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Upper bounds on ideals in the computably enumerable Turing degrees

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ABSTRACT

We study ideals in the computably enumerable Turing degrees, and their upper bounds. Every proper Σ_4^0 ideal in the c.e. Turing degrees has an incomplete upper bound. It follows that there is no Σ_4^0 prime ideal in the c.e. Turing degrees. This answers a question of Calhoun (1993) [2]. Every proper Σ_3^0 ideal in the c.e. Turing degrees has a low$_2$ upper bound. Furthermore, the partial order of Σ_3^0 ideals under inclusion is dense.

1. Introduction

Let $(U, \leq, \vee)$ be an upper semilattice. A set $I \subseteq U$ is an *ideal* if I is closed downwards, and I is closed under the join operation $\vee$. We study ideals in the c.e. Turing degrees. Our motivation is manifold. Firstly, collections of ideals form natural extensions of the degree structure. Secondly, some important degree classes are ideals, such as being half of a minimal pair in the c.e. degrees, or being the degree of a K-trivial set in the Δ_2^0 degrees. The latter example shows that the notion of an ideal can be seen as an abstract framework for certain lowness properties, i.e. properties saying that a set is close to being computable.

An *upper bound* of an ideal I is a degree $\mathbf{b}$ such that $I \subseteq [\mathbf{0}, \mathbf{b}]$. Our leading question is the following:

Let I be a proper ideal with a certain type of effective presentation. What can we say about upper bounds of I?

It was motivated by the view of ideals as abstract lowness properties, for in that case one would expect upper bounds that are far from Turing complete. Indeed, we use a general result in this direction to show that some c.e. low$_2$ set is Turing above all the K-trivial sets. A further result enables us to answers a question of Calhoun [2] in the negative: there is no Σ_4^0 proper prime ideal in the c.e. degrees.

Terminology for ideals. Fix an upper semilattice $(U, \leq, \vee)$. The ideal *generated* by a set $S \subseteq U$ consists of the elements of U that are below finite joins of elements in S. The set of ideals of U is a lattice, where the meet of I, J is the intersection, and the join is the ideal generated by $I \cup J$. Thus, $I \vee J = \{x \in U \colon \exists y \in I\, \exists z \in J\, [x \leq y \vee z]\}$. An ideal I is called *proper* if $I \neq U$.

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Table 1
Bounds for a proper ideal I in the c.e. degrees.

Effectivity condition on I	Bound for I	Reference
Uniformly superlow generated	Superlow	Proposition 3.1 (ii)
Uniformly low generated	Low	Proposition 3.1 (i)
Σ_3^0	Low$_2$	Theorem 4.4
Σ_4^0	Incomplete	Theorem 2.1

Each $u \in U$ determines the ideal $\hat{u} = \{x \colon x \leq u\}$, called a *principal ideal*. The map $u \to \hat{u}$ is a usl embedding of U into its ideal lattice.

Describing ideals. There are two interrelated approaches for describing a certain ideal I in the c.e. degrees. Similar observations apply to the Δ_2^0 degrees.

(a) One can generate I by using a uniformly c.e. sequence; such an ideal is said to be *uniformly generated*. The class of uniformly generated ideals is closed under join in the lattice of ideals.
(b) One can describe its index set $\Theta I = \{e\colon \text{the degree of } W_e \text{ is in } I\}$ within the arithmetical hierarchy. If ΘI is Σ_k^0 etc. we say that I is a Σ_k^0 *ideal*. Each principal ideal is Σ_4^0. For $k \geq 4$ the class of Σ_k^0 ideals forms a sublattice of the lattice of ideals.

Each Σ_3^0 ideal is uniformly generated by Yates [23]. Each uniformly generated ideal is Σ_4^0. We will see that the converse implications fail.

Main results. By the Thickness Lemma (see [20]) every proper uniformly generated ideal has an incomplete upper bound. We strengthen and vary this in several ways in order to address our leading question.

- Firstly, each proper Σ_4^0 ideal in the c.e. degrees has an incomplete upper bound (Theorem 2.1). This strengthens the Thickness Lemma result for uniformly generated ideals. The proof makes essential use of the fact that there is a high non-cuppable degree (see [11]).
- Secondly, each proper Σ_3^0 ideal in the c.e. degrees has a low$_2$ upper bound (Theorem 4.4). We first prove in Lemma 4.1 the useful fact that each uniformly c.e. subsequence of a proper Σ_3^0 ideal is uniformly low$_2$.

A summary of the results is given in Table 1. This relies on some definitions. Recall that a set A is *low* if $A' \leq_T \emptyset'$ and *superlow* if $A' \leq_{tt} \emptyset'$ (equivalently, $A' \leq_{wtt} \emptyset'$). An index for reduction procedures showing these properties is called a *lowness index* and a *superlowness index* of A, respectively.

Definition 1.1. We say that a uniformly c.e. sequence of sets (A_i) is *uniformly low* if given an input e one can compute a lowness index for $\oplus_{i \leq e} A_i$; it is *uniformly superlow* if from e one can compute a superlowness index for $\oplus_{i \leq e} A_i$. We will also apply these definitions to sequences of c.e. degrees.

Definability and global properties. In earlier investigations of ideals, researchers focused on definability, and on the global properties of ideal lattices. A few proper ideals are known to be first-order definable without parameters in the c.e. degrees. The classic examples are: the ideal of cappable degrees (i.e. the halves of minimal pairs), and its subideal, the non-cuppable degrees. Nies [14] showed that the ideal generated by a definable set is also definable. Applying this, Yu and Yang [24] found further examples, for instance, the ideal generated by the non-bounding degrees. It is still unknown whether infinitely many ideals are definable without parameters.

