h-Monotonically Computable Real Numbers

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Received xx yyyyyyy xxxx, revised xx yyyyyyy xxxx, accepted xx yyyyyy xxxx
Published online xx yyyyyy xxxx

Key words h-monotone computable reals; ω-monotone computable reals.
MSC (2000) 03F60,03D55

Let $h : \mathbb{N} \to \mathbb{Q}$ be a computable function. A real number x is called h-monotonically computable (h-mc, for short) if there is a computable sequence (x_s) of rational numbers which converges to x h-monotonically in the sense that $h(n)|x - x_n| \geq |x - x_m|$ for all n and $m > n$. In this paper we investigate classes h-**MC** of h-mc real numbers for different computable functions h. Especially, for computable functions $h : \mathbb{N} \to (0, 1)_{\mathbb{Q}}$, we show that the class h-**MC** coincides with the classes of computable and semi-computable real numbers if and only if $\sum_{i\in\mathbb{N}}(1 - h(i)) = \infty$ and the sum $\sum_{i\in\mathbb{N}}(1 - h(i))$ is a computable real number, respectively. On the other hand, if $h(n) \geq 1$ and h converges to 1, then h-**MC** $=$ **SC** no matter how fast h converges to 1. Furthermore, for any constant $c > 1$, if h is increasing and converges to c, then h-**MC** $= c$-**MC**. Finally, if h is monotone and unbounded, then h-**MC** contains all ω-mc real numbers which are g-mc for some computable function g.

1 Introduction

In mathematics, real numbers are usually represented by Cauchy sequences of rational numbers. Naturally, in order to discuss the computability of a real number x, we consider only the computable sequences of rational numbers which converges to x. However, the limits of computable sequences of rational numbers do not match very well the intuition of "computable real numbers". For example, if x is the limit of a computable sequence (x_s) of rational numbers, we can actually say nothing about the approximation errors $|x - x_s|$ in any effective sense. In other words, we do not know any effective lower or upper bound of x. As a result, only the limits of computable sequences of rational numbers which converge *effectively* are called *computable*. Here a sequence (x_s) converges to x *effectively* means $|x - x_s| \leq 2^{-s}$ for all s (see [10, 5, 9]). This is also equivalent to Turing's original definition of *computable numbers* in [13] that $x \in [0; 1]$ is computable if it has a computable decimal expansion, i.e., $x = \sum_{n=0}^{\infty} f(n) \cdot 10^{-n}$ for some computable function $f : \mathbb{N} \to \{0, 1, \cdots, 9\}$. On the other hand, the limits of computable sequences of rational numbers are called *computably approximable*. The class of computable and computably approximable real numbers are denoted by **EC** (stands for **E**ffectively **C**omputable) and **CA**, respectively. Here we consider only the real numbers of the unit interval $[0, 1]$ because any other real numbers y can be decomposed as $y = x \pm n$ for $x \in [0, 1]$ and $n \in \mathbb{N}$ and x and y have obviously the same effectivity in any reasonable sense.

In some sense, a sequence (x_s) converges optimally if it converges effectively, because we have a full and effective control on the error-estimation to its limit. As observed by Specker [12], even monotone sequences do not necessarily converge effectively. This leads to a weaker version of computable real numbers. We call a real number x *left computable (right computable)* if it is the limit of an increasing (decreasing) computable sequence

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of rational numbers. Left and right computable real numbers are called *semi-computable*. The classes of right computable, left computable and semi-computable real numbers are denoted by **LC**, **RC** and **SC**. It is easy to see that a real number x is left computable if and only if it has a c.e. left Dedekind cut $L_x := \{r \in \mathbb{Q} : r < x\}$. Thus, left computable real numbers are also called *computably enumerable* or c.e. for short in literature (see e.g., [4, 2]). The difference of any two left computable real numbers is called *weakly computable* or d-c.e. (see Rodney G. Downey [4]). Ambos, Weihrauch and Zheng [1] show that x is weakly computable if and only if there is a computable sequence (x_s) of rational numbers converging to x weakly effectively in the sense that the sum $\sum_{i=0}^{\infty} |x_i - x_{i+1}|$ is finite and the class **WC** of all weakly computable real numbers is in fact the closure of **SC** under the arithmetical operations $+, -, \times$ and $\div$.

Notice that any semi-computable real number x has an effective approximation (x_s) with continuously improving approximation-errors. That is, $|x - x_n| \geq |x - x_m|$ hold for any natural numbers $n < m$. As a generalization of this property, Calude and Hertling [3] introduced the notion of monotonically computability. According to Calude and Hertling, a sequence (x_s) converges to x *monotonically* if there is a constant c such that

$$(\forall n, m \in \mathbb{N})(n < m \Longrightarrow c \cdot |x - x_n| \geq |x - x_m|). \tag{1}$$

Naturally, a real number x is called *monotonically computable* if there is a computable sequence (x_s) of rational numbers which converges to x monotonically. Obviously, for $c < 1$, condition (1) implies the computability of the limit x. In general, Calude and Hertling showed in [3] that if a computable sequence (x_s) converges monotonically (even for some $c \geq 1$ of (1)) to a computable real number x, then it converges computably in the sense that there is a computable function $d : \mathbb{N} \to \mathbb{N}$ such that $|x - x_s| \leq 2^{-t}$ for any $s \geq d(t)$. Certainly, we are more interested in monotonically computable real numbers which are not computable. In this case, different constants c reflect the different "non-computability" of the limits. To study more precisely this kind of difference, let's call a real number x c-*monotonically computable* (c-mc for short) if there is a computable sequence (x_s) of rational numbers which satisfies condition (1) and converges to x. Thus, a real number x is computable if and only if it is c-mc for some $c < 1$ and x semi-computable if and only if it is 1-mc. More interesting properties of c-mc real numbers are shown in [8, 6]. For example, the dense hierarchy theorem of c-mc real numbers holds, i.e., for any constants $c_1 > c_2 \geq 1$, there exists a c_1-mc real number which is not c_2-mc. And every c-mc real number is also weakly computable but there is a weakly computable real number which is not c-mc for any constant c, that is, the class of monotonically computable real numbers is properly contained in the class **WC**.

The condition (1) has been extended further by the first and second authors in [6] to the following

$$(\forall n, m \in \mathbb{N})(n < m \Longrightarrow h(n) \cdot |x - x_n| \geq |x - x_m|) \tag{2}$$

for a function $h : \mathbb{N} \to \mathbb{Q}$. That is, the ratios of approximation errors are bounded by h. In this case, the sequence (x_s) is called h-*monotonically convergent* and the corresponding limit x is called h-monotonically computable (h-mc for short). x is called ω-*monotonically computable* (ω-mc) if it is h-mc for some computable function h. The classes of c-mc, h-mc and ω-mc real numbers are denoted by c-**MC**, h-**MC** and ω-**MC**, respectively. Different from the class of monotonically computable real number class, the class ω-**MC** is not contained in the class **WC** any more and it actually incomparable with the class **WC** as shown in [6].

In this paper we will compare the classes **EC** and **SC** with classes h-**MC** for different computable functions h. Especially, for computable functions $h : \mathbb{N} \to \mathbb{Q}$ such that $h(n) \leq 1$ for all n, we show that h-**MC** $=$ **EC** if and only if $\sum_{i\in\mathbb{N}}(1 - h(i))$ is infinite; h-**MC** $=$ **SC** if and only if the sum $\sum_{i\in\mathbb{N}}(1 - h(i))$ is a computable real number and **EC** $\subsetneq$ h-**MC** $\subsetneq$ **SC** otherwise. On the other hand, if $h : \mathbb{N} \to \mathbb{Q}$ is a computable function such that $h(n) \geq 1$ for all n and $\lim_{n\to\infty} h(n) = 1$, then the class h-**MC** coincides always with **SC** no matter how fast the function h converges 1.

For other constant $c > 1$, the situation is different. We will show that, if h is a monotone computable function which approaches to c from below, then h-**MC** $=$ c-**MC**. And for any computable functions $h_1, h_2 : \mathbb{N} \to (c, \infty)_{\mathbb{Q}}$, if $\lim_{n\to\infty} h_1(n) = \lim_{n\to\infty} h_2(n) = c$, then h_1-**MC** $=$ h_2-**MC**. Also for the ω-monotone computability, we have a very simple characterization that, if h is a unbounded monotone computable function, then h-**MC** $=$ ω-**MC**.

In the next section, we explain some useful notions and notations. In Section 3 we recall some basic properties of h-mc real numbers in general. Section 4 discusses the relationships between h-**MC** and **EC** and shows a criterion that h-**MC** coincides with **EC** iff the sum $\sum_{i\in\mathbb{N}}(1 - h(i))$ is infinite. The relationships between h-mc and semi-computable real numbers are discussed in Section 5. Here we achieve also a criterion that h-**MC** = **SC** if and only if $\sum_{i\in\mathbb{N}}(1-h(i))$ is a computable real number. The classes h-**MC** for the computable functions h which converge to a constant $c \geq 1$ are investigated in Section 6. In the last section we look at the class of ω-mc real numbers.

