APPROXIMATION REPRESENTATIONS FOR Δ_2 REALS

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Abstract. We study Δ_2 reals x in terms of how they can be approximated symmetrically by a computable sequence of rationals. We deal with a natural notion of 'approximation representation' and study how these are related computationally for a fixed x. This is a continuation of earlier work; it aims at a classification of Δ_2 reals based on *approximation* and it turns out to be quite different than the existing ones (based on information content etc.)

1. Introduction

There are many ways to study real numbers from an effectiveness point of view. Most of the work has been done in classical computability theory (see Odifreddi [5], [6]) and in particular the study of degree structures and hierarchies of reals (i.e. sets). Other work is on randomness, see e.g. [1]. Another approach is concerned much with what ways (in some sense related to effectiveness) a real number can be approximated by a sequence of rationals. This approach is more in the framework of *computable analysis*, see e.g. Calude, Coles, Hertling, Khoussainov[3], Calude, Hertling[4] and Rettinger and Zheng[8], Zheng[7] for hierarchies of reals. In Barmpalias[2] we initiated the study of Δ_2 reals x by means of the structure of the sets

$$A_z = \{i \mid z_i < x\}$$

where z is a computable sequence of rationals (z_i) with limit x, under strong reducibilities. It is well known that the limits of a computable sequences of rationals are exactly the Δ_2 reals, and so our approach is restricted to this important class, the reals T-reducible to $\mathbf{0}'$. In fact we are only interested in sets A_z that are bi-infinite (something we assume from now on). In this case we call z a *symmetric approximation* to x.

Definition 1. *If* $\lim z = x$ *is a symmetric approximation to* x*, the set* A_z *is called an approximation representation of* x*.*

We often say just *representation* for short. It is clear that such a set *represents* a particular computable approximation of a real. We have a

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correspondence of a Δ_2 real with all its representations and conversely a representation may be a representation of many different reals; see figure 1.

FIGURE 1. Reals and representations

A basic fact is

Proposition 1 (Barmpalias[2]). *Every representation of a real is Turing equivalent with it.*

So all representations of a real lie in the same T-degree and that is why we are interested in strong reducibilities $\leq_{\mathrm{r}}$. Under $\leq_{\mathrm{r}}$ the r-degrees of representations of a real x form a substructure $\mathcal{D}_x^{\mathrm{r}}$ of the r-degrees within the Turing degree of x. We call this the *approximation structure* of x.

Moreover, a representation of x is c.e. iff x is c.e. (i.e. the limit of a computable increasing sequence of rationals). The main results in [2] were about c.e. reals, and so all the representations we considered were c.e. We constructed a c.e. x such that an infinite antichain is embeddable in $\mathcal{D}_x^{\mathrm{wtt}}$. The same method can be used to embed an infinite computably independent set of sets $\{A_i\}$ (i.e. with $A_n \not\leq_{\mathrm{wtt}} \oplus_{i\neq n} A_i$) and so construct an x such that every countable partial ordering is embeddable in $\mathcal{D}_x^{\mathrm{wtt}}$. In contrast we constructed a non-computable c.e. x such that all of its representations are m-equivalent (i.e. $\mathcal{D}_x^{\mathrm{m}}$ consists of a single element).

By exploring the variety of representations that a c.e. (and in general Δ_2) real can have, from a computational point of view (e.g. strong reducibilities), we aim at a classification of these reals according to their approximation properties. This approach is natural since approximation is a characterising feature of the Δ_2 class. Also it is a different way of looking at those reals, and we very much like to establish connections with existing classifications.

In this paper we continue in the line of Barmpalias[2] but looking at more advanced questions. In the next section we give a characterisation of representations in terms of cuts or linear orderings. We show that representations are exactly the cuts of computable orderings of $\mathbb{N}$, of order type $\omega + \omega^*$. So a Δ_2 real naturally defines a class of such cuts (i.e. its representations) and most of the results below can be stated in terms of cuts. In the same section we also mention that no representation lies on a proper class of the difference hierarchy and that there are reals that have different wtt-degree than any of their representations.

In section 3 we look at the question of how the representations of two T-equivalent reals are related. We construct T-complete x_1, x_2 and a representation A of x_1 such that every representation of x_2 is wtt-incomparable to A. So the two structures are not necessarily related computationally (appart from the fact that they lie in the same T-degree). The proof uses an infinite injury argument, and it is the first one we use for the construction of representations.

In section 4 we look at density: given $A_1 <_{\text{wtt}} A_2$ representations of a c.e. x is there a representation of it A with $A_1 <_{\text{wtt}} A <_{\text{wtt}} A_2$? Although the wtt-degrees of c.e. sets are dense, it turns out that a negative answer is true. We use an infinite injury tree argument to construct suitable x, A_1, A_2 and support this claim.

We assume that the reader is familiar with computability theory and in particular with priority arguments on a tree. We follow the standard notation in computability theory. For background in computable analysis the references given above are useful.

2. Some facts about representations

Let (z_n) be a computable (say injective) sequence of rationals. This sequence defines a computable linear ordering $\prec_z$ on $\mathbb{N}$: $n \prec_z m \iff z_n < z_m$. If (z_n) converges symmetrically to some x, A_z is a bi-infinite cut[1] of that computable ordering. It is known (see Odifreddi[5]) that the cuts of computable linear orderings of $\mathbb{N}$ are exactly the semi-recursive sets. We recall the following

Definition 2 (Jockusch)**.** *A set A is semirecursive if there is a computable f such that*

- $f(x, y) \in \{x, y\}$
- $x \in A \;\vee\; y \in A \Rightarrow f(x, y) \in A$.

So approximation representations are semi-recursive sets, but the converse doesn't hold, as the following proposition shows. Let ω^* be the inverse of the usual ordering ω of the naturals. It is not difficult to show that *any linear ordering of $\mathbb{N}$ in which every element has either finitely many predecessors or finitely many successors, is isomorphic to $\omega + \omega^*$*. Also, *any linear ordering of $\mathbb{N}$ which has a unique bi-infinite cut is isomorphic to $\omega + \omega^*$*. We also have

Proposition 2. *A set of naturals is an approximation representation of some Δ_2 real iff it is the bi-infinite cut of a computable linear ordering of $\mathbb{N}$, of order type $\omega + \omega^*$.*

Proof. Suppose we are given such an ordering $\prec$ and C its unique bi-infinite left cut; we will define a (symmetrically) convergent computable sequence z such that $A_z = C$. We define z_0 in the middle of $(0, 1)$ and suppose that for all $i < s$, $z_i \downarrow$. Let a_s be the largest z-term with index $< s$ and $\prec s$; and $a_s = 0$ if such doesn't exist. Also let b_s be the smallest z-term with index

[1] *cut* of a linear ordering $<$ of $\mathbb{N}$ is a downwards or upwards $<$-closed subset of $\mathbb{N}$. We often identify a cut with its complement.