Nies [14] shows that the Σ_k^0 ideals, for fixed $k \geq 7$, are uniformly definable with parameters in the c.e. degrees. He also proves that the single ideal of non-cuppable degrees is definable in the ideal lattice, as well as in each lattice of Σ_k^0 ideals, for fixed $k \geq 6$: it is the infimum of all maximal ideals.

Prime ideals. Let U be an usl with least element 0. An ideal $I \subseteq U$ is called a *prime ideal* if below any two elements of $U - I$ there is a further element of $U - I$.

Calhoun [2] constructed a uniformly Δ_5^0 sequence of incomparable prime ideals in the c.e. Turing degrees. There is no easy way to decrease the complexity in his construction. On the other hand, by [1] the cappable degrees form a prime ideal in the c.e. degrees. Schwarz [19] showed that this ideal is Π_4^0 complete. Prompted by this, Calhoun [2] asked whether there is a Σ_4^0 prime ideal in the c.e. degrees. Using Theorem 2.1, stating that every proper Σ_4^0 ideal I in the c.e. degrees has an incomplete upper bound, we answer this question in the negative: by Welch [22], for every c.e. degree $\mathbf{b} < \mathbf{0}'$ there is a minimal pair of c.e. degrees $\mathbf{a}_0, \mathbf{a}_1$ such that $\mathbf{a}_i \not\leq \mathbf{b}$. So, I cannot be a prime ideal.

By Welch's result the Π_4^0 ideal of cappable degrees does not have an incomplete upper bound. So Theorem 2.1 is optimal in terms of quantifier complexity. In this context, it is unknown whether the ideal generated by the cappable degrees and the K-trivial c.e. degrees (or the strongly jump traceables) is prime. This ideal is proper because no cappable degree is low cuppable. We also do not know of a further proper Π_4^0 prime ideal.

Ideals and randomness. Recent interest in ideals of the c.e. degrees, or of the Δ_2^0 degrees, stems from the discovery of natural ideals for these degree structures. They often arise via concepts related to randomness, a field full of surprising coincidence results for degree classes. Frequently, such results state in fact the coincidence of ideals.

For the following, we refer the reader to [17] for definitions. The K-trivial sets were introduced by Chaitin [3]. He proved that each K-trivial set is Δ_2^0, while Solovay [21] constructed a K-trivial set that is not computable. The K-trivial sets coincide with the sets that are low for ML-randomness, the sets that are low for K, and the bases for ML-randomness [15,9]. In [5,15] it is shown that the K-trivial sets induce an ideal $\mathbf{K}$ in the Δ_2^0 degrees. Each K-trivial set is Turing (even truth-table) below a c.e. K-trivial set [15]. Thus, $\mathbf{K}$ is fully determined by its intersection with the c.e. degrees.

K-trivial sets are computationally weak. This intuition leads to several results on upper bounds of the ideal $\mathbf{K}$. For example, Nies [16] showed that $\mathbf{K}$ does not have a low c.e. upper bound. On the other hand, Theorem 4.4 implies that $\mathbf{K}$ has a low$_2$ upper bound. Miller and Nies [12] asked whether there is a low upper bound at all for the K-trivial degrees. Kučera and Slaman [10] answered this question in the affirmative. They also gave a characterization of the ideals in the Turing degrees which have a low upper bound.

A further natural ideal in the c.e. degrees is the ideal $\mathbf{S}$ induced by the strongly jump traceable c.e. sets, introduced in [6]. The ideal $\mathbf{S}$ is a proper subideal of $\mathbf{K}$ by [4]. By [13] $\mathbf{S}$ is Π_4^0 complete. $\mathbf{S}$ coincides with the degrees of c.e. sets below all superlow ML-random sets, and also with the degrees of c.e. sets below all superhigh ML-random sets [18,7].

Some Σ_3^0 ideals lie strictly between $\mathbf{S}$ and $\mathbf{K}$. For instance, let Y be a superlow Martin-Loef random set, and let B be a c.e. K-trivial set such that $B \not\leq_T Y$ (see [17, Ex. 8.5.25]). Then the c.e. sets Turing below Y induce an ideal as desired. However, currently no "natural" ideal is known to lie properly between $\mathbf{S}$ and $\mathbf{K}$. A promising candidate is the ideal induced by the c.e. sets Turing below each of the a.e. dominating Martin-Loef random sets. This ideal is contained in $\mathbf{K}$, and known to differ from $\mathbf{S}$ by [8].

Some open questions on ideals.

The following questions are currently open, but not necessarily hard. A few further questions relating to particular results are scattered through the text.

1. Is every Σ_4^0 ideal the intersection of the principal ideals that it is contained in? (This would strengthen Theorem 4.4.) Is every Π_4^0 ideal the intersection of the Σ_3^0 ideals containing it?
2. Study the quotient usls of the c.e. degrees modulo the ideals $\mathbf{K}$ and $\mathbf{S}$. Are they dense?
3. Given $k \geq 4$, is the class of principal ideals definable in the lattice of Σ_k^0 ideals?
4. Is the ideal of capable degrees definable in the lattices of Σ_k^0 ideals, for $k \geq 5$?
5. [12] Are there c.e. degrees $\mathbf{a}$, $\mathbf{b}$ such that $\mathbf{K} = [\mathbf{0}, \mathbf{a}] \cap [\mathbf{0}, \mathbf{b}]$?

2. Bounds for Σ_4^0 ideals

Theorem 2.1. *Every proper Σ_4^0 ideal $\mathbb{I}$ of the c.e. degrees has an incomplete upper bound.*

Proof. We build a Turing incomplete c.e. set B such that the degree of B is an upper bound for $\mathbb{I}$.

