RANDOM REALS AND LIPSCHITZ CONTINUITY

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Abstract. Lipschitz continuity is used as a tool for analyzing the relationship between incomputability and randomness. Having presented a simpler proof of one of the major results in this area—the theorem of Yu and Ding that there exists no cl-complete c.e. real—we go on to consider the global theory. The existential theory of the cl degrees is decidable but this does not follow immediately by the standard proof for classical structures such as the Turing degrees since the cl degrees is a structure without join. We go on to show that strictly below every random cl degree there is another random cl degree. Results regarding the phenomenon of quasi-maximality in the cl degrees are also presented.

1. Introduction

In randomness and incomputability we have two fundamental measures of complexity and it therefore seems an important and basic question to ask how these two measures of complexity are related. In answering this question the reducibility with which we will principally be concerned will, of course, be the Turing reducibility. The Turing reducibility, however, does not preserve randomness and it is clear that all Turing degrees will contain reals which are very far from being random—reals with initial segments of very low algorithmic complexity. The suggestion is therefore that it may be useful to study reducibilities which relate more directly to randomness, reducibilities in particular which preserve randomness, in order to make the relationship between these two measures of complexity clearer. Such considerations lead us to consider the (perhaps unfortunately named) strong weak truth table reducibility, which was originally introduced by Downey, Hirschfeldt and La Forte [7].

Definition 1.1. *Given reals $\alpha, \beta \in 2^\omega$ we say that α is strong weak truth table reducible to β ($\alpha \leq_{sw} \beta$) if there exists a Turing functional Γ and a constant c such that $\Gamma^\beta = \alpha$ and the use of this computation on any argument n is bounded by $n + c$.*

Proposition 1.1. *(Downey, Hirschfeldt, La Forte* [7]*) The sw reducibility preserves randomness: if $\alpha \leq_{sw} \beta$ and α is (Martin-Löf) random then β is random.*

Having defined the reducibility we can then go on, as always, to consider the induced degree structure. The degrees are the equivalence classes under

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the sw reducibility and the ordering that we consider on these degrees is that induced by the sw reducibility on the reals. It is with the theory of this structure that we shall be concerned in this paper. The sw degrees possess the nice property that any degree either contains *only* random or *no* random reals. The sw reducibility is a generalization of the so-called *ibT* reducibility.

Definition 1.2. *(Soare*[13]*) We say that* α *is identity bounded reducible to* β *(*$\alpha \leq_{ibT} \beta$ *for short) if there is a Turing functional* Γ *such that* $\Gamma^{\beta} = \alpha$ *and the use of the computations is bounded by the identity function i.e. on each argument* n *the* β*-queries are for numbers* $\leq n$*. The induced degrees are called ibT degrees.*

The *ibT* reducibility is closely related to a 'domination' reducibility which was used by Nabutovsky, Soare and Weinberger in their applications of computability to differential geometry (see Soare[13]). In fact, the sw reducibility is also related very directly to Lipschitz continuity.

Definition 1.3. *A partial operator* Γ *from a (pseudo) metric space* (X, d) *to itself is Lipschitz continuous if there is a constant* C *such that*

$$d(\Gamma(x), \Gamma(y)) \leq C \cdot d(x, y) \tag{1}$$

for all x, y *in the domain of* Γ*.*

Definition 1.4. *Let* $(2^{<\omega}, d)$ *be the pseudo metric space which is the set of finite binary strings with the pseudo metric* d *defined as follows. For incompatible* $\sigma, \tau \in 2^{<\omega}$*,* $d(\sigma, \tau) = 2^{-n}$ *where* n *is the least bit on which* σ *and* τ *differ. For compatible* σ, τ*,* $d(\sigma, \tau) = 0$*.*

Definition 1.5. *(A.E.M. Lewis, G. Barmpalias* [3]*) Let us say* α *is computably Lipschitz reducible to* β *(*$\alpha \leq_{cl} \beta$*) if there exists a Turing functional* Γ *such that* $\Gamma^{\beta} = \alpha$ *and which is Lipschitz continuous when considered as a map* $(2^{<\omega}, d) \mapsto (2^{<\omega}, d)$*.*

Our motivation in making this definition can be seen in the following proposition.

Proposition 1.2. *(A.E.M. Lewis, Barmpalias* [3]*) For all* $\alpha, \beta \in 2^{\omega}$*,* $\alpha \leq_{cl} \beta$ *iff* $\alpha \leq_{sw} \beta$*.*

Preferring this terminology we shall talk in terms of the cl degrees rather than the sw degrees in all that follows. The results of this paper are divided into three sections. In section 2 we shall give a simpler proof of one of the major results in this area, theorem 1.1 below.

