TRACING AND DOMINATION IN THE TURING DEGREES

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Abstract. We show that if $\mathbf{0}'$ is c.e. traceable by $\mathbf{a}$, then $\mathbf{a}$ is array non-computable. It follows that there is no minimal almost everywhere dominating degree, in the sense of Dobrinen and Simpson [DS04]. This answers a question of Simpson and a question of Nies [Nie09, Problem 8.6.4]. Moreover, it adds a new arrow in [Nie09, Figure 8.1], which is a diagram depicting the relations of various 'computational lowness' properties. Finally, it gives a natural definable property, namely non-minimality, which separates almost everywhere domination from highness.

1. Introduction

In recent years, research in algorithmic randomness has enriched classical computability theory with new notions and concepts, which give new insights to the subject. A well known example is the 'lowness' notion of K-triviality, which was studied in [DHNS03, Nie05] and turned out to be degree theoretic. In fact, it was shown that the K-trivial sets form an ideal in the Turing degrees. Other examples are 'highness' notions like almost everywhere domination, which was introduced by Dobrinen and Simpson in [DS04]. This was motivated by some questions on the reverse mathematics of measure theory. Recall that, given functions $f, g : \mathbb{N} \to \mathbb{N}$, we say that f dominates g if $f(n) \geq g(n)$ for almost all $n \in \mathbb{N}$.

Definition 1.1 (Dobrinen and Simpson [DS04])**.** A Turing degree $\mathbf{a}$ is called almost everywhere (a.e.) dominating, if for almost all $X \in 2^{\omega}$ and all functions $g \leq_T X$, there is a function $f \leq_T \mathbf{a}$ which dominates g.

Kurtz [Kur81] showed that $\mathbf{0}'$ is a.e. dominating. This notion is very related to the highness property from classical computability theory: recall that a set A is *high* if $A' \geq_T \emptyset''$. This means that if we can answer $\Sigma^0_1(A)$ questions, then we can answer any $\Sigma^0_1(\emptyset')$ question. In this sense, A is close to the halting problem $\emptyset'$, hence the name 'high'. Martin [Mar66] showed that A is high iff it can compute a function which dominates all computable functions. Hence, it is easy to see that every a.e. dominating degree is high. Toward a characterization of the a.e. dominating degrees,

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Dobrinen and Simpson asked if this notion is equivalent to either highness or Turing completeness. In [BKHLS06, CGM06] it was shown that the class of a.e. dominating degrees lies strictly in between high and complete degrees, even in the local structure of computably enumerable degrees. Also, Kjos-Hanssen, Miller and Solomon [KHMS10] (also see [Sim07] or [Nie09, Section 5.6]) showed that a degree $\mathbf{a}$ is a.e. dominating iff every Martin-Löf random sequence relative to $\mathbf{a}$ is 2-random (i.e. Martin-Löf random relative to $\mathbf{0}'$). Thus a.e. domination can also be viewed as a notion from algorithmic randomness.

There has been an interest in clarifying the connections of this highness property with concepts from classical computability theory. For example, what role it plays in the partial ordering of the Turing degrees and whether it can be expressed purely in degree theoretic terms, without resorting to measure or randomness.[1] In this respect, the following questions were raised.

Recall from [TZ01] that a sequence of sets (T_i) is a trace for a function f, if $f(n) \in T_n$ for all $n \in \mathbb{N}$. We say that (T_i) has bound h, if $|T_n| < h(n)$ for all $n \in \mathbb{N}$. A degree $\mathbf{a}$ c.e. traceable, if there is a computable function h such that every function $f \leq_T \mathbf{a}$ has a c.e. trace with bound h.

- (Simpson, 2006) Is there a minimal a.e. dominating degree?
- (Nies [Nie09, Problem 8.6.4]) Is there a c.e. traceable a.e. dominating degree?

Our main result (Theorem 1.1) shows that each a.e. dominating degree is array non-computable, which answers these questions in the negative.[2] Moreover, it adds a new arrow in [Nie09, Figure 8.1], which is a diagram depicting the relations of various 'computational lowness' properties.