A c.e. degree $\mathbf{h}$ is called *non-cuppable* if $\mathbf{h} \vee \mathbf{w} < \mathbf{0}'$ for all c.e. $\mathbf{w} < \mathbf{0}'$. By a result of Miller [11] there is a high non-cuppable c.e. degree $\mathbf{h}$. The ideal generated by $\mathbb{I} \cup \{\mathbf{h}\}$ is proper. Hence, replacing $\mathbb{I}$ by this ideal if necessary, we may assume that $\mathbb{I}$ already contains the high degree $\mathbf{h}$.

Our proof combines techniques reminiscent of the Thickness Lemma [20] with the idea of using such a high member of the given ideal as to reduce the relative arithmetical complexity of its index set. Let $\mathcal{I}$ be the set of representatives of the degrees in $\mathbb{I}$. Also, let (W_e) be an effective list of all the c.e. sets. Since $\mathcal{I}$ is Σ_4^0, there exists a Π_3^0 relation P such that $W_e \in \mathcal{I} \iff \exists n P(e, n)$. Let H be a set in $\mathbf{h}$. Since $H' \geq_T \emptyset''$, we have $\Sigma_1^0(\emptyset') \subseteq \Delta_2^0(H)$ and therefore $\Sigma_4^0 \subseteq \Sigma_3^0(H)$. Hence there exists a uniformly c.e. sequence of operators $(V_{e,n})$ such that

$$W_e \in \mathcal{I} \iff \exists n\, V_{e,n}^H = \mathbb{N}$$

and for all sets X and $e, n \in \mathbb{N}$ the set $V_{e,n}^X$ is an initial segment of $\mathbb{N}$. Let $\langle \cdot, \cdot \rangle : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ be a computable bijection such that $\langle e, n \rangle \geq e, n$ for all $e, n \in \mathbb{N}$.

In order to build B as desired, it suffices to meet the requirements

$$C_{\langle e,n \rangle} : V_{e,n}^H = \mathbb{N} \;\Rightarrow\; W_e \leq_T B \oplus H.$$

To make B Turing incomplete, it suffices to meet the requirements

$$N_m : \emptyset' = \Phi_m^B \;\Rightarrow\; \exists k \exists e_0, \ldots, e_{k-1} [\emptyset' \leq_T \oplus_{i<k} W_{e_i} \oplus H \;\wedge\; \forall i\, W_{e_i} \in \mathcal{I}].$$

This condition says that, if B is complete, then the ideal given by $\mathcal{I}$ is not proper. The sets W_{e_i}, $i < k$, will be the members of $\mathcal{I}$ that are coded into B through higher priority requirements. Here the priority ordering of the requirements is $C_0, N_0, C_1, N_1, \ldots$.

Strategies. To satisfy a requirement $C_{\langle e,n \rangle}$, we explicitly define a family of codes $(c_{e,n}(i))$. Let $\mathbb{N}^{[\langle e,n,i \rangle]} = \{\langle e, \langle n, \langle i, x \rangle \rangle \rangle \mid x \in \mathbb{N}\}$. For each $i, s \in \mathbb{N}$, if $i \in V_{e,n}^H[s]$ let $c_{e,n}(i)[s]$ be the least number in $\mathbb{N}^{[\langle e,n,i \rangle]}$ which is larger than the H-use of '$i \in V_{e,n}^H[s]$' and larger than the stage $t < s$ where this computation converged. If $i \notin V^H_{e,n}[s]$ let $c_{e,n}(i)[s]$ be undefined. In the construction, if i is enumerated in W_e at stage s and $c_{e,n}(i)[s]\downarrow$, we will enumerate $c_{e,n}(i)[s]$ into B. If $V^H_{e,n} = \mathbb{N}$ this coding gives a reduction of W_e to $B \oplus H$: to decide whether $i \in W_e$ we compute a stage s such that $i \in V^H_{e,n}$ where H has the final value up to the use, and output 'yes' iff $c_{e,n}(i)[s]$ is in B or $i \in W_e[s]$.

If $W_e \notin \mathcal{I}$ then merely a computable set is coded into B. Indeed, in this case $V^H_{e,n}$ will be a finite set (an initial segment of $\mathbb{N}$) so almost all markers will be either eventually permanently undefined or infinitely often defined on values that tend to infinity. By the rules that govern the movement and definition of the values of the markers $c_{e,n}(i)$, for each $n \in \mathbb{N}$ we can calculate a stage by which either n has been enumerated in B or no marker will ever take a value $\leq n$ at any later stage. This shows that if $W_e \notin \mathcal{I}$ then the corresponding strategy enumerates a computable set in B. This feature will be crucial in meeting the N_m requirements.

The length-of-agreement function for $\Phi^B_m = \emptyset'$ is

$$\ell_m[s] = \max\{k\colon \Phi^B_m[s] \restriction k = \emptyset'_s \restriction k\}.$$

The strategy for N_m at stage s is to restrain B up to the use of the computations $\Phi^B_m(x)$ for $x < \ell_m[s]$. Thus the restraint of N_m at stage s is

$$r_m[s] = \max\{\text{use } \Phi^B_m(x)[s]\colon x < \ell_m[s]\}.$$

We use the Lachlan hat trick (see [20]) for the functionals $\Phi^B_m, m \in \mathbb{N}$. Thus $\Phi^B(n)[s]\downarrow = k$ with use u only if $\langle B_s \restriction u, k\rangle \in \Phi_s$ *and* u is no larger than the number entering B at stage s. In particular, if $\Phi^B(n)[s]\downarrow$ with use u and $B_{s+1} \restriction u \neq B_s \restriction u$ then $\Phi^B(n)[s+1]\uparrow$. This will ensure that for each m_0, the combined restraint $\max_{m\leq m_0} r_m[s]$ has finite lim inf taken over all stages s.

Construction. At stage $s+1$, for each $e, n, t < s$ do the following. If $t \in W_e[s+1] - W_e[s]$ put $c_{e,n}(t)[s+1]$ into B, unless one of the following holds:

- $c_{e,n}(t)[s+1]\uparrow$.
- $B_s \cap \mathbb{N}^{[\langle e,n,t\rangle]} \neq \emptyset$.
- $c_{e,n}(t)[s+1] \leq r_m[s]$ for some $m < \langle e, n\rangle$.