Theorem 1.1. *(Yu, Ding* [15] *) There exists no cl-complete c.e. real.*

By extending the methods involved in the proof of this result Yu and Ding were, in fact, able to achieve the stronger result which is theorem 1.2. The proof we detail in section 2 can also be extended in precisely the same way in order to give this result.

Theorem 1.2. *(Yu, Ding* [15] *) There exist c.e. reals* α *and* β *such that for no c.e. real* γ *is it the case that both* $\alpha \leq_{cl} \gamma$ *and* $\beta \leq_{cl} \gamma$*.*

In section 3 we show that the existential theories of the cl and the ibT degrees are decidable. Moreover, the existential theory of the cl (ibT) degrees of c.e. sets and the existential theory of the cl (ibT) degrees of c.e. reals are also decidable. In fact, we show that all finite partial orders are embeddable in these structures. These results do not follow immediately by the standard proof for classical structures such as the Turing degrees, since this proof requires the existence of a join (or, at least, an 'upper bound' operator). This difficulty, however, is quite easily overcome by considering sets of reals for which we do have a join (or, for the case of ibT, an 'upper bound') operator.

Theorem 1.3. *All finite partial orders are embeddable in the cl and ibT degrees of c.e. sets or c.e. reals. Thus, the existential theories of these structures are decidable.*

In section 4 we shall consider the quasi-maximality phenomenon in the cl degrees. The existence of quasi-maximal degrees was first proved in [3].

Theorem 1.4. *(A.E.M. Lewis, Barmpalias* [3]*) There is a quasi-maximal cl-degree* $\mathbf{a}$*: if* $\alpha \in \mathbf{a}$ *then any* $\beta \geq_{cl} \alpha$ *is Turing below* α*.*

This may be seen as quite a striking result since we are not generally used to degree structures possessing anything like maximal elements in the global sense. In the cl degrees, however, quasi-maximality is widespread and relates very directly to randomness:

Theorem 1.5. *(A.E.M. Lewis, Barmpalias* [3]*) Every random real is of quasi-maximal cl degree.*

The following corollary then follows immediately from the existence of low random reals and from the fact that every Turing degree above $0'$ contains a random real.

Corollary 1.1. *(A.E.M. Lewis, Barmpalias* [3]*)*

1. *There exist low reals which are of quasi-maximal cl degree.*
2. *There exists no cl-complete* Δ_2 *real.*
3. *Every Turing degree above* $0'$ *contains a real of quasi-maximal cl degree.*

Theorem 1.6 tells us that, unfortunately, quasi-maximality does not characterize randomness.

Theorem 1.6. *(A.E.M. Lewis, Barmpalias* [3] *) There exists* α *of quasi-maximal cl degree which is not random.*

Upon realizing that quasi-maximality does not characterize randomness it is natural to ask whether maximality might provide such a characterization. Such hopes, however, are in vain.

Theorem 1.7. *(A.E.M. Lewis, Barmpalias* [3] *) No random real is of maximal cl degree.*

Theorem 1.7, then, shows us that strictly above each random cl degree there is another random cl degree. In section 4 of this paper we shall prove the complimentary result:

Theorem 1.8. *Strictly below each random cl degree there is another random cl degree. Also, strictly below each random ibT degree there is another random ibT degree.*

We also show that it is not the case that the random cl degrees are precisely the quasi-maximal non-maximal cl degrees.

Theorem 1.9. *There exists α which is not random and which is of quasi-maximal non-maximal cl degree.*

We assume some background on computability theory and some knowledge of standard conventions, most of which can be found in Soare[14]. Basic knowledge of algorithmic randomness is also helpful. Section 4 requires some familiarity with global constructions in degree theory, and in particular the finite extension method. Also, familiarity with the classical proof of the embeddability of finite partial orders into the Turing degrees is helpful for section 3 .

2. Proof of theorem 1.1

Given a c.e. real α it suffices to construct c.e. reals β, γ such that

$$\mathcal{Q}_{\Phi,\Psi} : \beta \neq \Phi^{\alpha} \ \vee \ \gamma \neq \Psi^{\alpha}$$

for all partial computable cl-functionals Φ, Ψ.

Definition 2.1. *The positions on the right of the decimal point in a binary expansion are numbered as $1, 2, 3, \dots$ from left to right. The positions on the left of the decimal point are numbered as $0, -1, -2, \dots$.*

Consider the following cl-game between α and β. These numbers have initial values and during the stages of the game they can only increase. If β increases and i is the leftmost position where a β-digit change occurred, then α has to increase in such a way that some α-digit at a position $\leq i$ changes. This game describes a cl-reduction. If α has to code two reals β, γ then we get a similar game (where, say, at each stage only one of β, γ can change). We say that α follows the *least effort strategy* if at each stage it increases by the least amount needed. The following observation will be useful in what follows.