For the high degrees, there are several natural order theoretic properties which distinguish them in the structure of Turing degrees. As an example we mention that every high Δ^0_2 degree bounds a minimal degree, Cooper [Coo73]. In the computably enumerable degrees, Cooper [Coo74] showed that every high degree bounds a minimal pair.[3]

Almost everywhere domination can be expressed as a completeness notion with respect to the LR reducibility. This is a pre-ordering that is obtained by partially relativizing the notion of 'low for random' from [KT99]: $A \leq_{LR} B$ iff every Martin-Löf random relative to B is also Martin-Löf random relative

[1] This question can be seen as part of a larger program, which aims at characterizing notions from algorithmic randomness in computability theoretic and/or combinatorial terms. For example, a major open question in this area is whether the Martin-Löf random degrees are first order definable in the partially ordered structure of the Turing degrees, see [MN06, Question 2.3]. Another well known example, is the open question whether K-triviality (equivalently, low for Martin-Löf randomness) can be characterized in purely combinatorial terms, see [MN06, Question 3.1]. Such a characterization was found for the notion of 'low for Schnorr random' in [TZ01].

[2] for definitions and more details, see below.

[3] The latter is known to fail for lower jump classes, even for high$_2$. See [DLS93].

to A. A set A is LR-complete if $\emptyset' \leq_{LR} A$. In [KHMS10] (also see [Sim07]) it was shown that a set is LR-complete iff it is a.e. dominating.

There is, in fact, a whole array of highness notions that are obtained in this way and are motivated by different areas in computability theory. In connection to the study of mass problems, we mention the BLR-completeness of Cole/Simpson [CS07] and the equivalent JT-completeness. Let (Φ_e) be an effective list of all Turing functionals. Recall from [Nie06] that a set A is jump-traceable if the jump function $J^A(e) \simeq \Phi_e^A(e)$ has a c.e. trace with a computable bound. Let (W_e) be an effective list of all c.e. sets. Simpson [Sim07], partially relativizing the notion of jump traceability, gave the following definition: X is jump traceable by Y if there are computable functions f, g such that $J^X(e) \in W^Y_{f(e)}$ and $|W^Y_{f(e)}| < g(e)$ for all $e \in \mathbb{N}$. A set A is JT-complete, if $\emptyset'$ is jump traceable by A. The same can be said about c.e. traceability: X is c.e. traceable by Y if there is a computable function h such that every function $f \leq_T X$ has a Y-c.e. trace with bound h. A set A is $\emptyset'$-tracing if $\emptyset'$ is c.e. traceable by A. Clearly, these definitions also make sense for Turing degrees. By [Sim07, Kur81] (also see [Nie09, Chapter 8]) we have the following implications, where none of them can be reversed.

$$\text{Turing complete} \underset{\not\Leftarrow}{\Rightarrow} \text{LR-complete} \underset{\not\Leftarrow}{\Rightarrow} \text{JT-complete} \underset{\not\Leftarrow}{\Rightarrow} \emptyset'\text{-tracing.} \tag{1.1}$$

It is possible to show that *$\emptyset'$-tracing does not imply high*. Corollary 1 below shows that *high does not imply $\emptyset'$-tracing*. Therefore the two notions are incomparable. However it is well known (e.g. see Simpson [Sim07]) that JT-completeness implies highness.

Recall from [DJS96] that a degree $\mathbf{a}$ is called array computable if there exists a function $f \leq_{\mathrm{wtt}} \emptyset'$ which dominates all functions $g \leq_T \mathbf{a}$.

Definition 1.2. A degree $\mathbf{a}$ is weakly array computable if there exists a function $f \leq_{\mathrm{T}} \emptyset'$ which dominates all functions $g \leq_T \mathbf{a}$.

We notice that although weak array computability has not been defined explicitly in the literature, it has appeared implicitly in many arguments that are presented in terms of array computability or generalized low$_2$ sets (for example, see [Ler83, Chapter IV.3]). We show the following.

Theorem 1.1. *If $\mathbf{c}$ is c.e. and is c.e. traceable by $\mathbf{a}$, then no function that is computable in $\mathbf{c}$ dominates every function computable in $\mathbf{a}$.*

According to (1.1), we have the following.

Corollary 1. *If a degree is $\mathbf{0}'$-tracing (or JT-complete, or a.e. dominating) then it is not (weakly) array computable.*

Recall from [DJS96] that all minimal degrees are array computable.