$< s$ and $\succ s$; and $b_s = 1$ if such doesn't exist. Then define z_s in the middle of (a_s, b_s).

Since $\prec$ is computable, the definition of z is effective and z is computable. By induction $s \prec n \iff z_s < z_n$. We prove that z converges. For every $s \in C$ there are only finitely many $n \prec s$. So there must be an $s_1 \in C$ with $z_s < z_{s_1}$. So there is an increasing sequence (s_i) of elements in C such that (z_{s_i}) is increasing and so converging, say to a. Dual observations hold for $\overline{C}$ and so we get a decreasing and converging (say to b) (z_{n_i}) with $n_i \in \overline{C}$. It is enough to show that $a = b$. Indeed, otherwise the interval (a, b) would be proper and no term of the two sequences would appear in it. But according to the way we define z any large enough z_s will appear in (a, b), a contradiction.

So z converges and since A_z is a bi-infinite cut, it is identical to C. Finally it is obvious that an approximation representation A_z is a cut of a computable ordering of type $\omega + \omega^*$, which concludes the proof. □

Another interesting question is how a representation of a real x relates to x w.r.t. strong reducibilities. We observe

Proposition 3. *No wtt-complete c.e. real x has a representation $\equiv_{wtt} x$.*

Proof. In [2] we showed that any representation A_z of x is a hypersimple set. And it is known that no such sets are wtt-complete. □

Finally we note

Proposition 4. *Representations are either c.e. or co-c.e. or they don't belong to any finite level of difference hierarchy.*

To see this, first note that if A_z is not c.e. or co-c.e. then it is bi-immune (see [2]). Then the proposition follows from

Lemma 1. *No set in a finite level of the difference hierarchy is bi-immune.*

Proof. Suppose that A is properly n-c.e. and n is even. We show that $\overline{A}$ is not immune. Let $\lim_s \phi(m, s) = A(m)$, $\phi(m, 0) = 0$ and

$$|s \ : \ \phi(m, s) \neq \phi(m, s+1)| \leq n \tag{1}$$

for every m. Note that since A is properly n-c.e. (so not $(n-1)$-c.e.) (1) holds with equality for infinitely many m. To effectively generate an infinite subset of $\overline{A}$ we start looking for k on which ϕ changes exactly n times. We will find infinitely many such k and since n is even they must have $\lim_s \phi(k, s) = 0$ and so belong to $\overline{A}$. The case 'n-odd' is dual (showing that A is not immune). □

3. Two approximation structures in $\mathbf{0}'$

It is natural to ask what is the relation of the information content of a real and the variety of its representations. The following theorem shows that reals with the same information content may have quite unrelated approximation structures. This means that a classification of the Δ_2 reals based on their approximation structures is qualitatively quite different from classifications based on information content.

Theorem 1. *There exist Turing complete c.e. reals x_1, x_2 and a representation A_{z^1} of x_1 such that every representation of x_2 is wtt-incomparable with A_{z^1}.*

We will build x_1, x_2 by an approximation procedure, in the framework developed in [2]; we review it briefly. For the construction of a c.e. real x with requirements on its representations, we start defining the terms of a sequence z in decreasing order. On the other hand we have a non-decreasing sequence y which controls the enumeration in A_z, i.e. whenever we wish to enumerate $n \searrow A_z$ (say at s) we define $y_s = z_n$. All this action takes place within $(0,1)$ and we picture $(0, y_s)$ as the *black area* (see figure 2) which expands, but also tends to a limit (since y is bounded). Also, we always define the z-terms outside the black area (though they may enter it later on).

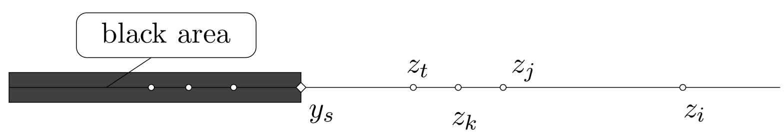

Figure 2: Construction

The convergence of z is guaranteed once we make sure that it steadily approaches $\lim y$ (at every stage s an interval (y_s, t) is suggested as appropriate for the definition of z_s; we always define it in the middle of the suggested interval). Eventually we will have $\lim y = \lim z$ and this c.e. real will satisfy the requirements.

The construction of z is most importantly a construction of a computable ordering of $\mathbb{N}$ which admits a unique bi-infinite cut. The properties of this ordering guarantee the satisfaction of the requirements.

The reals x_1, x_2 of theorem 1 will be constructed in a variation of this general framework. We lay out the requirements.

$$\begin{aligned}
\mathcal{R}:&\quad \lim y^1 = \lim z^1 := x_1 \ \&\ \lim y^2 = \lim z^2 := x_2\\
\mathcal{P}:&\quad K \leq_T x_1 \ \&\ K \leq_T x_2\\
\mathcal{Q}_e:&\quad \lim w^e = x_2 \Rightarrow \neg[A_{w^e} = \Phi_e^{A_{z^1}}; \phi_e] \ \vee\ A_{w^e} \text{ co-finite}\\
\mathcal{N}_e:&\quad \lim w^e = x_2 \Rightarrow \neg[A_{z^1} = \Phi_e^{A_{w^e}}; \phi_e]
\end{aligned}$$

where Φ_e, ϕ_e are effective enumerations of partial computable functionals and functions respectively and the expression $A = \Phi_B; \phi$ means that the use in these computations is bounded by ϕ (witnessing a wtt-reduction). $\mathcal{R}$ will be sorted out by the framework of the construction, as described above.

3.1. **$\mathcal{P}$-requirements.** To satisfy $\mathcal{P}$ we have to code K into x_1, x_2 *or* into sets equivalent with them. The easiest choice is to code it into A_{z^1}, A_{z^2} since these sets are directly involved in the construction (remember that any representation of a real *is* T-equivalent with it). Notice that the construction z^2 only makes the coding in $\mathcal{P}$ easier; z^2 is not involved in other requirements.

One can see that, if we are to satisfy all requirements, the coding in $\mathcal{P}$ will yield no stronger than T-reduction (i.e. *wtt*-reduction is not possible). Thus we enumerate Turing functionals Γ_1, Γ_2 such that

$$\Gamma_1^{A_{z^1}} = K \ \& \ \Gamma_2^{A_{z^2}} = K.$$

The uses γ_i will be increasing. They will always be defined on elements currently outside A_{z^i} and eventually rest on such an element. Also, at any stage

$$t < k \iff z^i_{\gamma_i(k)} < z^i_{\gamma_i(t)}.$$

3.2. **$\mathcal{Q}$-requirements.** The requirements most difficult to satisfy are the $\mathcal{Q}$ ones; these will bring an infinite injury character to the construction. The difficulty is that we don't have any control on the witnesses w_n, which can be enumerated without our will. The effective list of computable sequences w^e contains many inappropriate ones that we should reject in the first place, were we able to distinguish them in a computable way. Such are e.g. w's with A_w co-finite. For these w's our module will run forever, and we have to ensure that this feature does not harm other requirements (especially $\mathcal{P}$). Here is a strategy for $\mathcal{Q}$.