Lemma 2.1. (Passing through lemma) *Suppose that in some game (e.g. like the above) α has to follow instructions of the type 'change a digit at position $\leq n$'. Although $\alpha_0 = 0$, some α' plays the same game while starting with $\alpha'_0 = \sigma$ for a finite binary expansion σ. If α and α' both use the 'least effort' strategy described above and the sequence of instructions only ever demands change at positions $> |\sigma|$ then at every stage s,*

$$\alpha'_s = \alpha_s + \sigma. \tag{2}$$

Proof. By induction on s. For $s = 0$ the result is obvious. Suppose that the induction hypothesis holds at stage s. Then α'_s, α_s have the same expansions after position $|\sigma|$. At $s+1$, some demand for a change at some position $> |\sigma|$ appears and since α, α' look the same on these positions, α'_s will need to increase by the same amount that α_s needs to increase. So $\alpha'_{s+1} = \alpha_{s+1} + \sigma$ as required. □

Given $n > 0$ and $t \in \mathbb{Z}$ we are going to define the *Yu-Ding procedure* amongst α, β, γ with attack interval $(t-n, t]$. We assume that α, β, γ have initial value 0. Repeat the following instructions until $\beta(i) = \gamma(i) = 1$ for all $i \in (t-n, t]$.

s odd (1) let $\beta = \beta + 2^{-t}$ and let b equal the leftmost position where a change occurs in β.

(2) Add to α the least amount which causes a change in a digit at a position $\leq b$.

s even (1) let $\gamma = \gamma + 2^{-t}$ and let g equal the leftmost position where a change occurs in γ.

(2) Add to α the least amount which causes a change in a digit at a position $\leq g$.

It is not hard to see that the above procedure describes how α evolves when it tries to code β, γ via cl-reductions with identity use and it uses the *least effort strategy* (provided that the changes in β, γ occur at expansionary stages). Player α follows the *least effort strategy* when it increases by the least amount which can rectify the functionals holding its computations of β, γ.

Proposition 2.1. *Let $n > 0$. For any $k \in \mathbb{Z}$ the Yu-Ding procedure amongst α, β, γ with attack interval $(k, k+n]$ ends up with $\alpha = n2^{-k}$.*

Proof. By induction: for $n = 1$ the result is obvious. Assume that the result holds for n. Now pick $k \in \mathbb{Z}$ and consider the attack using $(k-1, k+n]$. It is clear that up to a stage s_0 this will be identical to the procedure with attack interval $(k, k+n]$. By the induction hypothesis $\alpha_{s_0} = n2^{-k}$ and $\beta(i) = \gamma(i) = 1$ for all $i \in (k, k+n]$, while $\beta(k) = \gamma(k) = 0$. According to the next step β changes at position k and this forces α to increase by 2^{-k} since α has no 1s to the right of position k. Then γ does the same and since α still has no 1s to the right of position k, α has to increase by 2^{-k} once again. So far

$$\alpha = n2^{-k} + 2^{-k} + 2^{-k} = n2^{-k} + 2^{-(k-1)}$$

and $\beta(i) = \gamma(i) = 0$ for all $i \in (k, k+n]$ while $\beta(k) = \gamma(k) = 1$. By applying the induction hypothesis again and the passing through lemma 2.1 the further increase of α will be exactly $n2^{-k}$. So

$$\alpha = n2^{-k} + 2^{-(k-1)} + n2^{-k} = (n+1)2^{-(k-1)}$$

as required. □

Now let us define the *Yu-Ding strategy* with attack interval $(t-n, t]$ to be the enumerations of β, γ as in the Yu-Ding procedure. In the context of a requirement $\mathcal{Q}_{\Phi,\Psi}$ we assume that each step is performed only when the reductions $\Phi^\alpha = \beta, \Psi^\alpha = \gamma$ are longer than ever before (i.e. at an expansionary stage).

Lemma 2.2. *In a game where α has to follow instructions of the type 'change a digit at position $\leq n$' (e.g. an cl-game between α and β, γ) the least effort strategy is a best strategy for α. In other words if a different strategy produces α' then at each stage s of the game $\alpha_s \leq \alpha'_s$.*

Proof. By induction on the stages s. We have that $\alpha_0 \leq \alpha'_0$. If $\alpha_s = \alpha'_s$ then it is clear from the definition of the least effort strategy that the induction hypothesis will hold at stage $s+1$. So suppose otherwise. Then $\alpha_s < \alpha'_s$ so that there will be a position n such that $0 = \alpha_s(n) < \alpha'_s(n) = 1$ and $\alpha_s \upharpoonright n = \alpha'_s \upharpoonright n$. Suppose that α, α' are forced to change at a position $\leq t$ at stage $s+1$. If $t < n$ it is clear that $\alpha_{s+1} \leq \alpha'_{s+1}$. Otherwise the leftmost change α can be forced to make is at position n. Once again $\alpha_{s+1} \leq \alpha'_{s+1}$. □

