$ is bounded by $K(n)+g(n)$. It is much easier to construct a Turing complete c.e. set (hence not K-trivial, by [DHNS03]) with this property rather than directly satisfying a list of requirements that guarantee that the constructed set is not K-trivial. Another example is the proof that there are no minimal pairs in the K-degrees from [Bar11b]. Given two c.e. sets W, V which are not K-trivial we wish to construct a c.e. set A which is not K-trivial and $A\leq_K W, A\leq_K V$. Again, it is much easier to ensure that A is Turing complete (hence, not K-trivial) than explicitly ensuring that A is not K-trivial. This shortcut is explained given that the argument that Turing complete sets are not K-trivial is rather involved. Other examples are based on the equivalence (2.4) where the notion on the right side involves oracle computations and the notion on the left side does not. We conclude that knowledge about the different 'faces' of the K-trivial sets can often aid and simplify arguments that involve K-triviality.

3.4. **Bounded by a random real.** Intuitively, a random real does not have much information. However in [Kuč85, Gác86] it was shown that every real is computable from a 1-random real (this fails for higher forms of randomness, like 2-randomness). Moreover in this computation the use for the calculation of the first n bits of the real from the random oracle can be bounded by $2n$.[5] Hence in the Turing and the weak truth table degrees, every degree is bounded by a 1-random degree. The same question has been considered for most of the degree structures that we consider in this survey. For the *ibT* degrees and the cl degrees it was shown to fail in [DH10, Section 9.13] (in this case, there is a Δ^0_2 degree which is not bounded by any 1-random degree). In [BL06a] it was shown that there is a c.e. real which is not $\leq_{\mathrm{cl}}$ bounded by any 1-random c.e. real. On the other hand note that every c.e. real is $\leq_r$-bounded by a 1-random c.e. real for $r\in\{S, rK, K, C\}$. The answer to the following question is not known.

Question 6. *Is every sequence reducible to a 1-random sequence with respect to $\leq_{rK}$ or $\leq_K$?*

The clause of Question 6 referring to $\leq_{rK}$ appeared in [RS06a]. Variations on this theme include the question of which c.e. sets are computable from incomplete 1-random reals, which was implicit already in [Kuč85]. It was solved by the cumulative results in [HNS07] and especially the recent [BRHN12] and [DM12].

A c.e. set is K-trivial iff it is computed by an incomplete 1-random real.

The 'only if' direction of this equivalence was a prominent problem in this area for a number of years and featured as an open question in a number of publications including [MN06, HNS07].

3.5. **Global structures of degrees of randomness.** A considerable difference between the degree structures in classical computability theory and those that are based on weak reducibilities is the existence of uncountable lower cones and (sometimes) degrees. In the case of the $\leq_K$, $\leq_C$ it is easy to see that 1-random reals bound uncountably many reals (and indeed, reals of every many-one degree)

---

[5] This is clear in [Gác86] but the argument in [Kuč85] can also give this refinement if some attention is given to the details.

[DDY04]. There are many other reals with this property. For example, in [BV11] it was observed that if $\lim_n(K(X\restriction_n) - 2K(n)) = \infty$ then there exist uncountably many reals $Y \leq_K X$ (and an analogous result holds for plain complexity). We may ask for a general characterization of the reals with this property (with respect to plain or prefix-free complexity).

Question 7. *Which reals have initial segment complexity that bounds the initial segment complexity of uncountably many reals?*

Such a characterization for the class of Δ^0_2 sets and the case of prefix-free complexity follows from two results in [BV11]. The first of these results is the observation that infinitely often K-trivial reals have countable lower cones with respect to $\leq_K$; the second one is (3.3).

A Δ^0_2 set X is infinitely often K-trivial iff $\{Z \mid Z \leq_K X\}$ is countable.

Recall that the analogue of Question 7 for $\leq_{LK}$ admits an elegant answer (see Section 2.4). We conjecture that the answer to Question 7 is exactly the reals which are not infinitely often K-trivial (or infinitely often C-trivial in the case of plain complexity).

A more striking difference between the K-degrees and degree structures in classical computability theory is the existence of an uncountable K-degree. In other words, there exists a real which has the same initial segment complexity (as measured by $\leq_K$) with uncountably many other reals. A real with this property was originally constructed by Joseph Miller (unpublished) and its construction may also be derived from an argument in [RS06b].

Turning to the basic algebraic properties of the structure of the K-degrees, a method for increasing or decreasing the prefix-free complexity of 1-random reals was presented in [MY10].

Theorem 3.5. *In the K-degrees of 1-random reals there are no maximal or minimal elements.*

As we will discuss in Section 3.6, this result can be localized inside the various levels of arithmetical complexity.