2 Preliminaries

In this section, we explain some basic notions and notations which will be used in this paper. By $\mathbb{N}, \mathbb{Q}$ and $\mathbb{R}$ we denote the classes of natural, rational and real numbers, respectively. Their positive subsets are denoted by $\mathbb{N}^+, \mathbb{Q}^+$ and $\mathbb{R}^+$, respectively. If $I \subseteq \mathbb{R}$ is an interval, we denote by $I_{\mathbb{Q}}$ the corresponding interval of rational numbers. For any sets A, B, we denote by $f : A \to B$ a total function from A to B and by $f :\subseteq A \to B$ a partial function with dom$(f) \subseteq A$ and range$(f) \subseteq B$.

We assume only very basic background on the classical computability theory (cf. e.g. [11, 14]). A function $f :\subseteq \mathbb{N} \to \mathbb{N}$ is called (partial) computable if there is a Turing machine which computes f. Suppose that (M_e) is an effective enumeration of all Turing machines. Let $\varphi_e :\subseteq \mathbb{N} \to \mathbb{N}$ be the function computed by the Turing machine M_e and $\varphi_{e,s} :\subseteq \mathbb{N} \to \mathbb{N}$ an effective approximation of φ_e up to step s. Namely, $\varphi_{e,s}(n) = m$ if the machine M_e with the input n outputs m in s steps and $\varphi_{e,s}(n)$ is undefined otherwise. Thus, (φ_e) is an effective enumeration of all partial computable functions $\varphi_e :\subseteq \mathbb{N} \to \mathbb{N}$ and $(\varphi_{e,s})$ a uniformly effective approximation of (φ_e). One of the most important properties of $\varphi_{e,s}$ is that the predicate $\varphi_{e,s}(n) = m$ is effectively decidable and hence in an effective construction we can use $\varphi_{e,s}$ instead of φ_e. The computability notions on other countable sets can be defined by some effective coding. For example, let $\sigma : \mathbb{N} \to \mathbb{Q}$ be an effective coding of rational numbers. Then a function $f :\subseteq \mathbb{Q} \to \mathbb{Q}$ is called computable if and only if there is a computable function $g :\subseteq \mathbb{N} \to \mathbb{N}$ such that $f \circ \sigma(n) = \sigma \circ g(n)$ for any $n \in$ dom$(f \circ \sigma)$. Other types of computable functions can be defined similarly. Of course, the computability notion on $\mathbb{Q}$ can also be defined directly based on the Turing machine. For the simplicity, we use (φ_e) to denote the effective enumeration of partial computable functions $\varphi_e :\subseteq \mathbb{N} \to \mathbb{Q}$ too. This should not cause confusion from the context.

A sequence (x_s) of rational numbers is called *computable* if there is a computable function $f : \mathbb{N} \to \mathbb{Q}$ such that $x_s = f(s)$ for all $s \in \mathbb{N}$. It is easy to see that, (x_s) is computable if and only if there are computable functions $a, b, c : \mathbb{N} \to \mathbb{N}$ such that $x_s = (a(s) - b(s))/(c(s) + 1)$ for all s.

In this paper, we consider only the h-monotonic computability for the computable functions $h : \mathbb{N} \to \mathbb{Q}$. Because of the density of $\mathbb{Q}$ in $\mathbb{R}$, all results can be extended to the cases of the computable functions $h : \mathbb{N} \to \mathbb{R}$. These results are omitted here for technical simplicity.

3 h-Monotone Computability

In this section, we look at some basic properties of h-monotonically computable real numbers for different computable functions $h : \mathbb{N} \to \mathbb{Q}$. First of all, by condition (2), it is easy to see that only rational numbers can be h-mc if $h(n) = 0$ for some natural number n. Therefore, we consider only computable functions $h : \mathbb{N} \to \mathbb{Q}^+$ in this paper.

For different constants c, the classes c-**MC** have been investigated in [8, 6]. We summarize some of the main results about c-**MC** as follows.

Theorem 3.1 (Rettinger and Zheng [6]**)**

1. **EC** $= c$-**MC** *for any* $0 < c < 1$ *and* **SC** $= 1$-**MC***;*
2. c_1-**MC** $\subsetneq c_2$-**MC** *for any constants* $c_2 > c_1 \geq 1$*; and*
3. $\bigcup_{c\in\mathbb{N}} c$-**MC** $\subsetneq$ **WC**.

Now we investigate the classes h-**MC** for computable functions $h : \mathbb{N} \to \mathbb{Q}$. Notice that, if (x_s) is a sequence of rational numbers such that $\lim_{s\to\infty} x_s = x$ and $x \neq x_s$ for all s, then the sequence (x_s) converges to x h-monotonically for the function $h : \mathbb{N} \to \mathbb{Q}$ defined by $h(n) := \sup\{|x - x_m| : m \geq n\}/|x - x_n|$. But, this function h is not computable in general. Thus, it makes sense only to discuss the h-monotone computability for computable functions h, that is, the class ω-**MC** of ω-mc real numbers. The very first question is, whether there exists a computably approximable real number which is not ω-mc at all? The positive answer is proved by the first and second authors in [6]. In fact, there exists even a weakly computable real number which is not ω-**MC**. On the other hand, the class ω-**MC** is not even contained in the class **DBC**, a proper superclass of **WC**, which is the class of all *divergence bounded computable* real numbers introduced in [7, 15]. Namely, x is divergence bounded computable means that there is a computable sequence (x_s) of rational numbers and a computable function f such that, for any n, there are at most $f(n)$ non-overlapping index pairs $i < j$ with $|x_i - x_j| \geq 2^{-n}$. That is we have

Theorem 3.2 (Rettinger, Zheng and Gengler [7]**)**

1. *There exists a weakly computable real number which is not ω-mc; and*

2. *There exists an ω-mc real number which is not divergence bounded computable,*

Thus, the class ω-**MC** is incomparable with both classes **WC** and **DBC**.

To compare the different classes h-**MC**, we have obviously h_1-**MC** $\subseteq h_2$-**MC** if $h_1(n) \leq h_2(n)$ for all n. This can be weakened slightly as follows.

Lemma 3.3 *If $h_1(n) \leq h_2(n)$ for almost all $n \in \mathbb{N}$, then h_1-**MC** $\subseteq h_2$-**MC**.*

This condition is of course not necessary, For instance, by Theorem 3.1.1. **EC** $=$ h-**MC** if there exists a constant c such that $h(n) < c < 1$ for all n. If we consider the computable functions $h_1, h_2 : \mathbb{N} \to \mathbb{Q}$ such that $h_1(n), h_2(n) \geq 1$ for all n, then we have another interesting sufficient condition for the inclusion h_1-**MC** $\subseteq h_2$-**MC** as follows.

Theorem 3.4 *Let $h_1, h_2 : \mathbb{N} \to [1, \infty)_{\mathbb{Q}}$ be computable functions. Then h_1-**MC** $\subseteq h_2$-**MC** holds if the following condition is satisfied.*

$$(\forall s \in \mathbb{N})(\exists^\infty t)(h_1(t) \leq h_2(s)). \tag{3}$$

P r o o f. Let $h_1, h_2 : \mathbb{N} \to [1, \infty)_{\mathbb{Q}}$ be two computable functions which satisfy condition (3). Suppose that x is h_1-mc and (x_s) is a computable sequence of rational numbers which converges to x h_1-monotonically. By condition (3), there is an increasing computable function $n : \mathbb{N} \to \mathbb{N}$ such that $h_1(n(s)) \leq h_2(s)$ for all $s \in \mathbb{N}$. Let (y_s) be a computable subsequence of (x_s) defined by $y_s := x_{n(s)}$ for any $s \in \mathbb{N}$. For all $s < t$, since $n(s) < n(t)$, we have $h_1(n(s))|x - x_{n(s)}| \geq |x - x_{n(t)}|$. This implies that

$$h_2(s)|x - y_s| = h_2(s)|x - x_{n(s)}| \geq h_1(n(s))|x - x_{n(s)}| \geq |x - x_{n(t)}| = |x - y_t|.$$

That is, the computable sequence (y_s) converges h_2-monotonically to x and hence x is also h_2-mc. Therefore, h_1-**MC** $\subseteq h_2$-**MC**. □

Notice that, if $c \geq 1$ is a constant and $h_1, h_2 : (c, \infty)_{\mathbb{Q}}$ are computable functions such that $\lim_{n\to\infty} h_1(n) = \lim_{n\to\infty} h_2(n) = c$, then they satisfy condition (3). Thus we have the following result.