Corollary 2. *Every a.e. dominating degree is array non-computable. In particular, it is not minimal.*

Corollary 2 contrasts the existence of a high minimal degree, which was shown in Cooper [Coo73]. In particular, the property of non-minimality separates almost everywhere domination (or JT-completeness) from highness.

The same question has been investigated for local structures of the Turing degrees. For example, in the Σ_1^0 structure of the Turing degrees Harrington (see [Mil81]) showed that some high degrees are non-cuppable. That is, their supremum with any incomplete c.e. degree is incomplete. The property of non-cupping was a candidate for separating a.e. domination from highness in the Σ_1^0 structure, until it was shown in [BM09] that there is a non-cuppable a.e. dominating c.e. degree. A very promising candidate for such a property is the existence of minimal pairs. Lachlan [Lac66] showed that there is a pair of high c.e. degrees that form a minimal pair. The existence of minimal pairs of c.e. almost everywhere dominating degrees has been the object of intense research in the past few years. However, it remains open. In Section 3 we show that there is a minimal pair of Δ_2^0 almost everywhere dominating degrees.

2. Proof of theorem 1.1

Let $\mathbf{c}, \mathbf{a}$ be as in the hypothesis of Theorem 1.1. Also let C be a c.e. set of degree $\mathbf{c}$ and A a set of degree $\mathbf{a}$. Then there exists a computable function f such that every C-computable function has an A-c.e. trace with bound f. Let $(E_{e,i})$ be an effective sequence of all c.e. operators such that $|E^X_{e,i}| < f(i)$ for all $X \in 2^\omega$ and $e, i \in \mathbb{N}$. This is an effective sequence of all c.e. traces relative to any oracle, with bound f. We obtain a universal trace by letting $V_i = \bigcup_{e<i} E_{e,i}$ (and hence, $V_i^X = \bigcup_{e<i} E^X_{e,i}$ for any $X \in 2^\omega$). Let $h(i) := if(i)$. Clearly, h is a bound for (V_i^X) for every $X \in 2^\omega$. Moreover, for every C-computable function g we have $g(i) \in V_i^A$ for almost all $i \in \mathbb{N}$. Since we have no effective way to locate A, we will work simultaneously for all (uncountably many) sets X such that C is c.e. traceable by X with bound f.

Let $g = \Phi^C$ be any given Turing C-computable function, where Φ is a Turing functional with use φ. Based on g, we will build a C-computable function Θ^C and a Turing functional Γ such that

- $\Gamma^X(n) \downarrow$ for all $n \in \mathbb{N}^{[e]}$
- $\Gamma^X(t) > \Phi^C(t)$ for some $t \in \mathbb{N}^{[e]}$

for all X, e such that $\Theta^C(e) \in V_e^X$. Since C is c.e. traceable by A, there will be some n such that Θ^C is traced by V_i^A on arguments $\geq n$. In that case Γ^A can easily be extended to a total A-computable function which is not dominated by g. Hence such a construction suffices for the proof of Theorem 1.1.

2.1. Requirements and plan.

It suffices to satisfy the following requirements.

$$R_e : \text{for all } X, \text{ if } \Theta^C(e) \in V_e^X \left\{ \begin{array}{ll} \text{(a)} & \forall n \in \mathbb{N}^{[e]},\ \Gamma^X(n)\downarrow \\ \text{(b)} & \exists n \in \mathbb{N}^{[e]},\ \Gamma^X(n) > \Phi^C(n) \end{array} \right\}.$$

In order to describe the idea behind the construction, assume that we only had to deal with one path X. Then we would choose an argument n and try to achieve $\Gamma^X(n) > \Phi^C(n)$. Before we enumerate such a Γ-axiom, we would define $\Theta^C(e)$ to be a *large*[4] number with use $\varphi(n)$ and wait until this value of $\Theta^C(e)$ appears in V_e^X. If later $C \upharpoonright \varphi(n)$ changes causing $\Gamma^X(n) \leq \Phi^C(n)$, we would repeat the same procedure on a different argument n'. Notice that after an unsuccessful round, $\Theta^C(e)$ is undefined due to the C change. Hence, in each round we always (re)define Θ_e^C. Moreover, after each unsuccessful round $|V_e^X|$ increases by one. Hence there can be at most $h(e)$ unsuccessful rounds, before we succeed.