In the K-degrees of 1-random reals every pair of degrees has a lower bound.

In fact, this result holds for every countable collection of 1-random K-degrees (instead of a pair). Also, it shows that almost all pairs of reals do not form minimal pairs in the K-degrees, contrasting the case of many other structures like the Turing degrees and the LK degrees (see Section 4.1). However minimal pairs in the K-degrees were constructed in [CM06, MS07, BV11].

There is a minimal pair in the K-degrees.

Despite the results on the K-degrees of 1-random reals in [MY10] the existence of maximal or minimal K-degrees is open.

Question 8. *Is there a maximal K-degree? Is there a minimal K-degree?*

A minimal rK degree was constructed in [RS06a] and a minimal C-degree was constructed in [MS07]. Another striking difference between the K-degrees and other degree structures we have seen is the existence of upper bounds.

In the K-degrees there is a pair of degrees with no upper bound.

In fact, as it was demonstrated in [MY08], this holds for the degrees of any two sets that are mutually 1-random relative to each other. A number of results concerning the interaction between the Turing degrees and the K-degrees or the C-degrees were presented in [MS07] in response to some questions in [MN06, Section 9].

3.6. Local structures of degrees of randomness.

In the rK, K and C degrees of c.e. sets the most striking result is the following consequence of Theorem 3.3.

The structures of rK, K and C degrees of c.e. sets have a greatest degree.

As we noted in Section 3.2, there is a well known set that can realize the role of the maximum in these degree structures; namely Kolmogorov's set of nonrandom strings (with respect to plain complexity). Another remark is that in stronger reducibilities than rK (like $\leq_S, \leq_{\mathrm{cl}}$) there is no maximum, not even maximal c.e. degrees [Bar05] (also see [ASDFM11] for a simpler proof). In the realm of c.e. reals, maximum degrees in $\leq_S, \leq_{rK}, \leq_K, \leq_C$ are exactly the degrees of 1-random c.e. reals (or, equivalently, of Ω numbers) by [Sol75, KS01, CHKW01]. In [DY04] it was shown that some pairs of c.e. reals do not have an upper bound with respect to $\leq_{\mathrm{cl}}$ in the c.e. reals, so in particular this structure does not have a maximum degree. An interesting open question is whether the latter structure has maximal degrees.

The results in [MY10] about the K-degrees of 1-random reals have effective versions which concern the Δ^0_2 substructure.

Theorem 3.6. *The K-degrees of 1-random Δ^0_2 reals have no maximal or minimal elements.*

In fact, every pair of 1-random Δ^0_2 reals is K-above another 1-random Δ^0_2 real. However there are pairs of Δ^0_2 reals with no upper bound in the K-degrees (e.g. any pair of relatively 1-random reals has this property).

The relations $\leq_S, \leq_{rK}$ imply $\leq_T$ while $\leq_C$ implies $\leq_T$ when it is restricted to sparse sets (e.g. sets of numbers of the form 2^{2^n}, see [MS07]). This relationship with Turing reducibility and the fact that there are minimal pairs of c.e. sets with respect to $\leq_T$ has the following consequence.

There are minimal pairs of c.e. sets with respect to $\leq_S$, $\leq_{rK}$ and $\leq_C$.

In contrast, with respect to $\leq_K$ not only pairs of nontrivial c.e. sets have a nontrivial lower bound, but also this bound can be chosen to be a c.e. set [Bar11b].

Theorem 3.7. *There are no minimal pairs in the structure of the K-degrees of c.e. sets.*

Moreover the same is true for the structure of K-degrees of c.e. reals. This contrast differentiates various local substructures of the K-degrees with the corresponding substructures with respect to the related measures $\leq_S, \leq_{rK}, \leq_C$ (see the discussion in Section 5).

However it is possible to construct minimal pairs of K-degrees of very low arithmetical complexity. The best result in this direction is from [BV11] where a Σ^0_2 nonzero K-degree is constructed which forms a minimal pair with every nonzero c.e. K-degree. The argument that is used does not allow for an improvement on the arithmetical complexity of the constructed set, so that the following remains an open problem.

Question 9. *Is there a pair of Δ^0_2 sets which form a minimal pair in the K-degrees?*

Concerning the c.e. K-degrees as a substructure of the K-degrees of Δ^0_2 sets we have the following result, which is a restatement of (3.1).

> *There is a Δ^0_2 nonzero K-degree which does not bound any nonzero c.e. K-degree.*

Another basic property of interest is density. The Sacks density theorem from [Sac64] asserts that the Turing degree of c.e. sets are dense. However density often fails for very strong or very weak reducibilities and the deeper reason is usually the non-existence of least upper bounds. The non-density of the *ibT* degrees of c.e. sets was shown in [BL06b]; the non-density of the cl and the Solovay degrees of c.e. sets was shown in [Day10]. The density of the rK and the K-degrees is unknown.