(1) Pick the least unused witness $n \notin A_w$ such that $w_n \downarrow < z^2_{\gamma_2(e+1)}$. *If $\gamma_2(e+1)$ changes during this cycle, $\mathcal{Q}$ is initialized and we start from (1).*
(2) Wait until $\Phi^{A_{z^1}}(n) \downarrow = 0; \phi(n) \downarrow$. *If in the meantime $n \searrow A_w$, go to (1).*
(3) Let k be the maximum such that $w_n < z^2_{\gamma_2(k)}$. If $z^1_t < z^1_{\gamma_1(k+1)}$ for some $t < \phi(n)$, $t \notin A_{z^1}$, define $y^1_s := z^1_{\gamma_1(k+1)}$.
    We ensure that the Γ_1-markers that sit (on z^1-terms) on the left of w are as many as the Γ_1-markers that sit on the left of z^2 terms involved in the use $\phi(n)$ and yet in the black area.
(4) Wait until $\Phi^{A_{z^1}}(n) \downarrow = 0; \phi(n)$ is restored. *(If in the meantime $n \searrow A_w$ go to (1).)*
    Then put $n \searrow A_w$ by defining $y^2_s := w_n$.
(5) If $\Phi^{A_{z^1}}(n) \downarrow$ is spoilt, go to (1).

As usual, s denotes the current stage of the construction in which this module works.

Analysis of outcomes. The *finite outcomes* are

- Stuck in (1)

- Stuck in (2) or infinitely many visits to (1), (2) but finitely many on the other steps *(count this as finite since no action is taken in the first two steps.)*
- Stuck in (4)
- Stuck in (5)

Note that each of these outcomes is not only successful for $\mathcal{Q}$ but also mean that $\mathcal{Q}$'s module stops interfering with the rest of the construction, from some point on. In particular it allows $\mathcal{P}$ to succeed (since it agitates each Φ-marker for only a finite time). Also any number of $\mathcal{Q}$'s can work together since $\mathcal{Q}_e$ can only agitate $\gamma_i(n)$ for $n > e+1$.

The *infinite outcomes* are

(a) We pass infinitely often from (3), (4) but only finitely often from (5). (*that is when almost every time we visit (4), the unwanted enumeration happens while waiting for* $\Phi^{A_{z^1}}(n)\downarrow$.)

(b) We reach and leave (5) infinitely often (because of a $A_{z^1} \upharpoonright \phi(n)$ enumeration.

The action involved in the infinite outcomes is *expansion of the* z^i*-black area.* In case (a) we only have expansion of the z^1-area while in case (b) expansion of both ones. This could interfere with $\mathcal{P}$ or even with other requirements. The idea for showing that it doesn't is to show that these actions, although apparently forced by steps (3),(4), they would anyway occur (sooner or later) by a $\mathcal{P}$-related action. Indeed, if for example we reach (5) and the computation is spoilt, this would be due to a $K \upharpoonright (k+1)$ enumeration. So even if we hadn't act under (3) or (4), this expansion of the black area would happen at the time of the $K \upharpoonright (k+1)$ enumeration; our actions are in acordance with $\mathcal{P}$. This way the impact $\mathcal{Q}$ has in the construction under an infinite outcome (given $\mathcal{P}$) is very little (namely it only affects the timing of the actions and not the actions themselves).

To illustrate this we prove the satisfaction of a single $\mathcal{Q}_e$ and $\mathcal{P}$ in a construction motivated only by these two requirements, and $\mathcal{Q}$ has an infinite outcome.

First we show that all Γ_i-markers eventually rest on $\overline{A_{z^i}}$ (i.e. outside the black area). Note that $\gamma_i(n)$ for $n \leq e+1$ won't be agitated by $\mathcal{Q}_e$. Now by induction: assume that for all $n < n_0$, $\gamma_i(n)$ eventually rest (say after stage s_0). From s_0 all of our w-witnesses will sit on the left of $z^2_{\gamma_2(n_0)}$; indeed, otherwise the module would terminate since the markers on the left of w are stable. So $\gamma_2(n_0)$ eventually rests on $\overline{A_{z^2}}$. According to step (3), $z^1_{\gamma_2(n_0)}$ will not be agitated again (so $\gamma_1(n_0)$ eventually rests).

Now the satisfaction of $\mathcal{Q}$ is evident, once we realize that supposing $\lim w = \lim y$ we get that either the module terminates or A_w co-finite. Indeed, if this didn't hold we would have infinitely many terms on the right of $z^2_{\gamma_2(e+1)}$; but since $z^2_{\gamma_2(e+2)} < z^2_{\gamma_2(e+1)}$ and sit outside the black area, this would contradict $\lim w = \lim y$.

We note that any number of $\mathcal{Q}$-requirements with any outcomes work well along with $\mathcal{P}$ and their satisfaction can be proved inductively as above. In particular no nesting of strategies is needed.

3.3. **$\mathcal{N}$-requirements.** $\mathcal{N}_e$ is easier to satisfy. After finitely many attempts we can ensure that our witnesses stay out of the black area as long as we want. This involves placing any witness z^1_t in a safe position, namely between $z^1_{\gamma_1(e)}$ and $z^1_{\gamma_1(e+1)}$. This will not cause any problems in the construction since we only work on $\mathcal{N}_e$ finitely often.

(1) Pick t big (so z^1_t currently undefined) and declare z^1_t witness (so give instruction for z^1_t's definition that $z^1_{\gamma_1(e+1)} < z^1_t < z^1_{\gamma_1(e)}$).

Wait until $\Phi^{A_w}(t)\downarrow = 0; \phi(t)\downarrow$. If in the meantime $t \searrow A_{z^1}$ or $\gamma_1(e+1)$ changes, start anew.

(2) If there are w_i, $i < \phi(t)$ outside the black area with $w_i < z^2_{\gamma_2(e)}$, put all these $i \searrow A_w$ (by defining $y^2_s := w_i$ where w_i is the maximum such w-term).

Wait until $\Phi^{A_w}(t)\downarrow = 0; \phi(t)$ is restored. If in the meantime $\gamma_1(e+1)$ changes (and so $t \searrow A_{z^1}$), go to (1).

(3) Put $t \searrow A_{z^1}$ (by defining $y^1_s := z^1_t$).

(4) If $\Phi^{A_w}(t)\downarrow = 0$ is spoilt, go to (1).