In reality, we have to deal with many paths X simultaneously. To achieve this, we will use compactness and focus on a Π_2^0 relation to measure our success in a universal way (not depending on a particular X). Given $e \in \mathbb{N}$, consider the Π_2^0 condition (2.1) and its negation (2.2).

$$\forall n, s_0\ \exists \sigma\ \exists s > s_0,\ \forall i \in \mathbb{N}^{[e]} \upharpoonright n\ [\Gamma^\sigma(i)[s]\downarrow\ \wedge\ \Phi^C(i)[s] \geq \Gamma^\sigma(i)[s]] \tag{2.1}$$

$$\exists n, s_0\ \forall \sigma\ \forall s > s_0,\ \exists i \in \mathbb{N}^{[e]} \upharpoonright n\ [\Gamma^\sigma(i)[s]\uparrow\ \vee\ \Phi^C(i)[s] < \Gamma^\sigma(i)[s]] \tag{2.2}$$

Relation (2.1) is a strong form of failure to satisfy R_e and is exactly the outcome we wish to avoid. On the other hand (2.2) does not, by itself, imply the satisfaction of R_e. However, given (2.2) the construction will be able to guarantee R_e. If a definition $\Gamma^\sigma(n)\downarrow$ is made at stage s of the construction, we set $\Gamma^\sigma(n)$ to be a *large* number. Since Φ^C is total, this means that the sets

$$T_n = \{X \mid \forall i \leq n\ [\Gamma^X(i)\downarrow\ \wedge\ \Phi^C(i) \geq \Gamma^X(i)]\} \tag{2.3}$$

are clopen (T_n consists of the reals extending one of the finitely many strings σ such that $\Gamma^\sigma(n)[s_0]\downarrow$ and $\Phi^C(n) \geq \Gamma^\sigma(n)$, where s_0 is the stage where the approximation to $\Phi^C(n)$ settles). If (2.1) holds, $T_n \neq \emptyset$ for all $n \in \mathbb{N}$ hence by compactness there is some X such that Γ^X is total and $\Gamma^X(n) \leq \Phi^C(n)$ for all $n \in \mathbb{N}^{[e]}$. This is exactly the outcome that the construction will prevent. For such reals X that seem to be a member of all T_n (notice that this is a Π_2^0 condition) we will pick a target argument n and decide not to define $\Gamma^X(n)$ unless $\Theta^C(e)$ is traced by V_e^X. Since we only work with finite approximations to reals, we will do this for strings σ. If σ seems to be extended by such a real X, we will pick a target n and enumerate it into a target set I_σ (in fact, all I_τ, $\tau \supset \sigma$). When we try to define $\Gamma(n)$ for some real extending σ and $n \in I_\sigma$, we are committed to refrain from the definition unless the appropriate tracing (as described above) takes place.

[4] that is, larger than any value of any parameter in the construction up to the current stage.

A graphical visualization of the argument is as follows. We start with the binary tree, and at each stage s we consider the strings of length s. For each finite path σ, we represent the numbers n in the target set I_σ by a dot on the nth digit of σ. A path through the binary tree may have many dots, because of previously assigned targets that were not successful. The largest number n_σ in I_σ (for a path σ of length s) is the *active* or *current* target for σ at stage s. This means that either $\Gamma^\sigma(n_\sigma)\uparrow$ or $\Gamma^\sigma(n_\sigma) > \Phi^C(n)$ at stage s. A dot on σ may be active for some $\tau \supset \sigma$ at some stage and non-active for another $\tau' \supset \sigma$. The construction will explicitly ensure that at each stage s, every path of length s has an active dot. Condition (2.2) requires that there is a single level on the binary tree, where every path of that level has a permanently active dot. This interpretation of the strategy is illustrated in Figure 1.

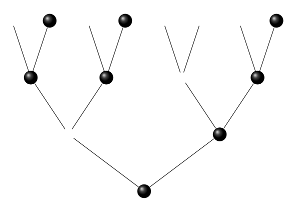

FIGURE 1. A graphical representation of the construction. Black dots along the paths σ represent positions in I_σ.