Question 10. *Is the structure of the rK or the K-degrees of c.e. sets dense?*

We note that all the above structures are downward dense by the splitting theorems that we discussed in Section 2.5. Moreover upward density holds for *ibT* and cl and Solovay degrees by [Bar05] (also see [ASDFM11] for a simpler proof).

4. Comparing the compressing power of oracles

The preorder $\leq_{LK}$ may be seen as a relaxation of Turing reducibility. Indeed, $X \leq_{LK} Y$ says that oracle Y can compress strings 'at the same rate' as oracle X (possibly even higher). In particular, if $X \leq_T Y$ then $X \leq_{LK} Y$ because every computation performed with oracle X can be simulated by a computation that uses oracle Y. Consequently every LK degree is partitioned into Turing degrees. These features of $\leq_{LK}$ beg for a comparison between the Turing degrees and the LK degrees (and, more generally, a comparison between $\leq_T$ and $\leq_{LK}$). Such issues were investigated in [BLS08a, BLS08b] (where they appear in terms of the equivalent preorder $\leq_{LR}$) both on a global and a local (e.g. restricted to c.e. or Δ^0_2 sets) setting. For example, given any set X there is another set Y such that $X \equiv_{LK} Y$ but the two sets are Turing incomparable. Alternatively, instead of Turing incomparability, we can require that $X <_T Y$. An analogous result was obtained for c.e. sets X, Y. In the c.e. case we may also require (instead of Turing incomparability) that $Y \leq_T X$ (and Y is noncomputable). In effect these arguments show how to perform various fundamental constructions from Turing degree theory *inside a single LK-degree.*

Turing reducibility can be characterized in terms of arithmetical definability while in the weaker $\leq_{LK}$ this important link with definability (upon which many arguments in the Turing degrees are based) is broken. This fact has some consequences on the global and local structures of the LK-degrees, especially in terms of differences with the corresponding Turing degree structures. A remarkable such difference is the existence of uncountable lower cones in the LK-degrees. These were discovered in [BLS08a, MY10] and the complete characterization of the sets X that have uncountably many LK-predecessors was given in [BL11a].

4.1. Global structure of the LK-degrees.

The characterization of the LK-degrees with the countable predecessor property is stated in Theorem 2.2. In particular, a set X has only countably many LK-predecessors if and only if the complexity function $K^X(\sigma)$ relative to X approaches $K(\sigma)$ within a constant distance infinitely often. However, the uncountable predecessor property holds 'almost nowhere' in a measure theoretic sense.

Almost all LK degrees have the countable predecessor property.

This and the next result is from [Mil10].

Almost all pairs of LK degrees have greatest lower bound zero.

Compactness arguments may be used in order to produce concrete constructions of minimal pairs which give more local results like the following from [BLN10].

Theorem 4.1. *There is a minimal pair of LK degrees below the degree of the halting problem.*

In [Bar10c] it was shown that in the LK degrees, every Δ^0_2 degree bounds a c.e. nonzero degree. By the downward density of the c.e. LK degrees (see the next section) it follows that (in contract to the Turing degrees) in the LK degrees no Δ^0_2 degree is minimal. However the existence of minimal degrees in this structure is an open question.

Question 11. *Is there a minimal LK-degree?*

In 2006 Simpson asked if there exists a minimal Turing degree which is LK-hard, i.e. is LK-above the halting problem. A negative answer was given in [Bar12].

4.2. **Local structure of the LK degrees.** An elementary difference between the local structures of the LK degrees and the Turing degrees is the existence of minimal pairs.

Theorem 4.2. *The LK degrees of c.e. reals (or c.e. sets or Δ^0_2 sets) have no minimal pairs.*

In fact, in [Bar10b] it was shown that in the LK degrees, (strictly) below every pair of Δ^0_2 nonzero degrees there is a nonzero c.e. degree. As usual, a degree (in any degree structure) is c.e. or Δ^0_2 if it contains a c.e. or Δ^0_2 set respectively. Not much is known about the existence of a least upper bound of pairs of LK degrees.

Question 12. *Do all or some pairs of LK degrees have a least upper bound?*

The usual join operator $\oplus$ in the Turing degrees fails very dramatically to be a supremum operator in the LK degrees. For example, given *any* $Z \geq \emptyset'$ there exist $X \leq_{LK} \emptyset'$ and $Y \leq_{LK} \emptyset'$ such that $X \oplus Y \equiv_T Z$ [BLS08b]. A related result is that every pair of low sets X, Y (i.e. such that $X' \equiv_T Y' \equiv_T \emptyset'$) have a low c.e. upper bound in the LK degrees [Dia12]; this result contrasts with the Sacks splitting theorem in the c.e. Turing degrees. Finally, there is the following question from [MN06, Question 9.12].