Although we implicitly assumed that the use in the functionals of $\mathcal{Q}$ is the identity function x, the case when it is $x+c$ is not very different. The Yu-Ding strategy with attack interval $(k,t]$ against $\mathcal{Q}'$ where the use of both functionals is (bounded by) $x+c$ gives the same result (assuming the least effort strategy on the part of α) as the Yu-Ding strategy with attack interval $(k+c, t+c]$ against $\mathcal{Q}$ where the use of both functionals is the identity. So, from lemma 2.2 and proposition 2.1 we get

Corollary 2.1. *If β, γ follow the Yu-Ding strategy in attacking $\mathcal{Q}$ (where the functionals have use bounded by $x+c$) with attack interval $(k, k+n]$ then either $\mathcal{Q}$ is satisfied or $\alpha \geq n2^{-(k+c)}$.*

The above corollary is all we need to prove the theorem. Assume an effective list of all requirements and successively assign attack intervals to them. If the attack interval for $\mathcal{Q}_i$ is $(k,n]$ define the one for $\mathcal{Q}_{i+1}$ to be $(n,t]$ where t is the least such that the estimation of corollary 2.1 gives $\alpha \geq 1$. Now assume that $\alpha \in [0,1)$ and apply the Yu-Ding strategy for each of the requirements on the relevant intervals in a global construction. There is no interaction amongst the strategies and the satisfaction of all the requirements follows from corollary 2.1.

3. The existential theory of the cl degrees

Given the lack of join in cl degrees, we wish to define a class of c.e. sets $\mathcal{A}$ such that the least upper bound of any two sets in this class exists and which contains a cl computably independent family of sets (in the sense that no set in that family can be cl-computed from the join of a finite number of sets in that family). The classical argument then suffices in order to show that the existential theory of the cl degrees, the existential theory of the cl degrees of c.e. sets and the existential theory of the cl degrees of c.e. reals are all decidable.

3.1. **Definition of $\mathcal{A}$.** Let $\mathcal{A}_0$ be the class of c.e. subsets of $I_0 = \{2^k \mid k \in \mathbb{N}\}$. Next, let $\mathcal{A}_1$ be the class of c.e. subsets of $I_0 \cup I_1$ where $I_1 = \{2^k - 1 \mid k > 1 + \log 1\}$. More generally, if

$$I_n = \{2^k - n \mid 2^k - n > 2^{k-1}\} = \{2^k - n \mid k > 1 + \log n\}$$

let $\mathcal{A}_k$ be the class of c.e. subsets of $\cup_{j \leq k} I_j$. These are the c.e. sets where the 1s in their characteristic sequence can only be *on* positions 2^n or in the previous k positions from them inside $(2^{n-1}, 2^n)$. We show that the class

$$\mathcal{A} = \cup_i \mathcal{A}_i$$

is closed under join. First note that $\mathcal{A}_0 \subset \mathcal{A}_1 \subset \dots$. We show that if $B, C \in \mathcal{A}_k$ then there is an $A \in \mathcal{A}_{2k+1}$ which is the least upper bound of B, C. We will only code B, C in A on arguments $\geq 2^{n_0} - k$ where n_0 is the least such that $2^{n_0} - 2k - 1 > 2^{n_0 - 1}$ i.e. $n_0 > 1 + \log(2k+1)$. The coding function for B will be the identity $I(x)$: whenever some $n \geq 2^{n_0} - k$ enters B we enumerate $n \searrow A$. The coding function for C will be $I(x) - k - 1$: whenever some $n \geq 2^{n_0} - k$ enters C we enumerate $n - k - 1 \searrow A$.

Now since $B, C \in \mathcal{A}_k$ (and the coding works for arguments $\geq 2^{n_0} - k$) the transposition $I(x) - k - 1$ will never produce numbers that fall into the first $k + 1$ positions from a power of 2. This means that we will never request the enumeration of a number that is already in A, so $B \leq_{\mathrm{cl}} A$ and $C \leq_{\mathrm{cl}} A$. To show that A is the least upper bound, suppose that $B \leq_{\mathrm{cl}} D$, $C \leq_{\mathrm{cl}} D$ with use on n bounded by $n + c$. To compute '$n \in A$?' from D first check whether n is a code for B or C, or it is not a code.

- If it is not a code, $n \notin A$ and if it is a code for B, $n \in A \iff n \in B$. The last clause can be cl-decided from D with use $n + c$.
- If it is a code for C, $n \in A \iff n + k + 1 \in C$. The last clause can be cl-decided from D with use $n + k + 1 + c$.