Corollary 3.5 *For any constant $c \geq 1$ and computable functions $h_1, h_2 : \mathbb{N} \to (c, \infty)_{\mathbb{Q}}$, if $\lim_{n\to\infty} h_1(n) = \lim_{n\to\infty} h_2(n) = c$, then h_1-**MC** $= h_2$-**MC**.*

4 Monotone Computability vs Computability

In this section, we will discuss the computability of h-mc real numbers for the computable functions $h : \mathbb{N} \to \mathbb{Q}$ with $h(n) \leq 1$ for all $n \in \mathbb{N}$ and show a necessary and sufficient condition for the equality h-**MC** $=$ **EC**.

Since h-**MC** $= \mathbb{Q}$ if $h(n) = 0$ for some n by condition (2), the class **EC** is not contained in every class h-**MC** for computable function h. However, the next lemma says that if $h(n) > 0$ for all n, then the class h-**MC** does contain already all computable real numbers, no matter how small the values of $h(n)$'s could be or even if $\lim_{n\to\mathbb{N}} h(n) = 0$.

Lemma 4.1 *If $h : \mathbb{N} \to \mathbb{Q}$ is a computable function such that $h(n) > 0$ for all $n \in \mathbb{N}$, then* **EC** $\subseteq h$-**MC**.

P r o o f. Let x be a computable real number and (x_s) a computable sequence which converges to x such that $2^{-s} \leq |x - x_s| \leq 2^{-s+1}$. This implies that $|x - x_s| \leq |x - x_t|$ for any $s \geq t$. Since $h(n) > 0$ for all $n \in \mathbb{N}$, we can define inductively an increasing computable function $g : \mathbb{N} \to \mathbb{N}$ by $g(0) := 0$ and

$$g(n+1) \quad := \quad (\mu s) \left(s > g(n) \ \& \ h(n) \cdot 2^{-g(n)} > 2^{-s+1} \right).$$

Then, for any $n < m$, we have $g(n+1) \leq g(m)$ and hence

$$h(n) \cdot |x - x_{g(n)}| \quad \geq \quad h(n) \cdot 2^{-g(n)} \geq 2^{-g(n+1)+1} \geq |x - x_{g(n+1)}| \geq |x - x_{g(m)}|.$$

That is, the computable sequence $(x_{g(s)})$ converges h-monotonically to x and hence x is h-mc. □

On the other hand, it is easy to see that any h-mc real number is computable if there is a constant $c < 1$ such that $h(n) \leq c$ for infinitely many $n \in \mathbb{N}$. Therefore, it suffices to consider only the computable functions $h : \mathbb{N} \to (0;1]_{\mathbb{Q}}$ such that $\lim_{n\to\infty} h(n) = 1$. Even for such kind of function h, the class h-**MC** can coincide with **EC** too. The next theorem shows a necessary and sufficient condition when this happens.

Theorem 4.2 *Let $h : \mathbb{N} \to (0;1]_{\mathbb{Q}}$ be a computable function. Then the class h-**MC** coincides with the class **EC** if and only if $\prod_{i=0}^{\infty} h(i) = 0$. Namely,*

$$h\text{-}\mathbf{MC} = \mathbf{EC} \iff \prod_{i=0}^{\infty} h(i) = 0.$$

P r o o f. "⇐": Suppose that $h : \mathbb{N} \to (0;1]_{\mathbb{Q}}$ is a computable function such that $\prod_{i=0}^{\infty} h(i) = 0$. We are going to show that h-**MC** $=$ **EC**. The inclusion **EC** $\subseteq h$-**MC** follows from Lemma 4.1. We show now the another inclusion h-**MC** $\subseteq$ **EC**.

Let x be an h-mc real number and (x_s) be a computable sequence of rational numbers which converges to x h-monotonically. Suppose without loss of generality that $|x - x_0| \leq 1$. Since $h(n) \cdot |x - x_n| \geq |x - x_m|$ for all $m > n$, we have $|x - x_n| \leq \prod_{i=0}^{n} h(i) \cdot |x - x_0| \leq \prod_{i=0}^{n} h(i)$. Because $\lim_{n\to\infty} \prod_{i=0}^{n} h(i) = \prod_{i=0}^{\infty} h(i) = 0$, we can define a strictly increasing computable function $g : \mathbb{N} \to \mathbb{N}$ inductively as follows.

$$\begin{cases} g(0) & := \quad 0 \\ g(n+1) & := \quad (\mu s) \left(s > g(n) \ \& \ \prod_{i=0}^{s} h(i) < 2^{-(g(n)+1)} \right). \end{cases}$$

Then the computable sequence (y_s) of rational numbers defined by $y_s := x_{g(s)}$ converges to x effectively and hence $x \in$ **EC**.

"⇒": Suppose that $\prod_{i=0}^{\infty} h(i) = c > 0$. Fix a rational number q such that $0 < q < c$. We will construct an increasing computable sequence (x_s) of rational numbers from the unit interval $[0;1]$ which converges h-monotonically to a non-computable real number x. For the h-monotonic convergence it suffices to satisfy

$$h(n)(x - x_n) \geq x - x_{n+1}$$

for all $n \in \mathbb{N}$. Since $x \leq 1$, this can be reduced further to $h(n)(1 - x_n) \geq 1 - x_{n+1}$ which is equivalent to

$$x_{n+1} \geq 1 - h(n)(1 - x_n). \tag{4}$$

To guarantee that x is not computable, it suffices to satisfy, for all $e \in \mathbb{N}$, the following requirements

$$R_e : \quad \left. \begin{array}{l} \varphi_e \text{ is an increasing total function,} \\ (\forall s)(|\varphi_e(s) - \varphi_e(s+1)| \leq 2^{-(s+1)}) \end{array} \right\} \Longrightarrow x \neq \lim_{s\to\infty} \varphi_e(s).$$

Thus, what we have to do is to construct an increasing computable sequence (x_s) which satisfies condition (4) and its limit x satisfies all requirements R_e for $e \in \mathbb{N}$.

Let's explain our idea to construct such a sequence (x_s) informally at first. As the first attempt, simply let $y_0 = 0$ and $y_{n+1} = 1 - h(n)(1 - y_n)$. In this case, we have $y_n = 1 - \prod_{i<n} h(i)$ for any $n \in \mathbb{N}$ and hence $\lim_{n\to\infty} y_n = 1 - c$. This sequence is an increasing computable sequence and satisfies condition (4) too. Nevertheless, its limit $\lim_{s\to\infty} y_s = 1 - \prod_{i\in\mathbb{N}} h(i)$ is not necessarily non-computable. To guarantee the

non-computability of x, we make an extra increment of $2\delta_e$ in the definition of y_{s+1}, if it is necessary, so that the requirement R_e is satisfied for all e. Here δ_e is a rational number with $0 < \delta_e < c$. Concretely, if there are natural numbers t, s such that $2^{-t} < \delta_e$ and $\varphi_e(t) < 1-h(s)(1-y_s)+\delta_e$, then we define $y_{s+1} := 1-h(s)(1-y_s)+2\delta_e$. In this case, we have $\lim_{n\to\infty} \varphi_e(n) \leq \varphi_e(t)+2^{-t} \leq \varphi_e(t)+\delta_e \leq 1-h(s)(1-y_s)+2\delta_e = y_{s+1} < \lim_{n\to\infty} y_n$. Otherwise, $\lim_{n\to\infty} \varphi_e(n) \geq \varphi_e(t) - 2^{-t} > \varphi_e(t) - \delta_e \geq \lim_{n\to\infty} y_n$ for some t with $2^{-t} < \delta_e$. In both cases, R_e is satisfied. This strategy can be implemented for each requirement independently. To guarantee that the sequence remains in the interval $[0;1]$, the δ_e's should be chosen in such a way that $\sum_{e\in\mathbb{N}} 2\delta_e \leq c$. Therefore, we can define simply $\delta_e := q \cdot 2^{-(e+2)}$.

The formal construction of the sequence (x_s):

Stage 0. Define $x_0 = 0$. All requirements are set into the state of "unsatisfied".

Stage $s+1$. Given x_s. We say that a requirement R_e *requires attention* if $e \leq s$, R_e is in the state of "unsatisfied" and the following condition is satisfied

$$(\exists t \leq s)\left(2^{-t} < \delta_e \ \&\ \varphi_{e,s}(t) \leq 1 - h(s)(1-x_s) + \delta_e\right) \tag{5}$$

Let R_e be the requirement of the highest priority (i.e., of the minimum index e) which requires attention at this stage. Then define

$$x_{s+1} := 1 - h(s)(1-x_s) + 2\delta_e$$

and set R_e to the state of "satisfied". In this case, we say that R_e *receives attention* at stage $s+1$.

Otherwise, if no requirement requires attention at stage $s+1$, then define simply

$$x_{s+1} := 1 - h(s)(1-x_s).$$

To show that our construction succeeds, we prove the following sublemmas.