Question 13. *Is the structure of the LK-degrees of c.e. sets dense?*

A partial positive answer was given in [BLS08b]. It was shown that if $A <_{LK} C$ and $A \leq_T C$ for two c.e. sets A, C then there is a c.e. set B such that $A <_{LK} B <_{LK} C$ (and $A <_T B <_T C$). This is a mere adaptation of the Sacks density argument, but it does imply downward and upward density of the c.e. LK degrees.

5. Natural separations of complexity measures

We have discussed a considerable number of reducibilities associated with algorithmic randomness, and their induced degree structures. Hence there are many possible comparisons that can be made between the first order theories of these structures. In the following we focus in certain pairs of structures which beg for a comparison. These are structures that are based on the same intuitive idea, i.e. they classify reals according to the same informal quality such as the amount of randomness or computational power. Interestingly, in some cases a local technical difference in the definitions, such as the choice of prefix-free machines in place of plain machines, is reflected in the associated first order theories in a very basic way. For example, it is often the case that the corresponding first order theories are different at the 2-quantifier level.

5.1. Plain and prefix-free complexity.

Comparisons between the plain and the prefix-free complexity of strings has been a topic of interest from the very beginnings of Kolmogorov complexity. For example, while prefix-free complexity is sub-additive (i.e. the complexity of the concatenation of two strings is less than the sum of the complexities of the two strings plus a constant), Martin-Löf showed that this is not true for the plain complexity. More strikingly by [MP02], for every d there are strings σ, τ such that $C(\sigma) > C(\tau) + d$ and $K(\tau) > K(\sigma) + d$.

We are interested in equally striking differences between the plain and the prefix-free initial segment complexity of infinite binary sequences. Consider the sentence "every c.e. set can be split into two disjoint c.e. sets of the same degree". By [Lac67] it is not true for the Turing degrees. Based on this result, it was observed that the situation in the case of C-degrees is the same [BHLM13]. However in the same paper it was shown that the sentence holds for the K-degrees. Hence this property separates the plain and the prefix-free initial segment complexity in a rather intrinsic way.

Our discussions of the two degree structures have pointed to another such difference in terms of a very basic algebraic property. Consider the sentence 'there exists a minimal pair'. By [MS07] this is true in the C-degrees of c.e. sets but by [Bar11b] it is not true in the K-degrees of c.e. sets. Moreover the same is true for the corresponding structures of c.e. reals. In logical terms, the two structures are not elementarily equivalent. The elementary difference we exhibited is a 2-quantifier sentence, so it lies at the lowest possible complexity class since the existential theories are the same (every finite partial order can be embedded in these structures).

5.2. Solovay degrees and K-degrees of left c.e. reals.

Solovay reducibility is arguably a very refined measure for the calibration of randomness amongst the c.e. reals. The main drawback is that only applies to reals which have computable approximations. Various proposals for an extension which applies to all reals were proposed in [DHL01]. One of these, already implicit in [Sol75], was $\leq_K$. The two measures model the same intuitive notion on c.e. reals, namely that one real is less random than another real. Yet the corresponding structures look dramatically different, even on a rather basic level. Since Solovay reducibility implies Turing reducibility, the Solovay degrees of the c.e. reals have minimal pairs. However by [Bar11b] this is not true for the K-degrees of c.e. reals.

5.3. **Stronger measures of randomness.** Two other measures of randomness that were proposed in [DHL01] were $\leq_{rK}$ and $\leq_{\mathrm{cl}}$. Since $\leq_{rK}$ implies $\leq_T$, the existence of minimal pairs of c.e. sets and c.e. reals also separates the corresponding structures with respect to $\leq_{rK}$ and $\leq_K$. On the other hand, the sentence 'there exists a maximum' separates the structures of the c.e. degrees with respect to $\leq_{rK}$ and $\leq_{\mathrm{cl}}$. The existence of a maximum in the former structure was established in [BHLM13] while the failure of this in the latter structure was observed in [Bar05, FL05]. This sentence also separates the corresponding structures of c.e. reals because 1-random c.e. reals are complete with respect to $\leq_{rK}$ (within the c.e. reals) but there is no $\leq_{\mathrm{cl}}$-complete c.e. real [DY04].