So, overall $A \leq_{\mathrm{cl}} D$ with use on n bounded by $n + k + 1 + c$. All this argument works for *ibT* computations instead of cl with the exception that the upper bound A of B, C defined above is no longer a *least* upper bound. But this does not make a difference in the argument below which shows that every finite partial order is embeddable in the c.e. degrees of the cl and *ibT* structures.

3.2. **A cl computably independent subclass of $\mathcal{A}$.** We must construct a sequence of c.e. sets (A_i) in $\mathcal{A}$. For each i let (F^i_j) be an enumeration of the least upper bounds (as defined above) of all finite classes of A_j sets with $j \neq i$. The requirements are:

$$\mathcal{Q}_{i,j,\Phi} : \Phi^{F^i_j} \neq A_i$$

where Φ runs over the cl functionals. But then the construction of all A_i can just be done according to the the usual Friedberg-Muchnik argument, choosing witnesses of the appropriate kind. The same holds if we consider *ibT* computations (with the same upper bound assignment).

3.3. **Embeddability of finite partial orders.** We follow Sacks' classical argument: given a finite partial ordering $\prec$ of (w.l.o.g.) $\mathbb{N}$ we assign to any point n in the domain, the (least) upper bound of all A_i such that $i \preceq n$ (as defined above). The proof that this is an isomorphism is as in the classical argument. This suffices for the proof of theorem 1.3

4. Global Quasi-Maximality

4.1. **Proof of theorem 1.8.** Suppose we are given a random real α. We are going to define a total computable tree Ψ which can be seen as a computable function which maps *each* finite binary string to another one of the same length, in such a way that if $\sigma \subset \tau$ then $\Psi^\sigma \subset \Psi^\tau$. This will clearly be a Turing functional with identity use and so, an *ibT* and cl computable functional. Hence we will be able to consider $\beta = \Psi^\alpha$ which (as shown

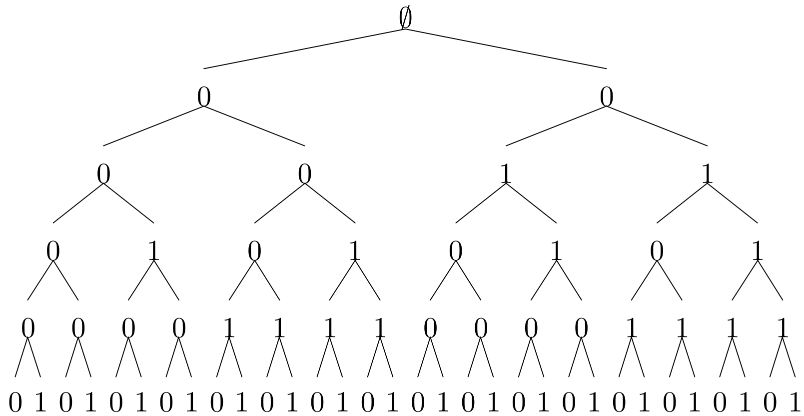

FIGURE 1. The tree Ψ. The label of a node is the sequence of digits we collect if we travel from the root to that node.

below) will be random and strictly below α thus proving the theorem. Let the functional Ψ be defined inductively as follows.

(i) For both strings τ of length 1 we define $\Psi^\tau = 0$.
(ii) If $|\tau|$ is of the form 2^n for some $n \geq 1$ then let τ_0 be the initial segment of τ of length $2^n - 1$. If there is a $\tau_1 \prec \tau_0$ (where $\prec$ is the lexicographical ordering) of length $2^n - 1$ such that $\Psi^{\tau_1} = \Psi^{\tau_0}$ then define $\Psi^\tau = \Psi^{\tau_0}1$ and otherwise define $\Psi^\tau = \Psi^{\tau_0}0$.
(iii) If $|\tau|$ is not of the form 2^n for any $n \geq 0$ then let τ_0 be the initial segment of τ of length $|\tau| - 1$. Let $c = \tau(|\tau| - 1)$ (i.e. the last bit of τ) and define $\Psi^\tau = \Psi^{\tau_0}c$.

It is important to have an intuitive picture of the above inductive definition (see figure 1). We begin by branching the empty sequence with two 0s. From then on, at levels 2^n (for any n) we extend with either two 1s or two 0s according to whether there is another node on the left which has the same label (i.e. is Ψ-mapped to the same string) with the node we are on or not. At all other levels we extend the strings as we would the identity (binary) tree (that is, a 0 on the left branch and a 1 on the right branch).

This tree is isomorphic to the binary tree I. We can say that the *names* of the nodes are the corresponding strings w.r.t. the definition of the binary tree and their *labels* are the corresponding strings w.r.t. the definition of Ψ. Then the map that Ψ defines is clear. It is not hard to show that Ψ has the following properties (use induction).