Claim 1. *Each requirement N_m is met.*

Only the finitely many coding strategies $C_{\langle e,n\rangle}$ of higher priority than N_m can enumerate into B and destroy computations Φ^B_m on arguments that are less than the current value of ℓ_m. We say that a strategy $C_{\langle e,n\rangle}$ is *active* if $V^H_{e,n} = \mathbb{N}$. For the $C_{\langle e,n\rangle}$ of higher priority than N_m which are not active let $t_{\langle e,n\rangle} \in \mathbb{N}$ be least such that $t_{\langle e,n\rangle} \notin V^H_{e,n}$. For these coding strategies $C_{\langle e,n\rangle}$ and all $t \geq t_{\langle e,n\rangle}$ we have that either $c_{e,n}(t)$ is redefined infinitely often (to larger and larger values) or that at some point it remains permanently undefined. Let s_0 be a stage where $W_e \restriction t_{\langle e,n\rangle}$ has settled for each non-active higher priority coding strategy. Suppose that $\Phi^B_m = \emptyset'$. To compute $\emptyset'(n)$ for each $n \in \mathbb{N}$ from the finitely many sets W_e in the active strategies of higher priority, find a stage $s > s_0$ where $\Phi^B_m(n)[s]\downarrow$ and for any $c_{e,n}(t)$, below the use of this computation such that $\langle e, n\rangle \leq m$, one of the following holds:

- it belongs to an active coding strategy $C_{\langle e,n\rangle}$ and $c_{e,n}(t)[s] \in B[s] \leftrightarrow t \in W_e$,
- it belongs to a non-active coding strategy $C_{\langle e,n\rangle}$ and $t \leq t_{\langle e,n\rangle}$.

Notice that such a stage s has to exist. According to the strategies, $\Phi^B_m(n)$ will be preserved after s, thus giving the correct value of $\emptyset'(n)$. This shows Claim 1.

Since $\mathcal{I}$ is a proper ideal it follows that for each m there is a least $k_m \in \mathbb{N}$ such that $\Phi^B_m(k_m) \neq \emptyset'(k_m)$.

Claim 2. *Each requirement $C_{\langle e,n\rangle}$ is met.*

Suppose that $V^H_{e,n} = \mathbb{N}$. For each $m < \langle e, n\rangle$ let u_m be the use of $\Phi^B_m \restriction k_m$. Let u be the maximum of all u_m for $m < \langle e, n\rangle$. There are infinitely many true stages in the enumeration of B (namely, stages s such that for the least number n enumerated into B at s we have $B[s] \restriction n = B \restriction n$). By the hat trick we have $r_m[s] = u_m$ for each true stage s and each $m < \langle e, n\rangle$.

We give a procedure which computes $W_e(i)$ for $i > u$ using $B \oplus H$ as an oracle. Notice that the $c_{e,n}(i)$, $i > u$, take values $> u$. Given $i > u$ find a stage s_0 such that $i \in V^H_{e,n}[s]$ with H correct up to the use. Notice that $c_{e,n}(i)[s_0]\downarrow$ and this parameter has reached its final value. Note that $i \in W_e$ iff $i \in W_e[s_0]$ or $c_{e,n}(i)[s_0] \in B$. For, we have $c_{e,n}(i)[s_0] > u$, and if i enters W_e at some stage $s_1 > s_0$ then we enumerate $c_{e,n}(i)[s_0]$ into B by the first true stage after s_1. $\square$

3. Bounds for uniformly low computably enumerable sequences

This section applies a technique of Robinson to exploit lowness of a c.e. set. In Definition 1.1 we defined uniformly (super)low c.e. sequences.

Proposition 3.1. (i) *Every uniformly low c.e. sequence of degrees has a low upper bound.*
(ii) *Every uniformly superlow c.e. sequence of degrees has a superlow upper bound.*

As the methods are standard, we only sketch the proof of Proposition 3.1(i). Let (A_i) be a uniformly low sequence of c.e. sets. We wish to construct a c.e. set A such that $A' \leq_T \emptyset'$ and $A_i \leq_T A$ for each i. For the coding of A_e into A we use the eth column of $\mathbb{N}$: we ensure that

$$P_e : n \in A_e \iff \langle e, n\rangle \in A \quad \text{for almost all } n.$$

To make A low we define a $\{0, 1\}$-valued computable function g such that $\lim_s g(e, s)$ exists for each e and

$$L_e : \Phi_e^A(e) \downarrow \iff \lim_s g(e, s) = 1.$$

Recall that (Φ_e) is an effective enumeration of all Turing functionals. We assume the hat trick for these functionals, as in Section 2. If $g(e, s) = 1$ at some stage s, the computation $\Phi_e^A(e)[s]$ halts with some use u. In this case the strategy L_e imposes a restraint u on A. This restraint has to be respected by all $P_j, j > e$. The priority ordering of the strategies is $P_0, L_0, P_1, L_1, \ldots$.

In the construction we enumerate a uniformly c.e. sequence U_e of sets of strings. The strings in U_e can be thought of as guesses for prefixes of $\oplus_{j\le e}A_j$. Since the sequence (A_i) is uniformly low, there is a computable $\{0, 1\}$-valued function $\widetilde{h}$ such that $\lim_s \widetilde{h}(e, t, s)$ exists for each $e, t \in \mathbb{N}$ and is 1 iff there exists a prefix of $\oplus_{j\le e}A_j$ in W_t. By the recursion theorem the construction can use an index of a computable function f such that $W_{f(i)} = U_i$ for all $i \in \mathbb{N}$. Let $h(e, s) = \widetilde{h}(e, f(e), s)$. Then $\lim_s h(e, s)$ equals 1 iff there exists a prefix of $\oplus_{j\le e}A_j$ in U_e. The objects h, (U_e) are used in the definition of g.