We may also examine the Solovay degrees of c.e. sets (sometimes called strongly c.e. reals) versus the K-degrees of c.e. sets. The Solovay degrees of c.e. reals coincide with the cl degrees of c.e. sets [SPK01]. Therefore there is no maximum in this structure. However there is a maximum in the structure of the K-degrees of c.e. sets. The same observation holds if we compare the Solovay degrees of c.e. sets with the rK-degrees and the C-degrees.

5.4. **Oracles for computation or mere compression.** Finally we wish to compare $\leq_T$ and a sort of extension of it in the form of $\leq_{LK}$. Not only does Turing reducibility imply $\leq_{LK}$, but also the underlying concept behind the latter is a 'relaxation' of the concept that lies behind the former. In other words instead of requiring that Y computes X, in some cases we are satisfied if Y merely computes enough information about X that can be used in order to perform *some* X-computable tasks. In our case this task is the discovery of algorithmic patterns in reals.

Recalling the results that we discussed in Section 4.2, consider the sentence 'there exists a minimal pair'. This is true in the c.e. Turing degrees but not in the c.e. LK-degrees. It is also true in the Turing degrees of Δ^0_2 sets but not in the LK-degrees of Δ^0_2 sets. These results reveal intrinsic differences between the two measures of relative complexity. Additional elementary differences between the two theories may be discovered upon the solution of some of the open questions of Section 4.2.

6. Conclusion

We have motivated and presented a number of ways one can follow in order to compare two infinite binary sequences in terms of their initial segment complexity or their power as oracles for the compression of reals and strings. More importantly, we compared these measures of complexity by providing results about the structures they induce and the properties they satisfy. A number of closely related measures of complexity were seen to diverge significantly in their basic properties. Such results pose a question as to which measures of complexity are faithful formalizations to the informal notions they represent. It is clear that different choices may be suitable for different situations and in this survey we did not give such recommendations. Rather, we provided a rich collection of recent results which highlight the differences of these measures of complexity.

We also pointed to a number of research problems which are likely to enhance our understanding of relative complexity, which is the theme of this survey. Most of these problems are rather concrete and seem tractable. Hopefully, they will stir some interest in this area, and their solutions will advance our understanding of weak reducibilities in general. On the other hand, a measure of how well we understand a (weak) reducibility is whether we can prove the decidability or undecidability of its induced degree structure. This question was not posed for the degree structures that we discussed, so we may pose it here. Undecidability proofs for degree structures are likely to involve known methods of embedding a different undecidable theory (e.g. see [DHL07]) as well as specific algebraic properties of the structure.

References

[ASDFM11] Klaus Ambos-Spies, Decheng Ding, Yun Fan, and Wolfgang Merkle. Maximal pairs of computably enumerable sets in the computable-Lipschitz degrees. Submitted, 2011.

[Bar05] George Barmpalias. Computably enumerable sets in the Solovay and the strong weak truth table degrees. In S. Barry Cooper, Benedikt Löwe, and Leen Torenvliet, editors, *CiE*, volume 3526 of *Lecture Notes in Computer Science*, pages 8–17. Springer, 2005.

[Bar10a] George Barmpalias. Compactness arguments with effectively closed sets for the study of relative randomness. *J. Logic Computation*, 2010. In press.

[Bar10b] George Barmpalias. Elementary differences between the degrees of unsolvability and the degrees of compressibility. *Ann. Pure Appl. Logic*, 161(7):923–934, 2010.

[Bar10c] George Barmpalias. Relative randomness and cardinality. *Notre Dame J. Formal Logic*, 51(2), 2010.

[Bar11a] George Barmpalias. On strings with trivial Kolmogorov complexity. *Int J Software Informatics*, 5(4):609–623, 2011.

[Bar11b] George Barmpalias. Universal computably enumerable sets and initial segment prefix-free complexity. Submitted, 2011.

[Bar12] George Barmpalias. Tracing and domination in the Turing degrees. *Ann. Pure Appl. Logic*, 163(5):500–505, 2012.

[BB12] Martijn Baartse and George Barmpalias. On the gap between trivial and nontrivial initial segment prefix-free complexity. *Theory of computing systems*, 2012. (in press).

[BD09] Laurent Bienvenu and Rod Downey. Kolmogorov complexity and Solovay functions. In *26th Annual Symposium on Theoretical Aspects of Computer Science (STACS 2009)*, pages 147–158. Dagstuhl Seminar Proceedings LIPIcs 3, 2009.

[BD12] George Barmpalias and Rodney G. Downey. Exact pairs for the ideal of the K-trivial sequences in the Turing degrees. Preprint, 2012.

[BDG10] George Barmpalias, Rodney Downey, and Noam Greenberg. Working with strong reducibilities above totally ω-c.e. and array computable degrees. *Transactions of the American Mathematical Society*, 362(2):777–813, 2010.