- For every τ, $\Psi^\tau \downarrow$ and is a string of the same length.
- For every string σ which begins with 0 there exist exactly two incompatible τ_0, τ_1 such that
$$\Psi^{\tau_0} = \Psi^{\tau_1} = \sigma.$$
- If $|\sigma| = 2^k + c < 2^{k+1}$ consider the two τ_i such that $\Psi^{\tau_0} = \Psi^{\tau_1} = \sigma$. Then τ_0, τ_1 differ at their c-th bit from the end, i.e. their $|\sigma| - c - 1$ bit. In particular, if σ is of length 2^k they differ on their last bit.

- Consider Ψ as a map from 2^ω to itself (i.e. between infinite binary sequences). Then for every infinite binary sequence α which begins with 0 there is a unique β such that $\Psi^\beta = \alpha$.

Now suppose given a random real α and let $\beta = \Psi^\alpha$. It is clear that α can *ibT* and cl compute β. We show first that β is random. So suppose otherwise. Then there exists a computably enumerable sequence of strings $\{\sigma_n\}_{n\in\omega}$ such that $\Sigma_n 2^{-|\sigma_n|} < \infty$ and such that for infinitely many n, $\sigma_n \subset \beta$. In order to define a Solovay test demonstrating that α is not random form a new sequence by replacing each σ_n which begins with a 0 with the two strings τ (of the same length) such that $\Psi^\tau = \sigma_n$. We are left, then, to show that $\alpha \not\leq_{cl} \beta$.

So suppose that, for all i, Ψ_i is the i-th cl functional with use $x + c_i$ on argument x (under an effective enumeration of all cl partial computable functionals). Given i such that $\Psi_i^\beta = \alpha$ we shall show that β is not random, giving the required contradiction. In what follows the *unpredictability paradigm* of randomness is most suitable. A *prediction function* for an infinite binary sequence β is a function which takes each initial segment of β and returns a guess about what the next bit will be. It can return 0 or 1 or even "no prediction". Given a partial computable prediction function f for β which:

(1) always predicts correctly or returns "no prediction" on β,
(2) predicts correctly infinitely many times, and such that
(3) if $f(\sigma)\uparrow$ for some $\sigma \in 2^{<\omega}$ then $f(\tau)\uparrow$ for all $\tau \supset \sigma$,

it is not hard to pass effectively to a Solovay test which 'captures' β i.e. shows that β is not random. In fact the following theorem holds.

Theorem 4.1. *(Chaitin [5]) Consider a total computable prediction function f, which given an arbitrary finite initial segment of a real β, returns either "no prediction", "the next bit is a 0", or "the next bit is a 1". If β is random and f predicts infinitely many bits of β then in the limit the relative frequency of correct and incorrect predictions tends to $\frac{1}{2}$.*

We note that we can replace the condition 'total computable prediction function' with 'partial computable prediction function which satisfies property (3) above (see the proof in [5]). The intuition is that according to the way in which we built Ψ, at levels $2^n - 1$ the next bit of β is given by the initial segment of α of length $2^n - 1$. So if β could cl-compute α then at appropriate levels we would be able to compute the next bit of β given the preceding initial segment of β. We chose levels 2^n in the definition of Ψ so that the distance between these levels increases (any other computable function with this property would do). We proceed in this way in order to deal with the constant advantage that the cl computations have over the use. If we only dealt with *ibT* functionals we would not need this property.

Given i such that $\Psi_i^\beta = \alpha$ we produce a prediction function f which always guesses correctly for β and which predicts infinitely many bits of β. Our function f operates as follows:

(1) For all σ which are not of length $2^n - 1$ for some $n > c_i$, f returns "no prediction".

(2) Suppose σ is of length $2^n - 1$ for some $n > c_i$. There exist precisely two strings τ such that $\Psi^\tau = \sigma$. Let these be τ_0 and τ_1, with τ_0 the leftmost. Run the computation Ψ_i^σ until we find that, either τ_0 is incompatible with Ψ_i^σ, or else τ_1 is incompatible with this string. This will happen since, according to the properties of Ψ mentioned above, τ_0, τ_1 will differ at position $2^{n-1} - 1$ which is less than $|\sigma| - c_i$. In the former case f returns "the next bit is a 1" and otherwise f returns "the next bit is a 0".

By the definition of Ψ and its properties the second clause of f always guesses correctly. If β was not strictly below α it would not be random, giving the required contradiction. We wish to note (after the comments of the referee) that the map Ψ preserves the so-called rK reducibility which is an alternative to the cl measure of relative randomness.