Construction. At stage s we attend to P_j, L_j for $j < s$ in the order of priority.

When we attend P_j, we enumerate into A the codes $\langle e, n\rangle$ for all $n \in A_s - A_{s-1}$ such that $\langle e, n\rangle$ is larger than the restraints of L_k, $k < j$.

When we attend L_j, we define $g(e, s)$ as follows. If $\Phi_e^A(e)$ is currently undefined, we let $g(e, s) = 0$. If it is defined with use u, we put the string $\sigma := \oplus_{j\le e}A_j[s] \upharpoonright u$ into U_e. Then we look for a stage $s' > s$ such that either $h(e, s') = 1$ or σ is not a prefix of $\oplus_{j\le e}A_j[s']$ and $h(e, s') = 0$. In the first case we define $g(e, s) = 1$ (and let L_e impose restraint u). In the second case we define $g(e, s) = 0$.

When we finish attending P_j, L_j for $j < s$, we proceed to the least stage s'' which is greater than all stages s' used while attending $L_j, j < s$. During the stages in between s and s'', the parameters of the construction remain frozen.

Verification. By the recursion theorem and the properties of h, it is not hard to see that $\lim_s g(e, s)$ exists for all e, and L_e is satisfied. In particular, the restraint imposed by L_e reaches a finite limit. Therefore, each P_e is satisfied.

(ii) Suppose the sequence (A_i) is uniformly superlow. Then there is a computable function d such that, for all e, $d(e)$ bounds the number of stages s such that $h(e, s) \neq h(e, s + 1)$. Notice that the number of stages s such that $g(e, s) \neq g(e, s + 1)$ is bounded by the number of stages s such that $h(e, s) \neq h(e, s + 1)$. Therefore this number is now bounded by $d(e)$. Hence A is superlow.

4. Every proper Σ_3^0 ideal has a low$_2$ upper bound

Besides proving the upper bound result for proper Σ_3^0 ideals in Theorem 4.4, in Proposition 4.2 we will separate the classes of Σ_3^0, uniformly generated, and Σ_4^0 ideals from the introduction.

We say that a uniformly c.e. sequence $(Y_k)_{k\in\mathbb{N}}$ is *uniformly low*$_2$ if, given input e, one can compute an index for a Turing reduction showing $(\oplus_{k\le e}Y_k)'' \le_T \emptyset''$. The results in this section rely on the crucial lemma that a uniformly c.e. subsequence of a proper Σ_3^0 ideal is uniformly low$_2$. Let the set $\mathtt{Tot}^B$ consist of the indices for Turing functionals which yield total functions with oracle B. Note that $\mathtt{Tot}^B$ is $\Pi_2^0(B)$ complete.

Lemma 4.1. *Let $\mathbb{I}$ be a proper Σ_3^0 ideal in the c.e. Turing degrees. If (Y_k) is a uniformly c.e. sequence such that $\deg(Y_k) \in \mathbb{I}$ for each k, then (Y_k) is uniformly low$_2$.*

Proof. Without loss of generality, we may assume that $Y_{k+1} \cap 2\mathbb{N} = Y_k$ for each k. Indeed, from the given uniformly c.e. sequence of sets we can effectively obtain the new one. Clearly any set in the given sequence is computable from a finite number of sets in the new sequence and vice versa. Hence the two sequences generate the same ideal. Then it is sufficient to show that $\mathtt{Tot}^{Y_k}$ is Σ_3^0 uniformly in k.

Without changing $\mathtt{Tot}^{Y_k}$, for the functionals (Φ_e) we may assume that $\Phi_e^X(v) \downarrow$ implies $\Phi_e^X(i) \downarrow$ for all $i < v$. Also, we may assume the hat trick as in Section 2. Since $\mathbb{I}$ is a proper Σ_3^0 ideal, it suffices to define a uniformly c.e. sequence $(U_{k,n})$ such that for each k, n we have

- if $n \in \mathtt{Tot}^{Y_k}$ then $\deg(U_{k,n}) \in \mathbb{I}$,
- if $n \notin \mathtt{Tot}^{Y_k}$ then $U_{k,n} =^* \emptyset'$.

At stage s, for each $n, k < s$ if $v \in \emptyset'_s$ and $\Phi_n^{Y_k}(v)[s] \uparrow$, enumerate v into $U_{k,n}$.

If $n \in \mathtt{Tot}^{Y_k}$ then $U_{k,n} \le_T Y_k$, so $\deg(U_{k,n}) \in \mathbb{I}$. On the other hand, by the conventions on (Φ_n), if $n \notin \mathtt{Tot}^{Y_k}$ then for almost every argument m the computation $\Phi_n^{Y_k}(m)[s]$ is undefined for infinitely many stages s. Hence $U_{k,n} =^* \emptyset'$. $\square$

Recall that every Σ_3^0 ideal in the c.e. Turing degrees is uniformly generated by [23].

Proposition 4.2. (i) *Some uniformly generated ideal in the c.e. Turing degrees is not Σ^0_3.*
(ii) *Some Σ_4 ideal in the c.e. Turing degrees is not uniformly generated.*
(iii) *Some Σ_4 ideal in the c.e. Turing degrees is not the join of two Σ_3 ideals.*

Proof. Part (i) follows from the result [23] that the principal ideal below a c.e. degree is Σ^0_3 iff the degree is low$_2$. Thus, the principal ideal below a non-low$_2$ c.e. degree is uniformly generated, but not Σ^0_3.