Sublemma 4.2.1 *For any $e \in \mathbb{N}$, the requirement R_e receives attention at most once and hence $\sum_{i=0}^{\infty} \sigma(i) \leq q$, where $\sigma(s) := 2\delta_e$ if the requirement R_e receives attention at stage $s+1$, and $\sigma(s) := 0$ otherwise.*

Proof of sublemma: By the construction, if a requirement R_e receives attention at stage s, then R_e is set into the state of "satisfied" and will never require attention again after stage s. That is, it receives attention at most once. This implies that, for any $e \in \mathbb{N}$, there is at most one $s \in \mathbb{N}$ such that $\sigma(s) = 2\delta_e$. Therefore, $q = \sum_{e=0}^{\infty} 2\delta_e \geq \sum_{i=0}^{\infty} \sigma(i)$.

Sublemma 4.2.2 *The sequence (x_s) is an increasing computable sequence of rational numbers from the interval $[0;1]$ and it converges h-monotonically to some $x \in [0;1]$.*

Proof of sublemma: At first we prove by induction on n the following claim

$$(\forall n \in \mathbb{N})\left(x_n \leq 1 - \prod_{i<n} h(i) + \sum_{i<n} \sigma(i)\right). \tag{6}$$

For $n = 0$, the claim (6) holds trivially, because $\prod_{i\in\emptyset} \cdots = 1$ and $\sum_{i\in\emptyset} \cdots = 0$ by convention.

Assume by induction hypothesis that the claim holds for n. Then we have

$$\begin{aligned} x_{n+1} &= 1 - h(n)(1-x_n) + \sigma(n) \\ &\leq 1 - h(n) \cdot \prod_{i<n} h(i) + h(n) \cdot \sum_{i<n} \sigma(n) + \sigma(n) \\ &\leq 1 - \prod_{i<n+1} h(i) + \sum_{i<n+1} \sigma(n). \end{aligned}$$

That is, the claim holds also for $n+1$ and this completes the proof of the claim. Since $\prod_{i<n} h(i) \geq \prod_{i=0}^{\infty} h(i) > q \geq \sum_{i=0}^{\infty} \sigma(i) \geq \sum_{i<n} \sigma(i)$ for any $n \in \mathbb{N}$, it follows that $x_n < 1$ for any $n \in \mathbb{N}$. Furthermore, because of

$$\begin{aligned} x_{n+1} - x_n &= 1 - h(n)(1-x_n) + \sigma(n) - x_n \\ &\geq 1 - h(n)(1-x_n) - x_n = (1-h(n))(1-x_n) > 0 \end{aligned}$$

for any $n \in \mathbb{N}$, the sequence (x_s) is a strictly increasing computable sequence of rational numbers from $[0; 1]$.

Besides, by the construction, the sequence (x_s) satisfies obviously the condition (4). Therefore, it converges to some $x \in [0; 1]$ h-monotonically and hence x is an h-mc real number.

Sublemma 4.2.3 *For any $e \in \mathbb{N}$, the requirement R_e is eventually satisfied and hence x is not a computable real number.*

Proof of sublemma. For any $e \in \mathbb{N}$, suppose that the premises of the requirement R_e are satisfied. Namely, φ_e is an increasing total function and satisfies the condition that $|\varphi_e(s) - \varphi_e(s+1)| \leq 2^{-(s+1)}$ for all s. Then the limit $z_e := \lim_{t\to\infty} \varphi_e(t)$ exists and $|z_e - \varphi_e(s)| \leq 2^{-s}$ holds for any $s \in \mathbb{N}$. We consider the following two cases:

Case 1. The requirement R_e receives attention at some stage $s + 1$. According to condition (5), there is a natural number $t \leq s$ such that $2^{-t} < \delta_e$ and $\varphi_e(t) \leq 1 - h(s)(1 - x_s) + \delta_e$. This implies that $\lim_{n\to\infty} \varphi_e(n) \leq \varphi_e(t) + 2^{-t} \leq 1 - h(s)(1 - x_s) + 2\delta_e \leq x_{s+1} < x$. That is, $\lim_{t\to\infty} \varphi_e(t) \neq x$ and hence R_e is satisfied in this case.

Case 2. Suppose that the requirement R_e does not receive attention at all. We will show that R_e is satisfied too. By Sublemma 4.2.1, we can choose an s_0 large enough such that no requirement R_i for $i < e$ receives attention after stage s_0. For a contradiction assume that $z_e = \lim_{t\to\infty} \varphi_e(t) = x$. From the hypothesis of R_e, φ_e is an increasing total function. Choose $t, s_1 > s_0$ such that $2^{-t} < \delta_e$ and $\varphi_{e,s_1}(t)$ is defined. Then there exists an $s_2 \geq \max\{e, t\}$ such that $\varphi_{e,s_2}(t) < x_{s_2}$. Since $0 < h(s_2), x_{s_2} < 1$, we have $x_{s_2} < 1 - h(s_2)(1 - x_{s_2})$. This implies that $\varphi_{e,s_2}(t) \leq 1 - h(s_2)(1 - x_{s_2}) + \delta_e$. That is, condition (5) is satisfied at stage $s_2 + 1$. Therefore, R_e will require and also receive attention at stage $s_2 + 1$, because no requirement R_i for $i < e$ requires attention at this stage. This is a contradiction.

In summary, x is an h-monotonically computable but not computable real number. This completes the proof of the theorem. □

Corollary 4.3 *Let $h : \mathbb{N} \to (0, 1]_{\mathbb{Q}}$ be a computable function. Then h-**MC** = **EC** if and only if the sum $\sum_{i\in\mathbb{N}}(1 - h(i))$ is infinite.*

Proof. It follows immediately from Theorem 4.2 and a classical result that the sum $\sum_{i\in\mathbb{N}}(1-h(i))$ is infinite if and only if the product $\prod_{i\in\mathbb{N}} h(i)$ is equal to 0. □

5 Semi-computability

In this section we discuss the relationship between h-monotone computability and semi-computability. We will show that, for any computable function $h : \mathbb{N} \to (0; 1]_{\mathbb{Q}}$, h-**MC** = **SC** if and only if the sum $\sum_{i\in\mathbb{N}}(1 - h(i))$ is a computable real number.

First of all we shown a simple sufficient condition that all h-mc real numbers are semi-computable.

Lemma 5.1 *Let $h : \mathbb{N} \to \mathbb{Q}$ be a computable function such that $h(n) \leq 1$ for infinitely many $n \in \mathbb{N}$. Then h-**MC** $\subseteq$ **SC**.*

Proof. Let $h : \mathbb{N} \to \mathbb{Q}$ be a computable function with $(\exists^\infty n)(h(n) \leq 1)$. Then there exists an increasing computable function $g : \mathbb{N} \to \mathbb{N}$ such that $h(g(n)) \leq 1$ for all n. Suppose that x is an h-mc real number and (x_s) is a computable sequence (x_s) of rational numbers which converges to x h-monotonically. Then the sequence (x_s) can be sped up by choosing a subsequence $(x_{g(s)})$. Actually, we can easily see that the computable sequence $(x_{g(s)})$ of rational numbers converges to x $h \circ g$-monotonically, i.e., x is $h \circ g$-mc. On the other hand, since $(\forall n \in \mathbb{N})(h \circ g(n) \leq 1)$, x is a semi-computable real number by Theorem 3.1. □

By Lemma 5.1, if $(\exists^\infty n)(h(n) \leq 1)$, then any h-mc real number x is semi-computable. However, a computable sequence which h-monotonically converges to x is not necessarily monotone and a monotone sequence converging to x does not automatically converge h-monotonically. But the next result shows that, for any such h-mc real number x, there exists a monotone computable sequence which converges to x h-monotonically.

Lemma 5.2 *Let $h : \mathbb{N} \to \mathbb{Q}$ be a computable function such that $h(n) \leq 1$ for infinitely many $n \in \mathbb{N}$. If x is an h-mc real number, then there is a monotone computable sequence which converges to x h-monotonically.*

P r o o f. Let (x_s) be a computable sequence of rational numbers which converges h-monotonically to x and $h : \mathbb{N} \to \mathbb{Q}$ a computable function such that $h(n) \leq 1$ for infinitely many n. Suppose that (n_s) is the strictly increasing sequence of all natural numbers n_s such that $h(n_s) \leq 1$ and denote by $n(s)$ the least $n_i \geq s$. Now we consider the computable sequence (y_s) defined by $y_s := x_{n(s)}$ for all s.

We show at first that the sequence (y_s) converges h-monotonically to x. For any $s < t$, if $n(s) = n(t)$, then there is an i such that $s < t \leq n_i$ and $n(s) = n_i$ and hence $h(s) > 1$. This implies that $h(s)|x - y_s| = h(s)|x - x_{n(s)}| > |x - x_{n(t)}| = |x - y_t|$. Otherwise, if $n(s) < n(t)$, then there is an $i \in \mathbb{N}$ such that $s \leq n_i < t$, $n(s) = n_i$ and $n(s) < n(t)$. In this case, we have $h(s) = h(n_i)$ if $s = n_i$ and $h(s) > 1 \geq h(n_i)$ otherwise, i.e., $h(s) \geq h(n(s))$. This implies that $h(s)|x-y_s| = h(s)|x-x_{n(s)}| \geq h(n(s))|x-x_{n(s)}| \geq |x-x_{n(t)}| = |x-y_t|$.