Definition 4.1. *(Downey, Hirschfeldt, LaForte [6, 7]) Let α and β be reals. We say that β is relative K reducible (rK-reducible) to α, if there exist a partial computable binary function f and a constant k such that for each n there is a $j \leq k$ for which $f(\alpha \restriction n, j) \downarrow = \beta \restriction n$.*

It is not hard to see that Ψ sends reals to reals of the same rK degree. In other words, if $\Psi^\alpha = \beta$ then α, β are rK-equivalent. So we get the following:

Corollary 4.1. *There are random reals α, β of the same rK degree which belong to different, comparable cl degrees. In fact, every random rK degree contains such reals.*

As a final remark note that Ψ is a total computable functional and so whenever $\Psi^\alpha = \beta$, β is truth-table reducible to α.

4.2. **Proof of theorem 1.9.** Let us begin by reviewing the construction of a real of quasi-maximal degree. We make the following definitions.

(i) Let Ψ_i, the i^{th} cl functional, satisfy the condition that the use in computing argument n is $n + c_i + 1$ (should this computation converge).

(ii) For $\sigma \in 2^{<\omega}$ let $\Pi(\sigma, i)$ be the number of strings τ of length $|\sigma| + c_i$ such that $\sigma = \Psi_i^\tau$.

(iii) For $\sigma \in 2^{<\omega}$ let $\Upsilon(\sigma, i) = \min\{\Pi(\sigma', i) \mid \sigma' \supseteq \sigma\}$. Let $\Upsilon^\star(\sigma, i)$ be the least string $\sigma' \supseteq \sigma$ such that $\Pi(\sigma', i) = \Upsilon(\sigma, i)$.

Lemma 4.1. *For any σ, i we have $\Pi(\sigma 0, i) + \Pi(\sigma 1, i) \leq 2\Pi(\sigma, i)$.*

Proof. Consider the set of all one bit extensions of those strings τ of length $|\sigma| + c_i$ such that $\Psi_i^\tau = \sigma$. There are $2\Pi(\sigma, i)$ strings in this set. □

Lemma 4.2. *Given σ_0, i, let $\sigma_1 = \Upsilon^\star(\sigma_0, i)$. For all $\sigma_2 \supseteq \sigma_1$ we have $\Pi(\sigma_2, i) = \Upsilon(\sigma_0, i)$.*

Proof. By induction on the length of σ_2. So suppose given $\sigma_2 \supseteq \sigma_1$ such that $\Pi(\sigma_2, i) = \Upsilon(\sigma_0, i)$. Now if $\Pi(\sigma_2 0, i) < \Upsilon(\sigma_0, i)$ or $\Pi(\sigma_2 1, i) < \Upsilon(\sigma_0, i)$ this would contradict the fact that $\sigma_1 = \Upsilon^\star(\sigma_0, i)$. Thus by lemma 4.1, $\Pi(\sigma_2 0, i) = \Pi(\sigma_2 1, i) = \Upsilon(\sigma_0, i)$. □

Lemma 4.3. *Given σ_0, i, let $\sigma_1 = \Upsilon^\star(\sigma_0, i)$. For all $\alpha \supset \sigma_1$ and all β such that $\Psi_i^\beta = \alpha$ we have that $\beta \leq_T \alpha$.*

Proof. Given α and β as in the statement of the lemma, let T_n be all those strings τ of length $n+c_i$ such that Ψ_i^τ is the initial segment of α of length n and let $T=\bigcup_n T_n$. We say that a real lies on T if all but finitely many initial segments are in T. The following facts follow immediately from the fact that, by lemma 4.2, there are precisely the same number of strings (actually $\Upsilon(\sigma_0,i)$) in T_n for all sufficiently large n.

(i) There are a finite number of reals lying on T (at most $\Upsilon(\sigma_0,i)$).

(ii) We can compute (not just enumerate) T using an oracle for α.

By (i) there exists $\tau_0\subset\beta$ such that if $\beta'\neq\beta$ lies on T then $\tau_0\not\subset\beta'$. If we are given $\tau_1\supset\tau_0$ which is not an initial segment of β then using an oracle for α it follows by (ii) that we can find n such that there are no extensions of τ_1 in T_n. It is the fact that there may exist infinitely many $\tau_1\supset\tau_0$ in T which are not initial segments of β which means that we are not able to deduce $\beta\leq_{\mathrm{cl}}\alpha$.

□

Now lemma 4.3 means that in order to construct a real of quasi-maximal cl degree we can simply proceed with an argument by finite extension. We may define $\sigma_0=0$. Then we define σ_1 to be some extension of $\Upsilon^\star(\sigma_0,0)$. Then we proceed to define σ_2 as an extension of $\Upsilon^\star(\sigma_1,1)$, and so on. But we must also ensure that $\alpha=\bigcup_n\sigma_n$ is non-random and non-maximal. If we can find a way to satisfy these two additional requirements by finite extension then we will be able to combine all strategies into a single argument and so prove the theorem.