[BHLM13] George Barmpalias, Rupert Hölzl, Andrew E.M. Lewis, and Wolfgang Merkle. Analogues of Chaitin's Ω in the computably enumerable sets. *Information Processing Letters*, 2013. In press.

[BL06a] George Barmpalias and Andrew E. M. Lewis. A c.e. real that cannot be *sw*-computed by any omega number. *Notre Dame Journal of Formal Logic*, 47(2):197–209, 2006.

[BL06b] George Barmpalias and Andrew E. M. Lewis. The ibT degrees of computably enumerable sets are not dense. *Ann. Pure Appl. Logic*, 141(1-2):51–60, 2006.

[BL06c] George Barmpalias and Andrew E. M. Lewis. Random reals and Lipschitz continuity. *Math. Structures Comput. Sci.*, 16(5):737–749, 2006.

[BL07] George Barmpalias and Andrew E. M. Lewis. Randomness and the linear degrees of computability. *Ann. Pure Appl. Logic*, 145(3):252–257, 2007.

[BL11a] George Barmpalias and Andrew E.M. Lewis. Chaitin's halting probability and the compression of strings using oracles. *Proceedings of the Royal Society A*, 467:2912–2926, 2011.

[BL11b] George Barmpalias and Angsheng Li. Kolmogorov complexity and computably enumerable sets. Submitted, 2011.

[BLN10] George Barmpalias, Andrew E. M. Lewis, and Keng Meng Ng. The importance of Π^0_1 classes in effective randomness. *J. Symbolic Logic*, 75(1):387–400, 2010.

[BLS08a] George Barmpalias, Andrew E. M. Lewis, and Mariya Soskova. Randomness, Lowness and Degrees. *J. Symbolic Logic*, 73(2):559–577, 2008.

[BLS08b] George Barmpalias, Andrew E. M. Lewis, and Frank Stephan. Π^0_1 classes, LR degrees and Turing degrees. *Ann. Pure Appl. Logic*, 156(1):21–38, 2008.

[BMN11] Laurent Bienvenu, Wolfgang Merkle, and André Nies. Solovay functions and K-triviality. In *STACS*, pages 452–463, 2011.

[BMN12] George Barmpalias, Joseph S. Miller, and André Nies. Randomness notions and partial relativization. *Israel J. Math.*, 2012. (in press).

[BN11] George Barmpalias and André Nies. Upper bounds on ideals in the computably enumerable Turing degrees. *Ann. Pure Appl. Logic*, 162(6):465–473, 2011.

[BRHN12] Laurent Bienvenu, Joseph S. Miller Rupert Hölzl, and Andrë Nies. Denjoy, demuth and density. In preparation, 2012.

[BS11] George Barmpalias and Tom F. Sterkenburg. On the number of infinite sequences with trivial initial segment complexity. *Theor. Comput. Sci.*, 412(52):7133–7146, 2011.

[BV11] George Barmpalias and C.S. Vlek. Kolmogorov complexity of initial segments of sequences and arithmetical definability. *Theoretical Computer Science*, 412(41):5656–5667, 2011.

[Cha75] Gregory J. Chaitin. A theory of program size formally identical to information theory. *J. Assoc. Comput. Mach.*, 22:329–340, 1975.

[Cha76] G. Chaitin. Information-theoretical characterizations of recursive infinite strings. *Theoretical Computer Science*, 2:45–48, 1976.

[CHKW01] C. Calude, P. Hertling, B. Khoussainov, and Y. Wang. Recursively enumerable reals and Chaitin Ω numbers. *Theoret. Comput. Sci.*, 255(1-2):125–149, 2001.

[CM06] Barbara F. Csima and Antonio Montalbán. A minimal pair of K-degrees. *Proc. Amer. Math. Soc.*, 134(5):1499–1502 (electronic), 2006.

[Day10] Adam R. Day. The computable lipschitz degrees of computably enumerable sets are not dense. *Ann. Pure Appl. Logic*, 161(12):1588–1602, 2010.

[DDY04] Decheng Ding, Rodney. Downey, and Liang Yu. The Kolmogorov complexity of random reals. *Ann. Pure Appl. Logic*, 129(1-3):163–180, 2004.

[DH10] Rodney Downey and Denis Hirschfeldt. *Algorithmic Randomness and Complexity*. Springer, 2010.

[DHL01] Rodney G. Downey, Denis R. Hirschfeldt, and Geoffrey LaForte. Randomness and reducibility. In Sgall et al. [SPK01], pages 316–327.