In the following we define and verify the strategy for R_e, which will only enumerate axioms for Γ on arguments in $\mathbb{N}^{[e]}$. The full construction is a straightforward combination of these strategies, where there is not interaction amongst different strategies. The module for R_e takes place on stages $s \in \mathbb{N}^{[e]}$. By speeding up the approximation to Φ, C we may assume that for all stages $s \in \mathbb{N}$ we have $\Phi^C(i)[s]\downarrow$ for all $i \le s$. This assumption is without loss of generality. At stage s, we consider all strings σ of length s, and decide if s belongs to I_σ. If it does not, then we trivially define $\Gamma^\sigma(s) = 0$. Otherwise we consider a definition subject to the tracing condition described above.

2.2. **Strategy for R_e.** At stage $s \in \mathbb{N}^{[e]}$ do the following for each string σ of length s:

(I) Determine if s is in I_σ.

- If $\exists i < s, i \in \mathbb{N}^{[e]}\ [\Gamma^\sigma(i)\uparrow\ \vee \Phi^C(i)[s] \le \Gamma^\sigma(i)]$, let $\Gamma^\sigma(s) = 0$.
- Otherwise let $s \in I_\tau$ for all $\tau \supseteq \sigma$.

Let $m_s = \max(\cup_{|\sigma|=s} I_\sigma)$.

(II) Attempt to define $\Gamma^\sigma(n)$ if $n \in I_\sigma \cap \mathbb{N}^{[e]}$ and $\Gamma^\sigma(n)[s]\uparrow$.

- If $\Theta^C(e)[s]\uparrow$ define $\Theta^C(e)[s]$ with use $C[s] \upharpoonright \varphi(m_s)[s]$ to be a *large* number.
- If $n \in I_\sigma \cap \mathbb{N}^{[e]}$, $\Gamma^\sigma(n)\uparrow$ (there is at most one such) and $\Theta^C(e)[s] \in V_e^\sigma$, define $\Gamma^\sigma(n)$ to be a *large* number.

2.3. **Verification for the R_e strategy.**

Lemma 2.1. *The functional* Γ *is consistent.*

Proof. At stage s the construction can only define $\Gamma^\sigma(n)$ for some $n \le s$ if

- σ is of length s.
- $\Gamma^\sigma(n)$ is currently undefined.

The consistency of Γ follows from this feature of the construction. □

Lemma 2.2. *At the end of each stage s, for each string σ of length s there exists exactly one number $n_\sigma \in I_\sigma$ (namely* $\max I_\sigma$*) such that*

$$\Gamma^\sigma(n_\sigma)[s]\uparrow \ \vee \ \Gamma^\sigma(n_\sigma) > \Phi^C(n_\sigma)[s]. \tag{2.4}$$

Proof. This follows by a straightforward induction on the construction of Section 2.2. □

Lemma 2.3. *If* (2.1) *holds, then $\Theta^C(e)$ is redefined infinitely many times and the sequence of values that it takes is increasing.*

Proof. Let $n_\sigma = \max I_\sigma$ for each string σ. The value of n_σ depends on the stage of the construction. If there was a stage s_0 such that $\Theta^C(e)$ was defined for the last time, according to the construction the C use of this definition will be $\varphi(m_{s_0})$. This is larger than $\varphi(n_\sigma)$ for all σ of length s_0. Given any string σ of length s_0 and $X \supset \sigma$, according to Lemma 2.2 and the fact that

$$C[s_0] \upharpoonright \varphi(n_\sigma[s_0]) = C \upharpoonright \varphi(n_\sigma[s_0])$$

one of the following is true

- $\Gamma^\sigma(n_\sigma) > \Phi^C(n_\sigma)$
- $\Theta^C(e) \notin V_e^X$, in which case $\Gamma^X(n_\sigma)\uparrow$
- $\Theta^C(e) \in V_e^X$, in which case $\Gamma^X(n_\sigma)$ will be defined after stage s_0 and thus, $\Gamma^X(n_\sigma) > \Phi^C(n_\sigma)$.

But in that case no more numbers will be enumerated in $I := \cup_{X\in 2^\omega} I_X$ after stage s_0, which contradicts (2.1). □

Lemma 2.4. *If Γ is constructed as in* Section 2.2*, then* (2.2) *holds.*

Proof. For a contradiction, suppose that (2.1) holds. Then, as explained in Section 2.1, the clopen sets T_n of (2.3) are non-empty and since $T_{i+1} \subseteq T_i$, by compactness there is some $X \in \cap_i T_i$. Let $I_X := \cup_{\sigma\subset X} I_\sigma$.