Note that the finiteness and success of this module depends solely on the success of $\mathcal{P}$ (that the Γ_1-markers eventually rest). As $\mathcal{Q}$-requirements respect $\mathcal{P}$ and $\mathcal{N}$ do as well (because $\mathcal{N}_e$ doesn't agitate $\gamma_i(n)$ for $n \leq e$) all strategies are compatible.

3.4. **Construction and Verification.** Let us divide $\mathcal{P}$ into $\mathcal{P}_1, \mathcal{P}_2, \dots$ where $\mathcal{P}_n$ denotes the requirement that $\gamma_1(n)$, $\gamma_2(n)$ both eventually rest (and of course Γ_1, Γ_2 hold correct computations). We agree on the following priority list of requirements:

$$\mathcal{P}_0 > \mathcal{N}_0 > \mathcal{Q}_0 > \mathcal{P}_1 > \mathcal{N}_1 > \dots$$

We also assume a uniform numbering of the requirements in this list, so that we can talk about the *i-th requirement* regardless its nature.

At each stage we enumerate one axiom for each Γ_i: find the least t such that $\Gamma_i^{A_{z^i}}(t)\uparrow$ and enumerate the axiom $\Gamma_i^{A_{z^i}}(t) = K(t)$ with big use $\gamma_i(t)$.

At stage $s+1$ we define z^i_s between y_s and the largest z^i_t, $t \leq s$ which lies outside the black area *unless* z_t is subject to a condition set by an $\mathcal{N}$-requirement. In the latter case we define it according to the condition. Note that we only specify where a term should be placed in relation with other defined terms. To make the construction definite, let the definitions be on the middle of the suggested interval.

At $s+1$ we also define y^i_{s+1} after a series of substages. At substage n we run the n-th strategy and get a temporary definition $y^i_{s+1}[n]$ of y^i_{s+1}. We do this for all $n \leq s$ and eventually define $y^i_{s+1} := y^i_{s+1}[s]$.

This concludes the construction but few explanatory words are appropriate. Every time we visit a strategy, we start from where we last stopped. Also the parameters we use have current value, as this was left by the last substage of the current stage (this also applies to the black area). Of course, in order to run a strategy, all parameters mentioned must be defined (otherwise we don't do anything more than deliver the parameters as we got them from the previous strategy, to the next one). Finally if we set a condition z^1_t according to $\mathcal{N}_e$ and $\gamma_1(e+1)$ changes before $z^1_t \downarrow$, we remove the condition since it was based on a value that changed.

Verification. We proceed inductively, supposing that for all $j < n$, $\mathcal{P}_j$, $\mathcal{N}_j$, $\mathcal{Q}_j$ are satisfied and the ones with finite outcome (including $\mathcal{P}_j$) have stopped acting after s_0. The construction carries on defining $\gamma_i(n)$ and $z^i_{\gamma_i(n)}$ outside the black area. And since $\mathcal{P}_j$, $\mathcal{N}_j$, $\mathcal{Q}_j$ for $j \geq n$ never force $\gamma_i(n) \searrow A_{z^i}$, $\gamma_i(n) \uparrow$ can only happen due to $\mathcal{N}_j$, $\mathcal{Q}_j$ for $j < n$ (given that $\gamma_i(j)$, $j < n$ have stabilised). Since $\mathcal{N}_j$, $j < n$ have ceased to act, they can't be responsible for $\gamma_i(n) \uparrow$ and the same holds for the $\mathcal{Q}_j$, $j < n$ with finite outcome.

Now we can prove that once $z^2_{\gamma_2(n)}$ is defined after s_0, it will stay outside the black area forever. Indeed, otherwise a $\mathcal{Q}_j$, $j < n$ with infinite outcome would come to a witness $w^j_t > z^2_{\gamma_2(n)}$, enumerate $t \searrow A_{w^i}$ under step (4) and hold $\Phi^{A_{z^1}}(t) \neq A_{w^i}(t)$ with use $A_{z^1} \restriction \phi(t)$ that can change only if one of $\gamma_1(k), k < n$ moves (due to the preliminary action of step (3)). By inductive hypothesis the disagreement would be preserved and $\mathcal{Q}_j$ would have finite action, contradiction.

So $z^2_{\gamma_2(n)}$ will eventually rest outside the black area and, according to the above,, no infinitary $\mathcal{Q}$ will pick a w-witness greater than $z^2_{\gamma_2(n)}$. Hence, according to step (3), no such requirement will move $\gamma_1(n)$. And due to the choice of s_0, no other requirement will agitate $\gamma_1(n)$, which will eventually stabilise, giving the success of $\mathcal{P}_n$.

Turning into $\mathcal{N}_n$, let $s_1 > s_0$ be large enough so that $\gamma_i(n)$ have stabilised. No lower priority requirement than $\mathcal{N}_n$ can enumerate $z^1_{\gamma_1(n+1)}$, and so an $\mathcal{N}_n$-witness z^1_t. Thus, only an infinitary higher $\mathcal{Q}_j$ could do that, under step (3) of its module. But again, if this happened we could show that $\mathcal{Q}_j$ has finite outcome: the witness it would hold when performing this enumeration would be greater than $z^2_{\gamma_2(n+1)}$ (otherwise it wouldn't enumerate $z^1_{\gamma_1(n+1)}$). So when it reached (5) (and it will reach it since s_1 is big enough), the computation would be preserved due to the choice of s_1, and the module would terminate; contradiction. Now if $\mathcal{N}_n$ doesn't reach (3), we're done. Otherwise the disagreement will be preserved due to the action in (2) and the choice of s_1.

As far as $\mathcal{Q}_n$ is concerned, if it gets stuck on a step of its module, it is obviously satisfied (as explained when we analysed its outcomes). Otherwise we will have

$$t \in \overline{A_{w^n}} \Rightarrow w^n_t > z^2_{\gamma_2(n+1)} \tag{2}$$

for all t. This concludes the induction step in an argument that shows the satisfaction of all $\mathcal{P}, \mathcal{N}$, and the $\mathcal{Q}$ with finite outcome. For the $\mathcal{Q}$ with infinite outcome it shows that (2) holds. Now we can see that these are also satisfied; indeed, supposing $\lim y^2 = \lim w^n$ we can see that there are only finitely many terms $w^n_t > z^2_{\gamma_2(n+1)}$ which also means that $\mathcal{Q}_n$ is satisfied. That is because the interval $(z^2_{\gamma_2(n+2)}, z^2_{\gamma_2(n+1)})$ is non-empty and lies outside the black area. So in this case $\overline{A_{w^n}}$ is co-finite, which is what we wanted.