For part (ii), consider a low$_2$ non-computable set A and an independent uniformly c.e. and uniformly low sequence (B_i) of sets such that $B_i \leq_T A$. (To be independent means that no member of the sequence is Turing reducible to a finite join of other members of the sequence. Such a sequence exists by a basic result in the theory of c.e. degrees; see Exercise 4.8 in [20].) Also, let I be a properly Σ^0_4 set of numbers. Now consider the ideal $\mathbf{I}$ generated by $\{\mathbf{b}_i \mid i \in I\}$, where $\mathbf{b}_i$ is the degree of B_i. The uniform lowness of the sequence (B_i) implies that $\mathcal{I}$ is Σ^0_4. On the other hand, since the sequence is independent, we have $i \in I \iff \mathbf{b}_i \in \mathbf{I}$. If $\mathbf{I}$ is uniformly generated, there is a uniformly c.e. sequence generating it. Notice that the ideal $\mathbf{I}$ is contained in the Σ^0_3 ideal of the c.e. sets that are computable in A. By Lemma 4.1 the uniformly c.e. sequence generating $\mathbf{I}$ has to be uniformly low$_2$. But then I is Σ^0_3, a contradiction.

Part (iii) is now an immediate consequence of the facts that each Σ^0_3 ideal is uniformly generated, and that the uniformly generated ideals are closed under join. □

We say that $(Y_k)_{k\in\mathbb{N}}$ is *uniformly monotonic* if $Y_k \leq_T Y_{k+1}$ for all $k \in \mathbb{N}$ and the indices for these reductions are given by a computable function. As is explained in Lemma 4.1, given a uniformly c.e. sequence of sets one can effectively obtain a uniformly monotonic and c.e. sequence of sets which generates the same ideal. Notice that a uniformly c.e. and uniformly monotonic sequence $(Y_k)_{k\in\mathbb{N}}$ is uniformly low$_2$ iff, given e, we can compute an index for a reduction showing $Y_e'' \leq_T \emptyset''$. The main work is to prove the following variant of Proposition 3.1.

Lemma 4.3. *Suppose that the uniformly c.e. sequence $(Y_k)_{k\in\mathbb{N}}$ is uniformly low$_2$ and uniformly monotonic. Then there is a low$_2$ c.e. set B such that $Y_k \leq_T B$ for each k.*

Combining Lemmas 4.1 and 4.3 yields the main result of this section.

Theorem 4.4. *Every proper Σ^0_3 ideal $\mathbb{I}$ in the c.e. Turing degrees has a low$_2$ upper bound.*

Proof. There is a Σ^0_3 set $I \subseteq \mathbb{N}$ such that $\mathbb{I} = \{\deg_T(W_i) \mid i \in I\}$. (Here $\deg_T(W)$ denotes the Turing degree of W.) By Yates [23] there is a uniformly c.e. sequence of sets (Y_k) whose degrees generate $\mathbb{I}$. By Lemma 4.1 this sequence is uniformly low$_2$. Moreover as explained in the proof of Lemma 4.1 we can assume that it is uniformly monotonic, without loss of generality. Now we apply Lemma 4.3. □

We prove Lemma 4.3 by building an appropriate c.e. set B. We meet the coding requirements

$$R_i: \text{ there is a Turing functional } \Gamma \text{ such that } Y_i = \Gamma^B.$$

We also meet requirements L_e which together ensure that $\mathtt{Tot}^B$ is Δ^0_3. For each e, requirement L_e will uniformly give a procedure for computing whether e is in $\mathtt{Tot}^B$ from $\emptyset''$.

We have a tree of strategies. The true path is Δ^0_3, and computes $\mathtt{Tot}^B$. At stage s we have an approximation δ_s to the true path. Let $\preceq$ denote the prefix relation amongst nodes on the tree. Given a node η, stage s is an η stage if $s = 0$ or $\eta \preceq \delta_s$.

Consider an L_e strategy α and an R_i strategy μ on the tree, such that $\alpha \preceq \mu$. Let Γ_μ be the Turing functional built by μ and let the use function of a functional be denoted by the corresponding lower case Greek letter. As usual, (Φ_e) denotes an effective sequence of all Turing functionals. If $\Phi_e^Y(x)[s]\downarrow$ then by convention $e, x < s$.

The basic idea is to ensure that $\gamma_\mu^B(x) \geq \phi_e^B(x)$, for each x. For, in this case we can expect the destructions of computations $\Phi_e^B(x)$ due to μ's coding into B to be controllable. However, if $\Phi_e^B(x)\uparrow$, this would make $\Gamma_\mu^B(x)$ undefined. So μ needs a guess at whether Φ_e^B is total. This will depend on the coding into B of the R_j strategies $\upsilon \prec \alpha$.

Let $\sigma * \tau$ denote the concatenation of strings σ, τ. Since the Y_k are uniformly low$_2$, we are able to express totality of Φ_e^B as a Σ^0_3 statement. The strategy μ has a guess at a witness n for this statement. Thus μ only has to respect α when $\alpha * n \preceq \mu$ for such a witness n.

We give some more detail. We write $\mu : R_j$ to indicate that μ is an R_j-strategy. Similarly for L_e. The priority ordering of the requirements is $R_0, L_0, R_1, L_1, \ldots$. The strategies $\mu : R_j$ have only one outcome, namely 0. The strategies $\alpha : L_e$ have outcomes

$$0 < 0' < 1 < 1' < \cdots < \mathtt{fin}.$$

Outcome $\mathtt{fin}$ indicates that the size of the domain of Φ_e^B only increases finitely often. An α-stage s such that $\alpha * \mathtt{fin} \not\preceq \delta_s$ is called an $\alpha * \infty$ stage. Outcome n' indicates that $\Phi_e^B(n)\uparrow$. Outcome n indicates that n is a witness for the Σ^0_3 statement related to the totality of Φ_e^B. The main technical definition is the following.