We claim that I_X is infinite. Indeed, otherwise there is some stage s_0 where Φ^C settles on all arguments in I_X, and is larger than Γ^X on all of these arguments. But since $X \in \cap_i T_i$ the construction of Section 2.2 would enumerate s_0 into $I_{X\restriction s_0}$, which is a contradiction; thus I_X has to be infinite.

For the final contradiction, we will show that V_e is unbounded. Let $t \in \mathbb{N}$. By Lemma 2.3 consider a stage s_1 where $\Theta^C(e)[s_1] > t$. Consider some $n \in I_X$ such that $n > s_1$ and let s_2 be the stage where $\Gamma^X(n)$ is defined. Then $\Theta^C(e)[s_2] \downarrow\in V_e^X$ and so, by Lemma 2.3 some number larger than t belongs to V_e^X. This completes the proof of the lemma. □

Lemma 2.5. *The set $I := \cup_{X\in 2^\omega} I_X$ is finite and $\Theta^C(e)$ is permanently defined .*

Proof. Since (2.2) holds by Lemma 2.4, I is bounded by the number n in the existential quantifier of (2.2). Then $\Theta^C(e)$ will only be defined with use at most $\varphi(n)$, so it will eventually settle. □

Lemma 2.6. *Requirement R_e in (2.1) is satisfied.*

Proof. Suppose that X is a real such that $\Theta^C(e) \in V_e^X$. Since I_X is finite, the parameter $n_{X\restriction t}$ will reach a limit n_x as $t \to \infty$. Now by (2.4) of Lemma 2.2 we have that $\Gamma^X(n_x)$ is defined and greater than $\Phi^C(n_x)$. By clause (II) of the construction this implies that, if $\Theta^C(e) \in V_e^X$ then Γ^X is total on $\mathbb{N}^{[e]}$. □

The global construction of Θ, Γ is a straghtforward combination of the R_e modules: at stage s, if $s \in \mathbb{N}^{[e]}$ run the strategy for R_e. The verification for R_e presented above implies that R_e is met for each $e \in \mathbb{N}$. This concludes the proof of Theorem 1.1.

3. A minimal pair of a.e. dominating degrees $\leq_T \emptyset'$

This section is devoted to a proof of the following fact.

Theorem 3.1. *In the Turing degrees, there exists a minimal pair of a.e. dominating degrees below $\mathbf{0}'$.*

We wish to construct two a.e. dominating sets A, B such that the following requirements are satisfied.

$$M_e: \ \Phi_e^A = \Phi_e^B \text{ total } \Rightarrow \Phi_e^A \text{ is computable}$$

where (Φ_e) is an effective list of all Turing functionals. It is not hard to see that if A, B are non-computable and all M_e are satisfied, then A, B form a minimal pair. In order to ensure that A, B are a.e. dominating, by [KH07, Sim07] it suffices to ensure for some c.e. operator $V : 2^\omega \to \mathcal{P}(2^{<\omega})$ we have

$$\mu(V^A) < 1 \ \text{ and } \ U^{\emptyset'} \subseteq V^A \tag{3.1}$$

$$\mu(V^B) < 1 \ \text{ and } \ U^{\emptyset'} \subseteq V^B \tag{3.2}$$

where $U^{\emptyset'}$ is the second member of the universal Martin-Löf test relative to $\emptyset'$. Notice that $\mu(U^{\emptyset'}) < 2^{-2}$. We define V in advance as follows: fix a computable function $f : \mathbb{N} \to 2^{<\omega}$ such that for all $\sigma \in 2^{<\omega}$ there exist infinitely many $m \in \mathbb{N}$ such that $f(m) = \sigma$. Then for all $X \in 2^{\omega}$ define $V^X = \{f(m) \mid X(m) = 1\}$, which is clearly a c.e. operator. The choice of V is such that, for any string σ and any clopen set C we can effectively choose $\tau \supset \sigma$ such that $V^{\tau} - V^{\sigma} = C$.