Notice that, if $y_s < y_{s+1}$, then $n(s) < n(s+1)$ and hence there is an $i \in \mathbb{N}$ such that $s = n_i < s+1$. This means that $h(s) = h(n_i) \leq 1$. Therefore we have $y_s < x$, because otherwise $x \leq y_s$ and so $h(s)|x - y_s| \leq y_s - x < y_{s+1} - x$ which contradicts the h-monotonic convergence of the sequence (y_s). Similarly, if $y_s > y_{s+1}$, then $y_s > x$. Now there are following four possibilities:

Case 1. The inequality $y_s < y_{s+1}$ hold for almost all $s \in \mathbb{N}$. In this case, we can easily transform the sequence (y_s) to a monotone one which converges h-monotonically to x.

Case 2. $y_s > y_{s+1}$ hold for almost all $s \in \mathbb{N}$. We can do similar to the case 1.

Case 3. For almost all s, $y_s = y_{s+1}$. In this case, the limit x is in fact a rational number and we are done.

Case 4. $(\exists^\infty s)(y_s < y_{s+1})$ and $(\exists^\infty s)(y_s > y_{s+1})$. In this case, we can define an increasing computable sequence and a decreasing computable sequence which converge both to x. For example, the increasing computable sequence (z_s) can be defined by $z_s := y_{g(s)}$ where $g : \mathbb{N} \to \mathbb{N}$ is defined inductively by

$$\begin{cases} g(0) & := \quad (\mu s)\,(y_s < y_{s+1}) \\ g(n+1) & := \quad (\mu s)\left(s > g(n) \;\&\; y_{g(n)} < y_s < y_{s+1}\right). \end{cases}$$

But in this case, x is a computable real number and hence there is an increasing computable sequence which converges to x h-monotonically by Lemma 4.1. □

As mentioned at the beginning of this section, if $c < 1$, then any c-mc real number is computable. It is natural to ask, for a function h with $0 < h(n) < 1$ for any $n \in \mathbb{N}$, is any h-mc real number computable too, or is there any 1-mc real number which is not h-mc for any computable function h with $(\forall n)(h(n) < 1)$? Next theorem gives a negative answer to both of these questions.

Theorem 5.3 *Every semi-computable real number is h-mc for a computable function $h : \mathbb{N} \to \mathbb{Q}$ such that $0 < h(n) < 1$ for any $n \in \mathbb{N}$.*

P r o o f. Suppose that x is a left computable real number and (x_s) is a strictly increasing computable sequence of rational numbers which converges to x. Let a be a rational number which is greater than x. Define a computable function $h : \mathbb{N} \to \mathbb{Q}$ by $h(n) := \frac{a-x_{n+1}}{a-x_n}$ for any $n \in \mathbb{N}$. Then $0 < h(n) < 1$ because $x_n < x_{n+1}$ for any $n \in \mathbb{N}$. Actually, the sequence (x_s) converges to x h-monotonically because

$$\begin{aligned} h(n) \cdot |x - x_n| &= (x - x_n)\left(\frac{a - x_{n+1}}{a - x_n}\right) > (x - x_n)\left(\frac{x - x_{n+1}}{x - x_n}\right) \\ &= (x - x_{n+1}) \geq |x - x_m| \end{aligned}$$

for any natural numbers n and $m > n$.

Similarly, if x is a right computable real number and (x_s) a strictly decreasing computable sequence of rational numbers which converges to x, then we define a computable function $h : \mathbb{N} \to \mathbb{Q}$ by $h(n) := \frac{x_{n+1}}{x_n}$ for any natural numbers n. Obviously, we have $0 < h(n) < 1$ for any $n \in \mathbb{N}$ and the sequence (x_s) converges to x h-monotonically because

$$\begin{aligned} h(n) \cdot |x - x_n| &= (x_n - x)\left(\frac{x_{n+1}}{x_n}\right) > (x_n - x)\left(\frac{x_{n+1} - x}{x_n - x}\right) \\ &= (x_{n+1} - x) \geq |x - x_m| \end{aligned}$$

for any natural numbers $m > n$. □

By Theorem 4.2, the condition $h(n) < g(n)$ for all n does not suffice to separate the class g-**MC** from h-**MC**. The next theorem gives a sufficient condition such that g-**MC** $= h$-**MC**.

Theorem 5.4 *Let $g, h : \mathbb{N} \to (0,1)_{\mathbb{Q}}$ be computable functions. If $h(n) \leq g(n)$ holds for all n and the sum $u := \sum_{i\in\mathbb{N}}(g(i) - h(i))$ is a computable real number, then g-**MC** $= h$-**MC**.*

P r o o f. Let $g, h : \mathbb{N} \to (0,1)_{\mathbb{Q}}$ be computable functions such that $h(n) \leq g(n)$ for all n. We need only to prove the non-trivial inclusion g-**MC** $\subseteq h$-**MC**. Let x be a g-mc real number. If x is computable, then it is also h-mc by Lemma 4.1. Suppose that x is left computable but not computable. Then, by Lemma 5.2, there exists an increasing computable sequence (x_s) of rational numbers which converges to x g-monotonically. That is, we have $g(n)(x - x_n) \geq x - x_{n+1}$ which is equivalent to

$$x \leq x_n + \frac{x_{n+1} - x_n}{1 - g(n)} \tag{7}$$

for all n. Let $v_n := \min\{x_m + \frac{x_{m+1}-x_n}{1-g(m)} : m \leq n\}$. Then we have $x \leq v_{n+1} \leq v_n$ and hence the limit $v := \lim_{n\to\infty} v_n$ exists and $x \leq v$. Thus, we have $v \leq x_n + \frac{x_{n+1}-x_n}{1-g(n)}$ which is further equivalent to

$$(1 - g(n))(v - x_n) \leq x_{n+1} - x_n \tag{8}$$

for all n. Since x is left computable but not computable and v is right computable, we have actually $x < v$. Therefore, we can choose a rational number r such that $0 < r < v - x$.

By assumption, the sum $u := \sum_{i\in\mathbb{N}}(g(i)-h(i))$ is computable and hence there exists a decreasing computable sequence (u_n) of rational numbers which converges to u. Now we can define a computable sequence (y_n) of rational numbers by $y_n := x_n - \delta_n$ for $\delta_n := r \cdot \left(u_n - \sum_{i<n}(g(i) - h(i))\right)$ for all n. Since $y_{n+1} - y_n = (x_{n+1} - x_n) + r \cdot (u_n - u_{n+1}) + (g(n) - h(n)) > 0$, the sequence (y_n) is increasing. Furthermore, we have $\lim_{n\to\infty} y_n = \lim_{n\to\infty} x_n$ because $\lim_{n\to\infty} \delta_n = 0$. In addition, for large n, we have $x - y_n \leq r$ and hence the following inequality holds

$$\begin{aligned}
y_{n+1} - y_n &= (x_{n+1} - x_n) + (\delta_n - \delta_{n+1}) \\
&= (x_{n+1} - x_n) + r \cdot ((u_n - n_{n+1}) + (g(n) - h(n))) \\
&\geq (1 - g(n))(v - x_n) + r \cdot (g(n) - h(n)) \\
&\geq r \cdot (1 - g(n)) + r \cdot (g(n) - h(n)) \\
&= r \cdot (1 - h(n)) \geq (x - y_n)(1 - h(n)).
\end{aligned}$$

Therefore, x is h-mc too. For right computable x, the proof is similar. □

The Theorem 5.4 can be easily strengthened to the following form.

Theorem 5.5 *Let $g, h : \mathbb{N} \to (0,1)_{\mathbb{Q}}$ be computable functions. If the sum $u := \sum_{i\in\mathbb{N}} |g(i) - h(i)|$ is a computable real number, then g-**MC** $= h$-**MC**.*

P r o o f. It suffices to apply Theorem 5.4 to functions g_1 and h_1 defined by $g_1(n) := \max\{g(n), h(n)\}$ and $h_1(n) := \min\{g(n), h(n)\}$, respectively, for all n. □

From the proof, Theorem 5.4 can not generalized directly to the case of constant function $g \equiv 1$. However, the result does hold correspondingly for $g \equiv 1$.