Each stage of the finite extension argument must consist of three steps. Given σ_s we define successively σ_s',σ_s'' and σ_s''', each string an extension of the last, before defining σ_{s+1} to be some extension of σ_s'''. First of all we define $\sigma_s'=\Upsilon^\star(\sigma_s,s)$ in order to ensure that the s^{th} quasi-maximality requirement is satisfied. Then we extend σ_s' to σ_s'' in such a way as to be able to ensure that α will not be random. How do we do this? For all σ, define $f(\sigma)=\mid\{n:\sigma(n)\downarrow=0\}\mid$. If α is a random real then, by theorem 4.1:

$$(\dagger)\lim_n\frac{f(\alpha\restriction n)}{n}\downarrow=\frac{1}{2}.$$

So if we define σ_s'' to be σ_s' concatenated with $2\cdot|\sigma_s'|$ many zeros then this will be sufficient to ensure that α is not random. This is indeed a successful finite extension strategy which deals with the non-randomness requirements.

Finally we have to make sure that there is some β which is strictly above the α we are constructing. We are going to do this by using the tree Ψ which we defined in the proof of theorem 1.8. However we define α, the fact that $\alpha(0)=0$ means that there will exist a unique β such that $\Psi^\beta=\alpha$. We must then extend σ_s'' to σ_{s+1} in such a way as to satisfy the non-maximality requirement

$$\mathcal{P}_s:\Psi_s^\alpha\neq\beta.$$

First find some σ_s''' extending σ_s'' such that $|\sigma_s'''|$ is of the form 2^k+2^{k-1} for some k such that $2^{k-1}>c_s$. There exist precisely two strings τ such that $\Psi^\tau=\sigma_s'''$. Let τ_0 be the leftmost and τ_1 the rightmost. If it is not the case that $\Psi_s^{\sigma_s'''}$ is defined and compatible with one of these strings then $\mathcal{P}_s$ is automatically satisfied. In this case we may define $\sigma_{s+1} = \sigma_s'''$. Otherwise proceed as follows. The length of $\Psi_s^{\sigma_s'''}$ is $|\sigma_s'''| - c_s$, $|\tau_i| = |\sigma_s'''|$ and the two τ_i differ at their $2^k - 1$ bit. So, since $c_s < 2^{k-1}$, $\Psi_s^{\sigma_s'''}$ must be compatible with at most one τ_i.

- If $\Psi_s^{\sigma_s'''}$ is defined and compatible with τ_0 then define σ_{s+1} to be an extension of σ_s''' of length 2^{k+1} which ends in a 1.
  Then the corresponding β must be compatible with τ_1 and so incompatible with $\Psi_s^{\sigma_s'''}$. So $\mathcal{P}_s$ is satisfied.
- If $\Psi_s^{\sigma_s'''}$ is defined and compatible with τ_1 then define σ_{s+1} to be an extension of σ_s''' of length 2^{k+1} which ends in a 0.
  Then the corresponding β must be compatible with τ_0 and so incompatible with $\Psi_s^{\sigma_s'''}$. So $\mathcal{P}_s$ is satisfied.

By induction all requirements are satisfied and this concludes the proof of the theorem. The nature of the argument—the fact that it is a proof by finite extension—gives us, in fact, the following stronger result:

Definition 4.2. *We say that S is a dense set of strings if every string has an extension belonging to S. A real α is weakly $n+1$-generic if $\{\sigma : \sigma \subset \alpha\}$ meets every dense Σ^0_{n+1} set of strings.*

Theorem 4.2. *Every real which is weakly 3-generic is a non-random real of quasi-maximal non-maximal cl degree.*

Proof. Given the proof of theorem 1.9 this can be seen immediately through consideration of the following dense Σ^0_3 sets of strings:

- For each $i \in \omega$ the set of strings $\{\Upsilon^\star(\sigma, i) : \sigma \in 2^{<\omega}\}$.
- If $i \in \omega$ let $\sigma^{\star_i}$ be σ concatenated with $i \cdot |\sigma|$ zeros. For each $i \in \omega$ the set of strings $\{\sigma^{\star_i} : \sigma \in 2^{<\omega}\}$.
- If $\sigma \in 2^{<\omega}$ and $i \in \omega$ then let $\sigma^{\dagger_i}$ be defined from σ the same way that we defined σ_{i+1} from σ_i'' in the proof of theorem 1.9 in order to meet requirement $\mathcal{P}_i$. For each $i \in \omega$ the set of strings $\{\sigma^{\dagger_i} : \sigma \in 2^{<\omega}\}$.

□

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