[DHL04] Rodney G. Downey, Denis R. Hirschfeldt, and Geoff LaForte. Randomness and reducibility. *J. Comput. System Sci.*, 68(1):96–114, 2004.

[DHL07] Rodney G. Downey, Denis R. Hirschfeldt, and Geoffrey LaForte. Undecidability of the structure of the Solovay degrees of c.e. reals. *J. Comput. Syst. Sci.*, 73(5):769–787, 2007.

[DHN02] Rodney G. Downey, Denis R. Hirschfeldt, and André Nies. Randomness, computability, and density. *SIAM J. Comput.*, 31(4):1169–1183, 2002.

[DHNS03] Rodney G. Downey, Denis R. Hirschfeldt, André Nies, and Frank Stephan. Trivial reals. In *Proceedings of the 7th and 8th Asian Logic Conferences*, pages 103–131, Singapore, 2003. Singapore Univ. Press.

[DHNT06] Rodney Downey, Denis R. Hirschfeldt, André Nies, and Sebastiaan A. Terwijn. Calibrating randomness. *Bull. Symbolic Logic*, 12(3):411–491, 2006.

[Dia12] David Diamondstone. Low upper bounds in the LR degrees. *Ann. Pure Appl. Logic*, 163(3):314–320, 2012.

[DM12] Adam R. Day and Joseph S. Miller. Private communication. August, 2012.

[DY04] Decheng Ding and Liang Yu. There is no *sw*-complete c.e. real. *J. Symbolic Log.*, 69(4):1163–1170, 2004.

[FL05] Yun Fan and Hong Lu. Some properties of sw-reducibility. *Journal of Nanjing University (Mathematical Biquarterly)*, 22:244–252, 2005.

[Gác86] Péter Gács. Every sequence is reducible to a random one. *Inform. and Control*, 70(2-3):186–192, 1986.

[GDO06] S. S. Goncharov, R. G. Downey, and H. Ono, editors. *Proceedings of the 9th Asian Logic Conference Held in Novosibirsk, August 16–19, 2005*. World Scientific Publishing Company, Singapore, 2006.

[HKM09] Rupert Hölzl, Thorsten Kräling, and Wolfgang Merkle. Time-bounded Kolmogorov complexity and Solovay functions. In *Mathematical foundations of computer science, 2009*, volume 5734 of *Lecture Notes in Comput. Sci.*, pages 392–402. Springer, 2009.

[HNS07] Denis R. Hirschfeldt, André Nies, and Frank Stephan. Using random sets as oracles. *J. Lond. Math. Soc. (2)*, 75(3):610–622, 2007.

[JS72] Carl G. Jockusch, Jr. and Robert I. Soare. Degrees of members of Π^0_1 classes. *Pacific J. Math.*, 40:605–616, 1972.

[KHMS06] Bjørn Kjos-Hanssen, Wolfgang Merkle, and Frank Stephan. Kolmogorov complexity and the recursion theorem. In *STACS*, pages 149–161, 2006.

[KHMS11] Bjørn Kjos-Hanssen, Wolfgang Merkle, and Frank Stephan. Kolmogorov complexity and the recursion theorem. *Trans. Amer. Math. Soc.*, 363, 2011.

[KHMS12] Bjørn Kjos-Hanssen, Joseph S. Miller, and Reed Solomon. Lowness notions, measure and domination. *J. Lond. Math. Soc.*, 2012. In press.

[Kol65] Andrey N. Kolmogorov. Three approaches to the definition of the concept "quantity of information". *Problemy Peredači Informacii*, 1(vyp. 1):3–11, 1965.

[KP54] S.C. Kleene and E. Post. The upper semi-lattice of degrees of recursive unsolvability. *Ann. of Math. (2)*, 59:379–407, 1954.

[KS01] Antonín Kučera and Theodore Slaman. Randomness and recursive enumerability. *SIAM J. Comput.*, 31(1):199–211, 2001.

[KS09] Antonín Kučera and Theodore Slaman. Low upper bounds of ideals. *J. Symbolic Logic*, 74(2):517–534, 2009.

[Kuč85] Antonín Kučera. Measure, Π^0_1-classes and complete extensions of PA. In *Recursion theory week (Oberwolfach, 1984)*, volume 1141 of *Lecture Notes in Math.*, pages 245–259. Springer, Berlin, 1985.

[Kum96] Martin Kummer. Kolmogorov complexity and instance complexity of recursively enumerable sets. *SIAM J. Comput.*, 25(6):1123–1143, 1996.

[Lac67] Alistair H. Lachlan. The priority method I. *Z. Math. Logik Grundlag. Math.*, 13:1–10, 1967.