Theorem 5.6 *Let $h : \mathbb{N} \to (0,1]_{\mathbb{Q}}$ be a computable function. If the sum $\sum_{i=0}^{\infty}(1 - h(i))$ is a computable real number, then **SC** $= h$-**MC**.*

P r o o f. We need only to show the non-trivial inclusion **SC** $\subseteq h$-**MC**. Let $h : \mathbb{N} \to (0,1]_{\mathbb{Q}}$ be a computable function such that the sum $u := \sum_{i=0}^{\infty}(1 - h(i))$ is a computable real number. Notice that, by condition (2), an increasing sequence (y_s) converges to x h-effectively if and only if

$$y_{s+1} - y_s \geq (x - y_s)(1 - h(s)) \tag{9}$$

hold for all s. Thus, if x is, say, left computable and (x_s) an increasing computable sequence of rational numbers which converges to x, then the sequence (y_s) defined by

$$y_s := x_s - \sum_{i>s}(1 - h(i)) \tag{10}$$

satisfies condition (9). However, if the sum $u := \sum_{i=0}^{\infty}(1 - h(i))$ is not computable, then (y_s) is not necessarily computable. Nevertheless, if u is computable, then there is a decreasing computable sequence (u_s) which converges to u. Now instead of (10), we can define a computable sequence (y_s) of rational numbers by

$$y_s := x_s - u_s + \sum_{i\leq s}(1 - h(i)) \tag{11}$$

and the sequence (y_s) is increasing and satisfies condition (9) too. That is, (y_s) converges to x h-monotonically and hence $x \in h$-**MC**. For right computable x, we can prove similarly the $x \in h$-**MC**. □

Notice that, for any computable functions $g, h : \mathbb{N} \to \mathbb{Q}$, if $h(n) \leq g(n) \leq 1$ for all n and $\sum_{i\in\mathbb{N}}(1 - h(n))$ is computable, then g-**MC** $=$ **SC** because **SC** $= h$-**MC** $\subseteq g$-**MC** $\subseteq$ **SC**. Actually, in this case, the sum $\sum_{i\in\mathbb{N}}(1 - g(i))$ is computable too as shown in the next lemma.

Lemma 5.7 *Let $h, g : \mathbb{N} \to (0, 1)_{\mathbb{Q}}$ be computable functions. If $g(s) \leq h(s)$ holds for all s and the sum $u := \sum_{i\in\mathbb{N}}(1 - g(i))$ is a computable real number, then the sum $v := \sum_{i\in\mathbb{N}}(1 - h(i))$ is computable too.*

Proof. Since the sum $u := \sum_{i\in\mathbb{N}}(1 - g(i))$ is computable, there exists an increasing computable sequence (u_s) of rational numbers converging to u such that $u - u_s \leq 2^{-s}$ for all s. Let $n(s)$ denote the least natural number t such that $\sum_{i=0}^{t}(1 - g(i)) \geq u_s$ and $v_s := \sum_{i=0}^{n(s)}(1 - h(i))$ for all s. Then (v_s) is an increasing computable sequence of rational numbers which satisfies the following

$$\begin{aligned} v - v_s &= \sum_{i>n(s)}(1 - h(i)) \leq \sum_{i>n(s)}(1 - g(i)) \\ &= u - \sum_{i\leq n(s)}(1 - g(i)) \leq u - u_s \leq 2^{-s} \end{aligned}$$

for all s. That is, the sequence (v_s) converges to v effectively and hence v is computable. □

Now we show that the computability of the sum $\sum_{i\in\mathbb{N}}(1-h(i))$ is in fact a necessary condition for the equality **SC** $= h$-**MC**.

Theorem 5.8 *Let $h : \mathbb{N} \to (0, 1)_{\mathbb{Q}}$ be a computable function. If $\sum_{i\in\mathbb{N}}(1 - h(i))$ is not a computable real number, then there is a left computable real number x which is not h-monotonically computable, i.e.,* **SC** $\not\subseteq$ h-**MC**.

Proof. Let $h : \mathbb{N} \to (0, 1)_{\mathbb{Q}}$ be a computable function such that $\sum_{i\in\mathbb{N}}(1 - h(i))$ is not computable. We are going to construct a computable nondecreasing sequence (x_s) of rational numbers which converges to x such that x is not h-mc. Since x is left computable, by Lemma 5.2, if x is h-mc, then there exists an increasing computable sequence of rational numbers which converges h-monotonically. Thus, x needs only to satisfy, for all $e \in \mathbb{N}$, the following requirements:

$$R_e \quad : \quad \left.\begin{array}{l} \varphi_e \text{ is an increasing total function and} \\ (\varphi_e(s)) \text{ converges } h\text{-monotonically to } y_e \end{array}\right\} \Longrightarrow y_e \neq x,$$

where (φ_e) is an effective enumeration of computable functions $\varphi_e :\subseteq \mathbb{N} \to \mathbb{Q}$.

We explain the idea how a single requirement R_e can be satisfied. Let $\Sigma_5 := \{0, 1, 2, 3, 4\}$ and $\mathbb{I}$ be set of all rational subintervals of the interval $[0, 1]$. We define at first an interval tree $I : \Sigma_5^* \to \mathbb{I}$ inductively by $I(\lambda) := [0, 1]$ and $I(wi) := [u_i, u_{i+1}]$ for all $w \in \Sigma_5^*$ and $i < 5$, where $u_0 < u_1 < u_2 < u_3 < u_4 < u_5$ is an equidistant subdivision of the interval $I(w)$. The interval $I(w)$ is denoted by $[a_w, b_w]$ for any $w \in \Sigma_5^*$. Then we have $a_w = \sum_{i<|w|} 5^{-(i+1)} \cdot w[i]$ and $b_w = a_w + 5^{-|w|}$. Let us begin with the interval $[0, 1]$ as the base interval of

R_e. Our goal is to find a subinterval $J \subseteq [0, 1]$ such that any point of J except endpoints satisfies the requirement R_e. In this case, the interval J is called a witness interval of R_e.

Assume that φ_e is a total function and $(\varphi_e(s))$ is an increasing sequence. At the beginning, let $I(11)$ be the first candidate of the witness interval of R_e. If the sequence $(\varphi_e(s))$ does not enter the interval $I(11)$, then we are done. Otherwise, suppose that there exists an s_0 such that $\varphi_e(s_0) \in I(11)$. If there exists a $t < s_0$ such that $h(t)(a_3 - \varphi_e(t)) < a_3 - \varphi_e(t+1)$ holds, then we change the witness interval to be $I(3)$. In this case, for any $x \in I(3)$, we have $h(t)(x - \varphi_e(t)) < x - \varphi_e(t+1)$ and hence $I(3)$ is a correct witness interval of R_e. Otherwise, if the inequality $h(t)(a_3 - \varphi_e(t)) \geq a_3 - \varphi_e(t+1)$ holds for all $t < s_0$, then we choose $I(131)$ to be a new candidate of witness interval. Analogously, if there exists an $s_1 > s_0$ such that $\varphi_e(s_1) \in I(131)$, then we choose either $I(3)$ or $I(1331)$ to be the new candidate of witness interval of R_e, depending on whether there exists a $t < s_1$ such that $h(t)(a_3 - \varphi_e(t)) < a_3 - \varphi_e(t+1)$ holds, and so on. Let's look at the possible outcome of our construction. If the interval $I(3)$ is chosen as a witness interval at some stage, then we are done because the sequence $(\varphi_e(s))$ does not converge to any element of $I(3)$.

Otherwise, if $I(3)$ has never been chosen as the witness interval of R_e, then there are two possibilities. Either there is a $k \in \mathbb{N}$ such that $I(13^k1)$ is chosen as a candidate of witness interval and there does not exist s such that $\varphi_e(s) \in I(13^k1)$ and hence $I(13^k1)$ is a correct witness interval of R_e or each of the following intervals $I(11), I(131), I(1331), \cdots$ is chosen to be a candidate of witness interval at some stage. In the latter case, there exists a computable increasing sequence (s_n) of natural numbers such that $\varphi_e(s_n) \in I(13^n1)$ for all n. This implies that the limit $y_e := \lim_{s\to\infty} \varphi_e(s) = 5^{-1} + 3 \cdot \sum_{i=2}^{\infty} 5^{-i}$ is a computable real number. In addition, we have the inequality $h(n)(a_3 - \varphi_e(n)) \geq a_3 - \varphi_e(n+1)$ holds for all $t \in \mathbb{N}$. That is, we have

$$(1 - h(n))(a_3 - \varphi_e(n)) \leq \varphi_e(n+1) - \varphi_e(n) \tag{12}$$

for all n. Notice that $a_3 - \varphi_e(n) > 5^{-1}$ for all n. Now we define a computable function $g : \mathbb{N} \to (0, 1)_{\mathbb{Q}}$ by

$$g(n) \quad := \quad 1 - 5 \cdot (\varphi_e(n+1) - \varphi_e(n))$$

for all n. By condition (12), we have $1 - g(n) = 5(\varphi_e(n+1) - \varphi_e(n)) \geq 5(1 - h(n))(a_3 - \varphi_e(n)) \geq 1 - h(n)$ and hence $g(n) \leq h(n)$ for all n. On the other hand, we have $\sum_{n\in\mathbb{N}}(1-g(n)) = 5\sum_{n\in\mathbb{N}}(\varphi_e(n+1)-\varphi_e(n)) = 5(y_e - \varphi_e(0))$. That is, the sum $\sum_{n\in\mathbb{N}}(1 - g(n))$ is a computable real number and hence $\sum_{n\in\mathbb{N}}(1 - h(n))$ is computable too by Lemma 5.7. This contradicts the hypothesis on h. This contradiction implies that only finitely many intervals can be chosen to be the candidate of a witness interval of R_e and the last one is a correct one.