Definition 4.1. Let $\alpha : L_e$. We say that x is *α-good* at an $\alpha * \infty$ stage $s > 0$ if

$$B \upharpoonright \phi_e^B(x)[t] = B \upharpoonright \phi_e^B(x)[s]$$

where t is the greatest $\alpha * \infty$ stage less than s, and for all υ where $\upsilon : R_j$ and $\upsilon \preceq \alpha$

$$\forall z\, [\gamma_\upsilon(z)[s] < \phi_e(x) \Rightarrow Y_j(z) = Y_j(z)[s]]. \tag{4.1}$$

Informally, this means that the $\phi_e^B(x)$ computations remain the same between stages t, s and thus we decide to preserve them. Note that, by the choice of (Y_k), if j is the largest index occurring in (4.1), Y_j can decide which arguments x are α-good.

By the recursion theorem applied to an index for the construction and the fact that the Y_k are uniformly low$_2$, there is a uniformly c.e. sequence $(S_{\alpha,n})$, where α ranges over the L strategies of the tree and $n \in \mathbb{N}$, of initial segments of $\mathbb{N}$ such that

$$\exists n\, S_{\alpha,n} = \mathbb{N} \iff \forall x \exists s > x\, [x \text{ is } \alpha\text{-good at } s]. \tag{4.2}$$

If α is an L_e node, then an α-stage s is called α-expansionary if the largest initial segment of $\mathbb{N}$ where Φ_e^B is defined has increased since the last α-stage. A number is called *large* at stage s of the construction if it is larger than any number that was mentioned in stages $< s$ of the construction.

4.1. Construction

At stage $s > 0$ determine δ_s. At the end of this stage, initialize all strategies β which lie to the left of δ_s (i.e. erase any computations that they have created). Suppose that we have determined $\eta = \delta_s \upharpoonright k$, where $k < s$.

Case $\eta : L_e$. Let $\alpha = \eta$. Let t be the greatest $\alpha * \infty$ stage less than s. If s is not expansionary, let $\delta_s(n) = \mathtt{fin}$. Otherwise let σ be the least possible outcome of α such that one of the following holds:

- $\sigma = n$ and $|S_{\alpha,n,s}| > |S_{\alpha,n,t}|$,
- $\sigma = m'$ and there exists $v \in \mathbb{N}$ such that $t < v < s$ and $\Phi_e^B(m)[v] \uparrow$.

Let $\delta_s(k) = \sigma$.

Case $\eta : R_i$. Let $\mu = \eta$. Let $t < s$ be the greatest μ-stage. Also let k_μ be the natural number coding the string μ.

1. (Correcting) If $\Gamma_\mu^B(x)[t] \downarrow$ and $\Phi_e^B(x + k_\mu)[t]$ was destroyed, then declare $\Gamma_\mu^B(x)[s]$ undefined.

2. (Coding into B) If $\Gamma_\mu^B(x)[t]$ is defined and $x \in Y_{i,s} - Y_{i,t}$, put $\gamma_\mu(x) - 1$ into B.

3. (Defining Γ) For each $x < s$ such that $\Gamma_\mu^B(x)$ is undefined, if $\Phi_e^B(x + k_\mu)[s] \downarrow$ define $\Gamma_\mu^B(x)$ with *large* use.

4.2. Verification

Notice that by the definition of *large* use in Step 3 of the construction, only undefined computations are declared undefined in Step 1.

First, we show that *there is a true path, namely a leftmost path TP such that $\delta_s \preceq TP$ for infinitely many stages s*. Suppose that $\alpha : L_e$ and, inductively, there exist infinitely many s such that $\alpha \preceq \delta_s$. Also suppose that α is the leftmost node with this property. We may suppose that there are infinitely many $\alpha * \infty$ stages. Given two nodes ζ, η on the tree, we say that $\zeta \leq_L \eta$ if ζ is either to the left of η or extended by η. If there is z such that $\Phi_e^B(z) \uparrow$ then there exist infinitely many s such that $\alpha \leq_L \delta_s * z'$. Otherwise Φ_e^B is total. Then for each x there is $s > x$ such that x is α-good at s. Thus, if n is a witness for (4.2), we have $\alpha \leq_L \delta_s * n$ for infinitely many stages s.

Next we look at the disturbance of Φ_e^B due to coding into B.

Lemma 4.5. *Suppose $\alpha : L_e$ for some α on TP such that there are infinitely many $\alpha * \infty$ stages. For each x, the computation $\Phi_e^B(x)$ is destroyed only finitely often by the coding strategies μ such that $\exists n\, \alpha * n \preceq \mu$.*

Proof. A computation $\Phi_e^B(x)$ existing at a stage can only be destroyed by the finitely many μ with code number $k_\mu < x$. Fix such $\mu : R_i$. Each time it destroys $\Phi_e^B(x)$ at some μ-stage s, some $y \leq x$ must enter Y_i by stage s and after the last stage μ was accessible. □

Now we verify that B is low$_2$.

Lemma 4.6. *Suppose $\alpha : L_e$ and $\alpha * m \preceq TP$ for some $m \in \mathbb{N}$. Then Φ_e^B is total. Since $\exists n\, \alpha * n' \preceq TP$ implies that Φ_e^B is partial, we have $B'' \leq_T TP \leq_T \emptyset''$.*

Proof. Let s_* be the least stage such that $\alpha * m$ is no longer initialized after s_*. Given x, we want to show that $\Phi_e^B(x) \downarrow$. By Lemma 4.5, pick $s_0 \geq s_*$ such that $\Phi_e^B(x)$ is no longer destroyed at stages $\geq s_0$ by any strategy μ such that $\alpha * n \preceq \mu$ for some n. By (4.2) there is an $\alpha * \infty$ stage $s \geq s_0$ such that

- $\Phi_e^B(x)[t] = \Phi_e^B(x)[s]$,
- x is α-good at t,

where t is the greatest $\alpha * \infty$ stage less than s. We claim that $\alpha * z' \not\preceq \delta_u$ for each $z \le x$ and each $u > s$, whence $\Phi_e^B(x)[s]$ is stable. Otherwise, let u be the least such, and let $v < u$ be the previous $\alpha * \infty$ stage. Then for some $z \le x$ the computation $\Phi_e^B(z)$ was destroyed by the action of some $\mu : R_i$ at a stage between v and u. However note that

- $\alpha \not<_L \mu$ by initialization of R_i at s,
- $\mu \not\preceq \alpha$ by definition of α-goodness,
- $\neg\exists n\, \alpha * n \preceq \mu$ by Lemma 4.5,
- $\neg\exists m\, \alpha * m' \preceq \mu$ by the minimality of u.