The only thing left to complete the verification is to show that $\mathcal{R}$ is satisfied. Fix $i \in \{1, 2\}$. According to the way that the terms of z^i are defined by the construction, it is enough to prove that

$$\lim y^i = \lim_s z^i_{\gamma_i(s)}. \tag{3}$$

Indeed, if we fix an s, almost all z^i-terms will be defined on the left of $z^i_{\gamma_i(s)}$ (because only finitely many terms which carry $\mathcal{N}$-conditions can be defined on the right of it). Now we will use the fact that we define z^i-terms in the middle of the suggested interval. The sequence $(z^i_{\gamma_i(s)})$ is decreasing and bounded; so $\lim_s z^i_{\gamma_i(s)}$ exists and is $\leq \lim y$ (as all of its terms are). Let $\lim y = x$ and consider the sequence recursively defined as

$$\begin{aligned} a_1 &= z^i_{\gamma_i(1)} \\ a_{s+1} &= x + \frac{a_s - x}{2} \end{aligned}$$

(intuitively, we start from $z^i_{\gamma_i(1)}$ and define the next term in the middle of the interval between x and the last term). It is straightforward that $\lim_s a_s = x$. If we prove that

$$a_s \geq z^i_{\gamma_i(s)}$$

for all s, using the fact that $z^i_{\gamma_i(s)} \geq x$ for all s, we get (3), i.e. what we need to finish. We prove it inductively: for $s = 1$ it is evident. Suppose that it holds for s. Note that when $z^i_{\gamma_i(s+1)}$ is defined, $z^i_{\gamma_i(s)}$ is already defined and so

$$z^i_{\gamma_i(s+1)} = y_t + \frac{y_t - z_k}{2}$$

for some t, k, with $z_k \leq z^i_{\gamma_i(s)}$ and $y_t \leq x$. By the induction hypothesis, we also have $z_k \leq a_s$, and so

$$z^i_{\gamma_i(s+1)} \leq x + \frac{x - a_s}{2} = a_{s+1}$$

and we are done.

4. Non-density of representations

It is natural to ask whether the wtt-degrees of representations of a fixed c.e. real are dense. The following theorem says that this is not always the case, and it is not obvious if we consider that the wtt-degrees of c.e. sets in general is dense.

Theorem 2. *There are c.e. reals y such that the wtt-degrees of the representations of y are not dense.*

We wish to construct two sequences z, x with the same limit and such that $A_z <_{\mathrm{wtt}} A_x$ and for every sequence w with the same limit and $A_z \leq_{\mathrm{wtt}} A_w \leq_{\mathrm{wtt}} A_x$, either $A_w \equiv_{\mathrm{wtt}} A_z$ or $A_w \equiv_{\mathrm{wtt}} A_x$.

An easy way to code A_z into A_x is to define each z-term on some x-term. This is what we'll do, and note that it implies $A_z \leq_{\mathrm{m}} A_x$.

The density requirement is the hardest, and we will split it into three. Given w, our first attempt will be to try to prevent $A_z \leq_{\mathrm{wtt}} A_w \leq_{\mathrm{wtt}} A_x$. For the first inequality we have $\mathcal{N}$, and $\mathcal{M}$ will work on preventing the second. If one of them fails to block the inequality, it will produce a certain infinitary outcome about w in relation with x or z. If they *both fail*, the information they give about w in relation with z *and* x, along with the work of a third requirement $\mathcal{Q}$ will deliver $A_w \equiv_{\mathrm{wtt}} A_z$ or $A_w \equiv_{\mathrm{wtt}} A_x$.

Along these lines we now formulate $\mathcal{N}$, $\mathcal{M}$. The usual way to block a wtt-inequality between representations (say $A_z \leq_{\mathrm{wtt}} A_w$) is to pick a witness z_i and wait until $\Phi^{A_w}(i)\downarrow;\phi$ (where Φ is a possible reduction). Then expand the black area up to the largest w-term less than z_i and wait until the computation is restored. If this happens, the use will be the same, and so no w-term in the use will be outside the black area and less than z_i. this means that now we can expand the black area up to z_i (thus diagonalising) and the computation will be preserved *unless a w-term below the use sits on z_i.*

So $\mathcal{N}$ will block $\leq_{\mathrm{wtt}}$ unless all of its z-witnesses sit on w-terms. And if we try as witnesses a cofinite subset of $\mathbb{N} - A_x$ (we have to employ witnesses that sit outside the black area), failing to block $\leq_{\mathrm{wtt}}$ will produce the outcome that almost all z-terms (outside the black area) sit on w-ones. Similarly, if $\mathcal{M}$ fails to block $A_w \leq_{\mathrm{wtt}} A_x$, this will be because almost every w-term (outside the black area) sits on an x-one.

4.1. Requirements.

To formalise these ideas, let Z be the set of the indices of the x-terms that happen to sit on z-terms (we know that every z-term is made to sit on an x-term). Similarly, with respect to the given w, let W be the set of indices of the x-terms that happen to sit on w-terms. Then we have

$$\mathcal{N}_w : A_z \leq_{\mathrm{wtt}} A_w \Rightarrow Z \cap \overline{A_x} \subseteq_* W$$

where $\subseteq_*$ means subset modulo finite sets. Now let X_w be the set of indices of w-terms that sit on x-ones. Note that this is a c.e. set, as well as Z, W that we considered above. Then we require

$$\mathcal{M}_w : A_w \leq_{\mathrm{wtt}} A_x \Rightarrow \overline{A_w} \subseteq_* X_w.$$

From the above it is clear that we are working modulo the black area. This means that we are only interested in elements sitting outside of it. This will continue to hold throughout the proof, since for the elements in the black area we can decide their luck by waiting long enough to appear there.

4.1.1. *$\mathcal{Q}$-requirements.* If both $\mathcal{N}, \mathcal{M}$ are satisfied by their second clause, we know that modulo (i.e. ignoring) the black area, almost every z-term sits on a w-term and almost every w-term sits on an x-term. The job of $\mathcal{Q}$ is to give Z a certain maximality property, but only modulo the black area. Indeed, it is not difficult to show that if Z were maximal then $A_x \leq_{\mathrm{m}} A_z$ [2] and so there is no hope for the requirement $A_z <_{\mathrm{wtt}} A_x$ to be satisfied. Given a sequence w as before, we want

$$\overline{A_x} \cap Z \subseteq_* \overline{A_x} \cap W \Rightarrow \overline{A_x} \cap Z =_* \overline{A_x} \cap W \;\vee\; \overline{A_x} \cap W =_* \overline{A_x}. \tag{4}$$

where W comes from w as before and $=_*$ is equality modulo finite sets. Note that when w runs over all computable sequences of rationals, $\{W\}$ is an effective enumeration of all c.e. sets. It is now not very hard to see that the satisfaction of (4) $\mathcal{N}_w, \mathcal{M}_w$ implies

$$A_z \leq_{\mathrm{wtt}} A_w \leq_{\mathrm{wtt}} A_x \Rightarrow A_w \leq_{\mathrm{m}} A_z \;\vee\; A_x \leq_{\mathrm{m}} A_w \tag{5}$$

which is what we want. Indeed, if we suppose $A_z \leq_{\mathrm{wtt}} A_w \leq_{\mathrm{wtt}} A_x$ then $\mathcal{N}_w, \mathcal{M}_w$ are satisfied by their second clauses. The second clause of $\mathcal{N}_w$ implies that the disjunction in (4) is true. For $A_w \leq_{\mathrm{m}} A_z$, using the second clause of $\mathcal{M}_w$, we only need to decide the luck of x_i with $i \in W$ (using A_z). This is possible if the first clause of the disjunction in (4) is true. If not, the second clause of that disjunction gives $A_x \leq_{\mathrm{m}} A_w$.