To satisfy all requirements R_e simultaneously, we apply a standard finite injury priority construction. We omit the details here. □

By Corollary 4.3, Theorem 5.6 and Theorem 5.8 we can summarize the possible class h-**MC** for computable function $h : \mathbb{N} \to (0, 1]$ as follows.

Corollary 5.9 *Let $h : \mathbb{N} \to (0, 1]_{\mathbb{Q}}$ be a computable function, then the following hold.*

1. *If $\sum_{n\in\mathbb{N}}(1 - h(n)) = \infty$, then h-**MC** = **EC**;*
2. *If $\sum_{n\in\mathbb{N}}(1 - h(n))$ is finite and computable, then h-**MC** = **SC**; and*
3. *If $\sum_{n\in\mathbb{N}}(1 - h(n))$ is finite but not computable, then **EC** $\subsetneq$ h-**MC** $\subsetneq$ **SC**.*

Thus, for computable functions $h : \mathbb{N} \to (0, 1]_{\mathbb{Q}}$, the corresponding class h-**MC** can coincide with **EC** or **SC** or locate strictly between **EC** and **SC**, depending on how fast the function h converges to 1 from below. However, if we consider a computable function $h : \mathbb{N} \to [1, \infty)_{\mathbb{Q}}$ which converges to 1 from above, the situation is completely different.

Theorem 5.10 *Let $h : \mathbb{N} \to [1, \infty)_{\mathbb{Q}}$ be a computable function. If $\lim_{n\to\infty} h(n) = 1$, then **SC** = h-**MC**.*

P r o o f. The inclusion **SC** $\subseteq$ h-**MC** follows immediately from the equation **SC** = 1-**MC** (Theorem 3.1.2) and the inclusion 1-**MC** $\subseteq$ h-**MC** (Lemma 3.3). Now we prove the other inclusion h-**MC** $\subseteq$ **SC**. By Corollary 3.5, if there exists a computable function $h : \mathbb{N} \to (1, \infty)_{\mathbb{Q}}$ which converges to 1 such that h-**MC** $\subseteq$ **SC**, then we are done. Let us consider the function h defined by $h(s) := 1 + 2^{-s}$ for all s .

Let $x \in [0,1]$ be an h-mc real number and (x_s) a computable sequence of rational numbers of the unit interval $[0,1]$ which converges to x h-monotonically. If $|x_s - x_{s+1}| \leq 2^{-s}$ holds for almost all s, then we are done because the limit x is computable and hence is also semi-computable in this case. Suppose that there are infinitely many s such that $|x_s - x_{s+1}| > 2^{-s}$. Since (x_s) converges to x h-monotonically, we have $h(s)|x - x_s| \geq |x - x_{s+1}|$ which is equivalent to

$$\begin{cases} x \leq x_s - \frac{x_{s+1}-x_s}{h(s)-1} \text{ or } x \geq x_s + \frac{x_{s+1}-x_s}{h(s)+1}, & \text{if } x_s < x_{s+1}; \\ x \leq x_s + \frac{x_{s+1}-x_s}{h(s)-1} \text{ or } x \geq x_s - \frac{x_{s+1}-x_s}{h(s)+1}, & \text{if } x_{s+1} < x_s. \end{cases} \tag{13}$$

for all s. Now we consider the following cases.

Case 1. There are infinitely many s such that $x_s < x_{s+1}$ and $|x_s - x_{s+1}| > 2^{-s}$. For these indices s, we have $x_s < x$ because of condition (13). Otherwise, if $x \leq x_s$, then $x \leq x_s - \frac{x_{s+1}-x_s}{h(s)-1} = x_s - (x_{s+1} - x_s) \cdot 2^s < x_s - 1 < 0$ which contradicts the choice of $x \in [0,1]$. Thus, we can construct an increasing computable subsequence of (x_s) which converges to x and hence x is left computable.

Case 2. There are finitely many s such that $x_s < x_{s+1}$ and $|x_s - x_{s+1}| > 2^{-s}$. Then, there must exist infinitely many s such that $x_s > x_{s+1}$ and $|x_s - x_{s+1}| > 2^{-s}$. In this case, we can show that x is right computable in a way similar to the case 1.

In both cases we can conclude that that x is semi-computable and hence h-**MC** $\subseteq$ **SC** holds. □

6 Around a Constant $c > 1$

In section 5, we investigated the h-mc real numbers for computable functions which approach 1. For such kind of functions h, we have h-**MC** $\subseteq$ **SC**. In this section we discuss the classes h-**MC** for computable functions $h : \mathbb{N} \to \mathbb{Q}$ which converge to a constant $c > 1$. We will show that h-**MC** $= c$-**MC** if h converges to c from below. If the computable functions $h, g : \mathbb{N} \to (c, \infty)_{\mathbb{Q}}$ converge to c from above, then we have g-**MC** $= h$-**MC**.

Different from the case of constant 1, the next theorem shows that, if $c > 1$ and $h : \mathbb{N} \to (0,c)_{\mathbb{Q}}$ is a computable function which converges to c from below, then the class h-**MC** is always the same no matter how fast the functions h approaches the constant c.

Theorem 6.1 *Let $c > 1$ be a constant and $h_1, h_2 : \mathbb{N} \to (0,c)_{\mathbb{Q}}$ be computable functions. If h_2 is nondecreasing and $\lim_{n\to\infty} h_1(n) = \lim_{n\to\infty} h_2(n) = c$, then h_1-**MC** $\subseteq h_2$-**MC**.*

P r o o f. Let $h_1, h_2 : \mathbb{N} \to (0,1)_{\mathbb{Q}}$ be computable functions, h_2 is nondecreasing and $\lim_{n\to\infty} h_1(n) = \lim_{n\to\infty} h_2(n) = c > 1$. Suppose, w.l.o.g., that $h_1(n), h_2(n) \geq 1$ for all $n \in \mathbb{N}$. We will show that h_1-**MC** $\subseteq$ h_2-**MC**.

Let x be an h_1-mc real number and (x_s) a computable sequence of rational numbers which converges h_1-monotonically to x. Since $h_1(n) < c = \lim_{s\to\infty} h_2(s)$ for all n, there is a strictly increasing computable sequence (n_i) of natural numbers such that $h_2(n_i) \geq h_1(i)$ for any $i \in \mathbb{N}$. Let $n(s)$ denote the largest i such that $n_i \leq s$ and define a computable sequence (y_s) of rational numbers by $y_s := x_{n(s)}$ for all s. We show that (y_s) converges to x h_2-monotonically.

For any natural numbers $s < t$, if $n(s) = n(t)$, then there is an $i \in \mathbb{N}$ such that $n_i \leq s < t < n_{i+1}$ and $n(s) = n(t) = i$. In this case, we have

$$h_2(s)|x - y_s| = h_2(s)|x - x_{n(s)}| \geq |x - x_{n(t)}| = |x - y_t|,$$

since $h_2(s) \geq 1$. Otherwise, if $n(s) < n(t)$, then there are $i < j$ such that $n(s) = i$, $n(t) = j$ and $n_i \leq s < n_j \leq t$. Because h_2 is increasing and $h_2(n_i) \geq h_1(i)$, we conclude that

$$h_2(s)|x - y_s| \geq h_2(n_i)|x - x_{n(s)}| \geq h_1(i)|x - x_i| \geq |x - x_j| = |x - y_t|.$$

Therefore, the sequence (y_s) converges to x h_2-monotonically and hence x is h_2-mc. □

Corollary 6.2 *Let $c > 1$ be a constant and $h_1, h_2 : \mathbb{N} \to (0,c)_{\mathbb{Q}}$ computable functions. If there exists a nondecreasing computable function $g : \mathbb{N} \to \mathbb{Q}$ such that the limit $\lim_{s\to\infty} g(s) = c$ and $h_1(s), h_2(s) \geq g(s)$ for all s, then h_1-**MC** $= h_2$-**MC**.*

The above proof does not work if h_1 or h_2 is the constant function c. But a corresponding result holds. To show that we need a new technique.

Definition 6.3 Let $h : \mathbb{N} \to \mathbb{Q}$ be a function. A sequence (x_s) of rational numbers is called *h-reduced* if, for all $i < j$, there does not exist natural numbers $s > j$ such that $h(i)|x_s - x_i| < |x_s - x_j|$.