Our argument is a finite extension construction relative to $\emptyset'$. For the satisfaction of M_e we will try to find an e-splitting extending the currently defined segments of A, B. Recall that an e-splitting is a pair of strings σ, τ such that $\Phi_e^{\sigma}(n) \neq \Phi_e^{\tau}(n)$ for some $n \in \mathbb{N}$. If there is no such e-splitting, it is easy to see that any total function computed by both Φ_e^A and Φ_e^B has to be computable. Thus in this case M_e is met. Otherwise we would like to extend the current segments of A, B with the e-splitting strings, thus meeting M_e in another way. However the e-splitting extensions may add too much measure in V^A, V^B, in which case we may refrain from doing so. In general, we will allow strategy M_e to add at most 2^{-e-2} measure in each of V^A, V^B.

The construction defines monotone sequences of strings $(\sigma_s), (\tau_s)$ and lets $A = \cup_s \sigma_s$, $B = \cup_s \tau_s$. At each stage s we also define segments of A, B in order to cover $U^{\emptyset' \restriction s}$ with V^A, V^B. Suppose that at some stage $s+1$ we do not find suitable extensions σ_{s+1}, τ_{s+1} for the satisfaction of M_e. In that case M_e is satisfied *unless* $\mu(V^A - V^{\sigma_s}) > 2^{-e-2}$ or $\mu(V^B - V^{\tau_s}) > 2^{-e-2}$. If we continuously check for the availability of suitable extensions for M_e, we claim that M_e will be satisfied. Indeed, there will be some stage s_0 such that $\mu(V^A - V^{\sigma_{s_0}}) \leq 2^{-e-2}$ and $\mu(V^B - V^{\tau_{s_0}}) \leq 2^{-e-2}$. If at s_0 we ask for a suitable e-splitting for M_e and we do not find it, we can employ the usual Kleene-Post argument to show that every total function computed by both Φ_e^A and Φ_e^B has to be computable. We say that strategy M_e *requires attention* at stage $s+1$ if we have not *acted* on it and there are $\sigma \supseteq \sigma_s$, $\tau \supseteq \tau_s$, $n \in \mathbb{N}$ such that $\Phi_e^{\sigma}(n) \neq \Phi_e^{\tau}(n)$ and $\mu(V^{\sigma} - V^{\sigma_s}) < 2^{-e-2}$, $\mu(V^{\tau} - V^{\tau_s}) < 2^{-e-2}$.

Construction. Let $\sigma_0 = \tau_0 = \emptyset$. At stage $s+1$ choose $\sigma \supset \sigma_s$, $\tau \supset \tau_s$ such that

$$V^{\sigma} - V^{\sigma_s} = V^{\tau} - V^{\tau_s} = U^{\emptyset' \restriction s+1} - U^{\emptyset' \restriction s}$$

Moreover, if there is a strategy that requires attention, choose the least one M_e and let σ_{s+1}, τ_{s+1} be an e-splitting extending σ, τ respectively, such that

$$\mu(V^{\sigma_{s+1}} - V^{\sigma}) < 2^{-e-2} \text{ and } \mu(V^{\tau_{s+1}} - V^{\tau}) < 2^{-e-2}.$$

Say that we have acted on M_e.

Verification. By construction, $\mu(V^A), \mu(V^B)$ are at most $\mu(U^{\emptyset'}) + \sum_e 2^{-e-2}$ so less than 1. Also, $U^{\emptyset'}$ is contained in both V^A and V^B. It remains to show that M_e is satisfied, for all $e \in \mathbb{N}$. Clearly, each M_e stops requiring attention after some stage. Fix e and choose some stage s_0 such that $\mu(V^A - V^{\sigma_{s_0}}) \leq 2^{-e-2}$, $\mu(V^B - V^{\tau_{s_0}}) \leq 2^{-e-2}$ and no M_i, $i < e$ requires attention after s_0.

Suppose that Φ_e^A, Φ_e^B are total and equal to the same function f. Then M_e never requires attention in the construction. To compute f on an argument n we just need to look for a string $\sigma \supseteq \sigma_{s_0}$ such that $\mu(V^\sigma - V^{\sigma_{s_0}}) < 2^{-e-2}$ and $\Phi_e^\sigma(n)\downarrow$. If $f(n)$ did not equal $\Phi_e^\sigma(n)\downarrow$ we would get the contradiction that at stage s_0 the construction acts on M_e.

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