This is a contradiction. □

Lemma 4.7. *For all $i \in \mathbb{N}$, requirement R_i is met.*

Proof. Let $\mu : R_i$, $\mu \preceq TP$. We show that Γ_μ^B is total. Fix y. Note that $\gamma_\mu^B(y)$ can only be moved by a strategy $\alpha : L_e$ such that $\exists n \in \mathbb{N},\ \alpha * n \preceq \mu$. But then Φ_e^B is total, so this happens only finitely often. Now it is clear that $\Gamma_\mu^B = Y_i$. □

This concludes the proof of Lemma 4.3.

5. Density results for Σ_3 ideals

The lattice of Σ_4^0 ideals in the c.e. degrees fails to be dense: for instance, each principal ideal $[\mathbf{0}, \mathbf{b}]$, where $\mathbf{b} \ne \mathbf{0}$, has a maximal subideal that is $\Delta_4^0(\mathbf{b})$. If we choose $\mathbf{b}$ low then this ideal is Σ_4^0.

In contrast, we have the following for the partial order of Σ_3^0 ideals in the c.e. degrees under inclusion.

Theorem 5.1. *The p.o. of Σ_3^0 ideals in the c.e. degrees is dense.*

Recall from Section 4 that a uniformly c.e. sequence $(Y_k)_{k\in\mathbb{N}}$ is *uniformly low*$_2$ if, given input e, we can compute an index for a Turing reduction $(\oplus_{k\le e} Y_k)'' \le_T \emptyset''$. Lemma 4.1 of Section 4 will also be crucial in the proof of Theorem 5.1.

The join of two ideals is the ideal generated by their union. Notice that the join of two Σ_3^0 ideals is not necessarily Σ_3^0. After all, by the Sacks splitting theorem every c.e. degree is the join of two low c.e. degrees. Lemma 4.1 shows that if two Σ_3^0 ideals are contained in another Σ_3^0 ideal, then their join is a Σ_3^0 ideal.

Now suppose that $\mathbb{I} \subset \mathbb{J}$ are Σ_3^0 ideals in the c.e. Turing degrees. As in Section 4, let (Y_k) be a uniformly c.e. sequence generating $\mathbb{I}$. Without loss of generality we may assume that $Y_k \le_T Y_{k+1}$ for all k. Also, let D be a c.e. set with degree in $\mathbb{J} - \mathbb{I}$. By Lemma 4.1 the sequence $(D \oplus Y_k)$ is uniformly low$_2$. We will split D into c.e. sets D_0, D_1 such that the following requirements are satisfied:

$$N_{e,k,j} : D_j \ne \Phi_e(Y_k \oplus D_{1-j})$$

for $e, k \in \mathbb{N}$ and $j = 0, 1$. Then the ideal generated by $(D_0 \oplus Y_k)$ lies strictly in between $\mathbb{I}$ and $\mathbb{J}$.

Notice that the relation $D_j = \Phi_e^{Y_k \oplus D_{1-j}}$ is $\Pi_2^0(D \oplus Y_k)$, uniformly on e, k, j, and the c.e. indices of the split D_0, D_1. Since the sequence $(D \oplus Y_k)$ is uniformly low$_2$, there is a uniformly c.e. sequence $(V_{e,k,j,n})$ such that each of $V_{e,k,j,n}$ is either $\mathbb{N}$ or finite and

$$D_j = \Phi_e^{Y_k \oplus D_{1-j}} \iff \exists n\, V_{e,k,j,n} = \mathbb{N}. \tag{5.1}$$

By the recursion theorem we may use c.e. indices for D_0, D_1 in the construction. Therefore, it can use the array $(V_{e,k,j,n})$ which refers to the sets D_0, D_1 that are being built. The priority list is determined by $N_{e,k,j} < N_{e',k',j'} \iff \langle e, k, j\rangle < \langle e', k', j'\rangle$. The *restraint* of $N_{e,k,j}$ at stage s is the greatest stage $< s$ where $|V_{e,k,j}|$ increased. Without loss of generality we assume that at each stage s exactly one number is enumerated into D.

Construction. At stage s, if x enters D pick the least $\langle e, k, j\rangle$ such that x is less than the restraint of $N_{e,k,j}$. Put x into D_j.

Verification. Notice that the construction ensures that D_0, D_1 form a c.e. splitting of D, even before the application of the recursion theorem. This shows that when we apply the recursion theorem to get the appropriate version of the construction, $(V_{e,k,j,n})$ has the intended meaning, i.e. (5.1) holds. The rest of the verification refers to the construction on the fixed point of the recursion theorem.

By (5.1) it suffices to show that each $V_{e,k,j,n}$ is finite. For a contradiction, let $\langle e, k, j, n\rangle$ be the least such that $V_{e,k,j,n}$ is infinite. Then by the construction, D_{1-j} will be computable. Indeed, a number x can only be enumerated into D_{1-j} by the first stage where $|V_{e,k,j,n}| > x$. By (5.1) we have $D \equiv_T D_j \le_T Y_k$, which is a contradiction. This concludes the verification.

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