Note that the $\leq_{\mathrm{m}}$ in (5) are in fact $\equiv_{\mathrm{m}}$. For (4) it is enough to satisfy

$$\mathcal{Q}_w : (\overline{Z} \cap \overline{A_x} \subseteq_* W) \;\vee\; (\overline{Z} \cap \overline{A_x} \cap W \text{ finite})$$

[2] consider the c.e. set $Z \cup A_x$; the maximality of Z gives $Z \cup A_x =_* Z$ or $Z \cup A_x =_* \mathbb{N}$, from which the claim follows.

4.1.2. *$\mathcal{P}$-requirements.* To guarantee the strictness of the inequality $A_z <_{\text{wtt}} A_x$ we have

$$\mathcal{P} : \Phi^{A_z} \neq A_x; \phi$$

where Φ runs over the partial computable functionals. This requirement along with $\mathcal{N}$, $\mathcal{M}$ (and no other) motivate the black area.

4.1.3. *A_z co-infinite.* Finally we want x, z to be symmetric approximations (i.e. A_x, A_z infinite and co-infinite) and while $\mathcal{P}$ implies this for x, it is not obvious by what we have said so far that the same holds for z. We can easily adjust the modules described below, such that they leave infinitely many z-terms outside the black area (by restraining a finite amount of $\mathcal{P}$, $\mathcal{N}$ and $\mathcal{M}$ action). But this is not necessary if we observe the following. Since A_x is semirecursive, it cannot be hh-simple and so it cannot be maximal. Consider a co-infinite c.e. W which contains A_x and the corresponding $\mathcal{Q}_w$. If A_z were cofinite, $\overline{A_x} \cap Z$ would be finite and thus the first clause (the hypothesis) of (4) would hold. By the properties of W there are infinitely many x-terms outside the black area which do not sit on w-terms. This means that the second clause of the disjunction in $\mathcal{Q}_w$ is false, and $\overline{A_x} \cap Z =_* \overline{A_x} \cap W$ must hold. But this is impossible since the first part is finite and the second infinite. So A_z will be co-infinite, provided that the requirements above are satisfied.

4.2. **Modules.** Above we showed that the requirements $\mathcal{P}, \mathcal{Q}, \mathcal{N}, \mathcal{M}$ are sufficient to imply the theorem. Before stating the strategies which will satisfy them, we say few things about the construction. As usual the black area is an increasing sequence, which we will keep implicit in this proof (e.g. expanding the black area up to a certain point means to define the current term of the sequence on that point). At the begining of stage s we define x_s between the end of the black area and the least x-term sitting outside of it. For the definition of z we have a set Z which is enumerated by various $\mathcal{Q}$-requirements and is, as before, the set of indices of x-terms which sit on z-ones. At the beginning of each stage we pick the least $n \in Z$ such that x_n doesn't sit on a z-term, and define $z_k = x_n$, where k is the least such that $z_k \uparrow$.

Hence there are *two sorts of enumerations* going on in the construction. One sort is those controlled by the black area (i.e. enumerations into A_z, A_x and the various A_w). The other is enumerations into Z and the various W. We only control (by $\mathcal{Q}$'s action) the one in Z; the one in W is done by the opponent. The two sorts of enumerations are unrelated, apart from the fact that Z-enumeration is done on the part (i.e. terms) that the the black area currently leaves unaffected.

The argument will be a tree construction, mainly because of the infinitary $\mathcal{Q}$ requirements. The black area expands according to the demand of the nodes of the tree, and at most one such expansion happens during a single stage s (and it happens in the end of it). In particular, at the end of s we let the least $\mathcal{P},\mathcal{N}$ or $\mathcal{M}$ currently accessible node which requires attention act (note that these are finitary). But since $\mathcal{Q}$ is infinitary, we let every accessible $\mathcal{Q}$-node act (and possible enumerate into Z) at the substage of S that it is accessed.

4.2.1. *$\mathcal{Q}$-module.* This requirement is interested in

$$\overline{A_x}\cap\overline{Z}=\{b_0<b_1<\dots\}.$$

The strategy follows the maximal set construction, when the last is done on a tree, and so it requires nesting. Suppose that $\mathcal{Q}$ is sitting on β. The possible outcomes are $\boxed{\text{i}}<\boxed{\text{f}}$. Let $INF(\beta)$ be the $\mathcal{Q}$-nodes γ with $\gamma*\boxed{\text{i}}\subseteq\beta$ and $FIN(\beta)$ the ones with $\gamma*\boxed{\text{f}}\subseteq\beta$. The outcome $\boxed{\text{i}}$ involves infinitary action and indicates

$$A_x\cap\overline{Z}\subseteq_* W_\beta$$

(where W_β is the c.e. set associated with the β's requirement); and $\boxed{\text{f}}$ indicates

$$(A_x\cap\overline{Z})\cap W_\beta \text{ finite.}$$

The module enumerates elements of $\overline{A_x}\cap\overline{Z}$ that have not appeared in W_β, into Z thus trying to make almost all b_n elements of W_β. But it acts only in expansionary stages which indicate that there is infinite potential in W_β. The level of b_n below which the work has already been done is

$$\ell(\beta)=\min\{n\mid b_n\in A_x\cap\overline{W}_\beta\ \wedge\ n>r(\beta)\}$$

where $r(\beta)$ is a finite restraint and the values of the parameters in the expressions are, as usual, subject to the current stage. If β is on the true path we will have $\lim_s\ell(\beta)[s]=\infty$ iff $\beta*\boxed{\text{i}}$ is on the true path. The strategy is the following:

Is there $n>\ell(\beta)$ with $b_n\in\overline{A_x}\cap(\cap_{\gamma\in INF(\beta)}W_\gamma)$?

- No: do nothing
- Yes: put $b_{\ell(\beta)},\dots b_{n-1}\searrow Z$.

Then access $\boxed{\text{i}}$ or $\boxed{\text{f}}$ depending on whether $\ell(\beta)$ has increased *(note that if it has acted under the 'yes' clause above, it has increased).*

Finally, the restraints will guarantee that $\overline{A_x}\cap\overline{Z}$ is infinite.