[Lev73] L. A. Levin. The concept of a random sequence. *Dokl. Akad. Nauk SSSR*, 212:548–550, 1973.

[LL99] James I. Lathrop and Jack H. Lutz. Recursive computational depth. *Inf. Comput.*, 153(1):139–172, 1999.

[LV97] Ming Li and Paul Vitányi. *An introduction to Kolmogorov complexity and its applications.* Graduate Texts in Computer Science. Springer-Verlag, New York, second edition, 1997.

[Mil04] J. Miller. Every 2-random real is Kolmogorov random. *J. Symbolic Logic*, 69:907–913, 2004.

[Mil10] Joseph S. Miller. The K-degrees, low for K degrees, and weakly low for K sets. *Notre Dame J. Formal Logic*, 50(4):381–391, 2010.

[ML66] Per Martin-Löf. The definition of random sequences. *Information and Control*, 9:602–619, 1966.

[MN06] Joseph S. Miller and André Nies. Randomness and computability: open questions. *Bull. Symbolic Logic*, 12(3):390–410, 2006.

[MP02] Andrei A. Muchnik and Semen Ye. Positselsky. Kolmogorov entropy in the context of computability theory. *Theor. Comput. Sci.*, 271(1–2):15–35, 2002.

[MS07] Wolfgang Merkle and Frank Stephan. On C-degrees, H-degrees and T-degrees. In *Twenty-Second Annual IEEE Conference on Computational Complexity (CCC 2007), San Diego, USA, 12–16 June 2007*, pages 60–69, Los Alamitos, CA, USA, 2007. IEEE Computer Society.

[MY08] Joseph S. Miller and Liang Yu. On initial segment complexity and degrees of randomness. *Trans. Amer. Math. Soc.*, 360(6):3193–3210, 2008.

[MY10] Joseph S. Miller and Liang Yu. Oscillation in the initial segment complexity of random reals. *Adv. Math.*, 2010. In press.

[Nie05] André Nies. Lowness properties and randomness. *Adv. Math.*, 197(1):274–305, 2005.

[Nie09] André Nies. *Computability and Randomness.* Oxford University Press, 2009.

[NST05] André Nies, Frank Stephan, and Sebastiaan A. Terwijn. Randomness, relativization and Turing degrees. *J. Symbolic Logic*, 70(2):515–535, 2005.

[Odi89] Piergiorgio Odifreddi. *Classical recursion theory. Vol. I.* North-Holland Publishing Co., Amsterdam, 1989.

[RS06a] Alexander Raichev and Frank Stephan. A minimal rk-degree. In Goncharov et al. [GDO06], pages 261–270.

[RS06b] Jan Reimann and Frank Stephan. Hierarchies of randomness tests. In Goncharov et al. [GDO06], pages 215–232.

[Sac63] Gerald E. Sacks. On the degrees less than $\mathbf{0}'$. *Ann. of Math. (2)*, 77:211–231, 1963.

[Sac64] Gerald E. Sacks. The recursively enumerable degrees are dense. *Ann. of Math. (2)*, 80:300–312, 1964.

[Soa87] Robert I. Soare. *Recursively enumerable sets and degrees.* Perspectives in Mathematical Logic. Springer-Verlag, Berlin, 1987. A study of computable functions and computably generated sets.

[Sol64] Ray J. Solomonoff. A formal theory of inductive inference. I. *Information and Control*, 7:1–22, 1964.

[Sol75] R. Solovay. Handwritten manuscript related to Chaitin's work. IBM Thomas J. Watson Research Center, Yorktown Heights, NY, 215 pages, 1975.

[SPK01] Jiri Sgall, Ales Pultr, and Petr Kolman, editors. *Mathematical Foundations of Computer Science 2001, 26th International Symposium, MFCS 2001 Marianske Lazne, Czech Republic, August 27-31, 2001, Proceedings*, volume 2136 of *Lecture Notes in Computer Science*. Springer, 2001.

[Ste11] Tom Sterkenburg. *Sequences with trivial initial segment complexity.* MSc Dissertation, University of Amsterdam, February 2011.

[Tur39] Alan M. Turing. Systems of logic based on ordinals. *Proc. London Math. Soc. (3)*, 45:161–228, 1939.

[vL87] Michiel van Lambalgen. *Random sequences.* PhD dissertation, University of Amsterdam, The Netherlands, 1987. ISBN 9729961506.

George Barmpalias: STATE KEY LABORATORY OF COMPUTER SCIENCE, INSTITUTE OF SOFTWARE, CHINESE ACADEMY OF SCIENCES, BEIJING 100190, P.O. BOX 8718, CHINA.

E-mail address: `barmpalias@gmail.com`

URL: `http://www.barmpalias.net`