Lemma 6.4 *Let $h : \mathbb{N} \to [1, \infty)_{\mathbb{Q}}$ be a computable function. If (x_s) is a computable sequence of rational numbers which converges h-monotonically to x, then there exists an h-reduced computable sequence of rational numbers which converges to x too.*

Theorem 6.5 *Let $c > 1$ be a constant and $h : \mathbb{N} \to (0, c)_{\mathbb{Q}}$ a computable function. If there is a nondecreasing computable function $g : \mathbb{N} \to (0, c)_{\mathbb{Q}}$ such that $g(n) \leq h(n)$ for all n and $\lim_{n\to\infty} g(n) = c$, then c-**MC** $=$ h-**MC**.*

Proof. Let $h : \mathbb{N} \to (0, c)_{\mathbb{Q}}$ be a computable function. Then the inclusion h-**MC** $\subseteq$ c-**MC** is trivial. By Corollary 6.2, it suffices to prove that for any c-mc real number x, there exists an increasing computable function $g : \mathbb{N} \to (0, c)_{\mathbb{Q}}$ with $\lim_{s\to\infty} g(s) = c$ such that x is g-mc.

Let $x \in [0, 1]$ be a c-mc real number and (x_s) a c-reduced computable sequence of rational numbers of $[0, 1]$ which converges to x c-monotonically. Assume w.l.o.g that x is not semi-computable. Thus we have $c \cdot |x - x_i| \geq |x - x_j|$ for any natural numbers $i < j$. In other words, the limit x does not belong to the open interval $I(i, j)$ which is defined by

$$I(i,j) \quad := \quad \begin{cases} \left(x_i - \frac{x_j - x_i}{c-1},\ x_i + \frac{x_j - x_i}{c+1}\right), & \text{if } x_i < x_j; \\ \left(x_i - \frac{x_j - x_i}{c+1},\ x_i + \frac{x_j - x_i}{c-1}\right), & \text{otherwise.} \end{cases}$$

The interval $I(i, j)$ is called a forbidden interval of index (i, j) (for $i < j$). Now we can prove the following claim.

Claim 6.5.1 *For any natural number n, there exist natural numbers i, j, k with $n < i < j < k$ such that $I(i, j) \cap I(j, k) \neq \emptyset$*

Proof. Assume by contradiction that the claim does not hold. That is, there is an n such that $I(i, j) \cap I(j, k) = \emptyset$ for any $n < i < j < k$. Choose an index pair $i < j$ large enough such that $\delta = |x_i - xj|$ very small. Suppose that $x_i < x_j$. Since $I(i, j) \cap I(j, s) = \emptyset$ for any $s > j$, then we have $(x_j - x_s)(c + 1) \leq \delta - \delta/(c + 1)$, i.e., $x_s \geq x_j - c\delta$, if $x_s < x_j$; and $(x_s - x_j)/(c - 1) \leq \delta - \delta/(c + 1)$, i.e., $x_s \leq x_j + \frac{c-1}{c+1}c\delta$, otherwise. This implies that the limit x locates in the interval $\left[x_j - c\delta,\ x_j + \frac{c-1}{c+1}c\delta\right]$. Since (x_s) converges to x, we can choose $\delta = |x_i, x_j|$ arbitrarily small. Thus, x is a computable real number which contradicts the hypothesis that x is not semi-computable. □ (claim)

By Claim 6.5.1, there is a computable sequence (i_n, j_n, k_n) of index-triples such that $i_n < j_n < k_n < i_{n+1}$ and $I(i_n, j_n) \cap I(j_n, k_n) \neq \emptyset$ for all n. Then, we define a computable sequence (y_n) of rational numbers by $y_n := (x_{i_n} + x_{j_n})/2$ for all n. Let $\epsilon_n := |x_{i_n} - y_{i_n}|/2$. Then we can define an increasing computable function $g : \mathbb{N} \to (0, c)_{\mathbb{Q}}$ such that $c - g(n) \leq c\epsilon_n/2$ for all n. Notice that, by assumption, we have $|x - x_n| \leq 1$ and $\epsilon_n \leq 1/2$ for all n. Thus, for any natural numbers $n > m$, we have

$$\begin{aligned} g(n)|x - y_n| \quad &= \quad (c - \delta_n)\left(\min\{|x - x_{i_n}|, |x - x_{j_n}|\} + \epsilon_n\right) \\ &= \quad c \cdot \min\{|x - x_{i_n}|, |x - x_{j_n}|\} + c\epsilon_n \\ &\quad\quad -\delta_n\left(\min\{|x - x_{i_n}|, |x - x_{j_n}|\} + \epsilon_n\right) \\ &\geq \quad \max\{|x - x_{i_m}|, |x - x_{j_m}|\} + c\epsilon_n - 2\delta_n \\ &\geq \quad \max\{|x - x_{i_m}|, |x - x_{j_m}|\} \\ &= \quad |x - y_m| + \epsilon_m \geq |x - y_m|, \end{aligned}$$

where $\delta_n := c - g(n)$. That is, the computable sequence (y_n) converges to x g-monotonically. Thus, x is g-mc. □

Similar to Theorem 6.1, for the case above a constant $c > 1$, we have only one class h-**MC** for any computable function $h : \mathbb{N} \to (c, \infty)_{\mathbb{Q}}$ such that $\lim_{s\to\infty} h(s) = c$.

Theorem 6.6 *Let $c > 1$ be a constant and $h_1, h_2 : \mathbb{N} \to (c, \infty)_{\mathbb{Q}}$ computable functions. If $\lim_{s\to\infty} h_1(s) = \lim_{s\to\infty} h_2(s) = c$, then h_1-**MC** $= h_2$-**MC**.*

Proof. Let $h_1, h_2 : \mathbb{N} \to (c, \infty)_{\mathbb{Q}}$ be computable functions such that $\lim_{s\to\infty} h_1(s) = \lim_{s\to\infty} h_2(s) = c$. For any $s \in \mathbb{N}$, since $h_2(n) > c$, there exist infinitely many t such that $h_1(t) \leq h_2(s)$. By Theorem 3.4, we have h_1-**MC** $\subseteq h_2$-**MC**. Similarly, we can show that h_2-**MC** $\subseteq h_1$-**MC** holds too. □

7 ω-Monotonically Computable Real Numbers

In the preceding sections we discussed the h-mc real numbers for computable functions h which converge to constants. All these real numbers are weakly computable as shown in Theorem 3.1. The situation is of course different if we consider also unbounded computable functions h, because it is shown in [8] that the class of ω-mc real numbers is incomparable with **WC**. In this section we will discuss the ω-mc real numbers. Naturally, we are interested in some kind of hierarchy of ω-**MC** by different classes h-**MC** according to the increasing speed of h. However we will see that this is impossible, because for any unbounded monotone computable function h we have already h-**MC** $= \omega$-**MC**.

Theorem 7.1 *Let $h : \mathbb{N} \to \mathbb{Q}^+$ be a monotone and unbounded computable function. Then h-**MC** $= \omega$-**MC**.*

Proof. Suppose that $h : \mathbb{N} \to \mathbb{Q}^+$ is a monotone and unbounded computable function. We are going to show that ω-**MC** $\subseteq h$-**MC**. Let x be an ω-mc real number. Then there is a computable function $g : \mathbb{N} \to \mathbb{Q}$ and a computable sequence (x_s) of rational numbers which converges g-monotonically to x. Since h is unbounded, we can define an increasing computable sequence (n_i) of natural numbers such that $h(n_i) \geq g(i)$ for all $i \in \mathbb{N}$. Let (y_s) be a computable sequence of rational numbers defined by $y_s := x_{n(s)}$ for any $s \in \mathbb{N}$, where $n(s)$ is the maximum i such that $n_i \leq s$. Then, the sequence (y_s) converges obviously also to x. Now we will show that the sequence (y_s) convergence actually h-monotonic to x.

For natural numbers $s < t$, if $n(s) = n(t)$, i.e., there is an i such that $n_i \leq s < t < n_{i+1}$ and $n(s) = i$, then $h(s)|x - y_s| = h(s)|x - x_i| \geq |x - x_i| = |x - y_t|$. Otherwise, if $n(s) < n(t)$, then there are $i < j$ such that $n_i \leq s < n_j \leq t$ and $n(s) = i < j = n(t)$. Since h is increasing, this implies that $h(s)|x - y_s| \geq h(n_i)|x - x_i| \geq g(i)|x - x_i| \geq |x - x_j| = |x - y_t|$. Therefore, x is h-monotonically computable. □

Corollary 7.2 *Let $h : \mathbb{N} \to \mathbb{Q}$ be a computable function. If there exists an unbounded monotone computable function g such that $g(n) \leq h(n)$ for all n, then h-**MC** $= \omega$-**MC**.*

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