4.2.2. *$\mathcal{P}$-module.* Suppose that $\mathcal{P}$ is sitting on β. The point here is that we need to impose suitable restraints, as we want each b_n to reach a final value. And indeed, b_n can be agitated either by a Z-enumeration or by an expansion of the black area (and so by P's action). Moreover we want a witness for P that is not a z-term (otherwise its enumeration may interfere with the use of the computation we want to preserve). We have two restraints; r for Z-enumeration and q for the expansion of the black area. In some of the strategies, r and q restraints are imposed by saying 'we r-restrain ...' etc. Let $s(\beta)$ (at a given stage) be the largest number that the nodes to the left of β have mentioned so far. Then *we define* $r(\beta)$ to be the least number greater than $|\beta|$, $s(\beta)$ and the numbers that are currently r-restrained by the nodes above β.

We also define $q(\beta)$ to be the least of $x_{b_{|\beta|}}$, $x_{s(\beta)}$ and the (rational) numbers that are currently q-restrained by the nodes above β. This restraint requires $(q(\beta), 1)$ to stay outside the black area. Note that some x_n which contribute to q may be currently undefined. In this case we use the convention that every $x_i \downarrow$ outside the black area with $i < n$ is q-restrained (this is reasonable since undefined terms will be later defined outside the black area). Note that $r(\beta), q(\beta)$ are the restraints that β should respect. The $\mathcal{P}$-strategy is the following:

(1) Pick a witness $n > r(\beta)$ with $n < \ell(\gamma)$ for all $\gamma \in INF(\beta)$ and x_{b_n} is not $q(\beta)$-restrained. Now r-restrain b_n and q-restrain x_{b_n}. *The requirement $n < \ell(\gamma)$ ensures that no higher node will put $b_n \searrow Z$. And for the lower nodes this is forbidden by The r-restraint we impose. Note that b_n will keep the current value until we reach step 4.*

(2) Wait until

$$\Phi^{A_z}(b_n) \downarrow = 0; \phi. \tag{6}$$

Output $\boxed{\text{w}}$. *If (6) never happens, x_{b_n} will stay outside the black area and the disagreement will witness the satisfaction of $\mathcal{P}$.*

(3) Expand the black area up to the maximum z-term in the use of (6), less than x_{b_n}; and wait until (6) is restored. Output $\boxed{\text{p}}$. *Because of the choice of n, this action respects the restraints of higher priority nodes. If (6) is never restored we win as before.*

(4) Expand the black area up to x_{b_n} and q-restrain the least z-term below the use, not in the black area. Output $\boxed{\text{d}}$. *We have created a disagreement which will be preserved due to the restraints we impose.*

4.2.3. *$\mathcal{N}$-module.* In $\mathcal{P}$-strategy we where able to pick a suitable witness and, by imposing restraints, keep it suitable until we diagonalise. In the $\mathcal{N}$-strategy we describe below we don't have this ability. We can try and find a suitable witness, but anytime after that, it may become unsuitable and so we have to change it. This situation may occur infinitely often and give us a useful infinitary outcome. The key idea is not to impose any restraint during these cycles. If $\mathcal{N}$ is attached to β, the strategy is the following:

(1) Pick the least $n \in Z$ with $x_n \downarrow < q(\beta)$ and $n \notin W \cup A_x$.

(2) Wait until one of the following happens:
   - $n \searrow W \cup A_x$
   - $\Phi^{A_w}(n) \downarrow = 0; \phi$

   Output $\boxed{\text{w}}$.

(3) If the first clause holds, go to step 1; otherwise proceed to the next step. *If the first clause fails and we get the computation, x_n will not be sitting on a w-term below the use (otherwise we would already have found this out and returned to step 1).*
(4) Restrain x_n with q. Expand the black area up to the maximum $w_i < x_n$ with $i < \phi(n)$ and wait until $\Phi^{A_w}(n) \downarrow= 0; \phi$ is restored. Output $\boxed{\text{p}}$. *If the computation is not restored we are done; otherwise the use will be the same, and so x_n will continue to be different than all w-terms below the use.*
(5) Expand the black area up to x_n and q-restrain the least w-term below the use, which lies outside the black area. Output $\boxed{\text{d}}$. *The disagreement will be preserved because of the remark in the previous step.*

If the module visits infinitely often steps 1,2,3, $\mathcal{N}$ is satisfied by its second clause and the outcome is $\boxed{\text{w}}$. If it gets stuck in step 2, it is satisfied by its first clause and the outcome is again w. If we get to 3 or 4 we are able to keep a suitable witness and so it is satisfied by its second clause and the outcome is $\boxed{\text{p}}$ or $\boxed{\text{d}}$ respectively.

4.2.4. *$\mathcal{M}$-module.* This is similar to the one for $\mathcal{N}$.

(1) Pick the least n with $w_n < q(\beta)$ and $n \notin A_w \cup X_w$.
(2) Wait until one of the following happens:
   - $n \searrow A_w \cup X_w$
   - $\Phi^{A_x}(n) \downarrow= 0; \phi$

   Output $\boxed{\text{w}}$.
(3) If the first clause holds, go to step 1; otherwise proceed to the next step. *If the first clause fails and we get the computation, w_n will not be sitting on an x-term below the use (otherwise we would already have found this out and returned to step 1).*
(4) Restrain w_n by q; expand the black area up to the maximum x-term below the use and smaller than w_n. Wait until $\Phi^{A_x}(n) \downarrow= 0; \phi$ is restored. Output $\boxed{\text{p}}$.
(5) Expand the black area up to w_n and q-restrain the least x-term below the use and $> w_n$. Output $\boxed{\text{d}}$.

4.3. **Construction.** Before stating the construction we give a brief account of the restraints we impose. The r-restraint is only taken into account by $\mathcal{Q}$ and $\mathcal{P}$ nodes; and only $\mathcal{P}$-nodes contribute to it (in a $\boxed{\text{w}}$-outcome). The q-restraint is taken into account by $\mathcal{N}$, $\mathcal{M}$, $\mathcal{P}$. And in fact these are the only modules that contribute to it.

We agree on a uniform labelling of the tree which is made out of the outcomes we defined in the modules above. Let this labelling be based on the following priority list

$$Q_0 > N_0 > M_0 > P_0 > Q_1 > \dots$$

The *construction* proceeds in stages, by accessing a branch of nodes of length s at stage s, according to their current outcome. While accessing the branch, we only execute the modules of the $\mathcal{Q}$-nodes or the $\mathcal{N}$ or $\mathcal{M}$ nodes that are in their first or second step (i.e. their infinitary part). For the other nodes we follow their last outcome. In the end of the stage we run the module of the highest accessible $\mathcal{P}$, $\mathcal{N}$ or $\mathcal{M}$ node that requires attention. These modules require attention when they are in a wait-type outcome (i.e. $\boxed{\text{w}}$, $\boxed{\text{p}}$) and they are ready to move on to the next step.

4.4. **Verification.** Obviously there is an infinite leftmost infinitely often visited path f. By induction we show that it is the true path, i.e. that every node on it (and its parameters) under any final (i.e. sitting on the true path) outcome behaves as described in the analysis of outcomes above, and so the requirement attached to it is satisfied. In other words that it satisfies the *working hypothesis* as we formulate it below.

- $\overline{A_x} \cap \overline{Z}$ *is infinite.*
- *For every* $\beta \subset f$, $r(\beta), q(\beta)$ *come to a limit.*
- *If* β *is a* $\mathcal{Q}$*-node and* $\beta * \boxed{i} \subset f$ *then* $\lim_s \ell(\beta)[s] = \infty$ *and* $|\overline{A_x} \cap \overline{Z} \cap W_\beta| = \infty$.
- *If* β *is a* $\mathcal{Q}$*-node and* $\beta * \boxed{f} \subset f$ *then* $\lim_s \ell(\beta)[s] < \infty$ *and* $|\overline{A_x} \cap \overline{Z} \cap W_\beta| < \infty$.

We can show straightaway that $\overline{A_x} \cap \overline{Z}$ is infinite. Indeed, if not there would be a least n such that $\lim_s b_n^s = \infty$. Because of the restraints r, q, after some stage no node to the right of f will be allowed to change the value of b_n. the same holds for the nodes to the left of f, because they are accessed only finitely many times. And again because of the restraints, only the nodes in $f \upharpoonright n$ (i.e. those of length $< n$) can agitate b_n. By finite induction it is easy to see that every $\mathcal{P}$, $\mathcal{N}$, $\mathcal{M}$ node in $f \upharpoonright n$ acts (i.e. expands the black area) only finitely often. So they stop agitating b_n and there must be a $\mathcal{Q}$-node β of maximal length in $f \upharpoonright n$ that enumerates the value of b_n into Z infinitely often. But this cannot be: when it does it again (after the nodes mentioned above have stopped agitating b_n and no node $<_L f \upharpoonright n$ becomes accessible) it will give b_n a value in $(\cap_{\gamma \in INF(\beta)} W_\gamma) \cap W_\beta$ and according to the module of $\mathcal{Q}$, none of the nodes $\subseteq \beta$ will enumerate the current value of b_n into Z (the ones with $\boxed{\text{f}}$-edges because they have stopped acting). So, since β was chosen maximal, b_n will not change again, a contradiction.

Suppose that $\beta \subset f$ and the working hypothesis holds for all $\gamma \subset \beta$; also that the corresponding requirements are satisfied. We show the same for β. Let us be in a final segment of stages such that no node $<_L \beta$ becomes accessible and $r(\beta)$, $q(\beta)$ have reached their final values.

4.4.1. *β in $\mathcal{Q}$ case.* First suppose that β is a $\mathcal{Q}$-node. We show that

- If $\beta * \boxed{\text{i}} \subset f$ then $|\overline{A_x} \cap \overline{Z} \cap W_\beta| = \infty$

- If $\beta * \boxed{\text{f}} \subset f$ then $|\overline{A_x} \cap \overline{Z} \cap W_\beta| < \infty$.

If the first clause didn't hold, there would be a least $n > r(\beta)$ such that $b_n \notin W_\beta$. After b_n takes its final value, β ceases to act (because any action would agitate b_n) which is a contradiction (since $\boxed{\text{i}}$ implies infinite action). Moreover $\lim_s \ell(\beta)[s] = \infty$ because otherwise there would be a $b_n \notin W_\beta$ with $n > r(\beta)$; and this cannot hold since, given that $|\overline{A_x} \cap \overline{Z} \cap W_\beta| = \infty$, β would change it to a value in W_β. In particular we get that $\overline{A_x} \cap \overline{Z} \subseteq_* W_\beta$ and so $\mathcal{Q}$ is satisfied.

The second clause is obvious if we consider the module of $\mathcal{Q}$. And of course $\lim_s \ell(\beta)[s] < \infty$ is also easy to see.

4.4.2. *β in $\mathcal{P}$ case.* Suppose that β is a $\mathcal{P}$-node. From what is said in the induction hypothesis about the $\gamma \in INF(\beta)$ it follows that β will find a suitable witness b_n. Now as long as we wait for $\Phi^{A_z}(b_n) \downarrow = 0; \phi$, b_n will not enter $A_x \cup Z$ (and nor do any b_t, $t < n$) because of r, q and the fact that the $\mathcal{N}, \mathcal{M}, \mathcal{P}$ nodes above have ceased to act. If we get stuck on step 2 (i.e. $\boxed{\text{w}}$) we are done. Otherwise we proceed to 3 and if we get stuck there (on $\boxed{\text{p}}$) we are done; if not, we end up in 4 where the computation is preserved due to q, and so we are done (on a $\boxed{\text{d}}$-outcome). It is easy to see that the restraints come to a limit.

4.4.3. *β in $\mathcal{N}$ or $\mathcal{M}$ case.* Suppose that β is an $\mathcal{N}$-node. If we never escape steps 1,2,3 we get $Z \cap \overline{A_x} \subseteq_* W$, a stable outcome $\boxed{\text{w}}$ and no restraints. If we manage to go to 4 but no further, we get a stable $\boxed{\text{p}}$ and a finite q-restraint. And if we make it to 4 we get a final $\boxed{\text{d}}$ and a finite q-restraint. The analysis for β an $\mathcal{N}$-node is similar.

To finish the proof we show that the sequence x converges. We know that $\overline{A_x}$ is infinite, and so that there exists an increasing sequence (n_i) such that the sequence (x_{n_i}) lie outside the black area. Also, according to the way we define the terms, (x_{n_i}) is decreasing, bounded and so it converges. The black area also converges at some y and, as in the proof of (1), it is enough to show that the two limits coincide. Define

$$\begin{aligned} a_1 &= x_{n_1} \\ a_{s+1} &= y + \frac{a_s - y}{2} \end{aligned}$$

It is straightforward that $\lim_s a_s = y$. If we prove that

$$a_s \geq x_{n_s}$$

for all s, using the fact that $x_{n_s} \geq y$ for all s, we get that $\lim_s x_{n_s} = y$ and finish. We prove it inductively: for $s = 1$ it is evident. Suppose that it holds for s. Let y_s be the right end of the black area at stage s. Note that when $x_{n_{s+1}}$ is defined, x_{n_s} is already defined and so

$$x_{n_{s+1}} = y_t + \frac{y_t - x_k}{2}$$

for some t, k, with $x_k \leq x_{n_s}$ and $y_t \leq x$. By the induction hypothesis, we also have $x_k \leq a_s$, and so

$$x_{n_{s+1}} \leq y + \frac{y - a_s}{2} = a_{s+1}$$

and we are done.

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