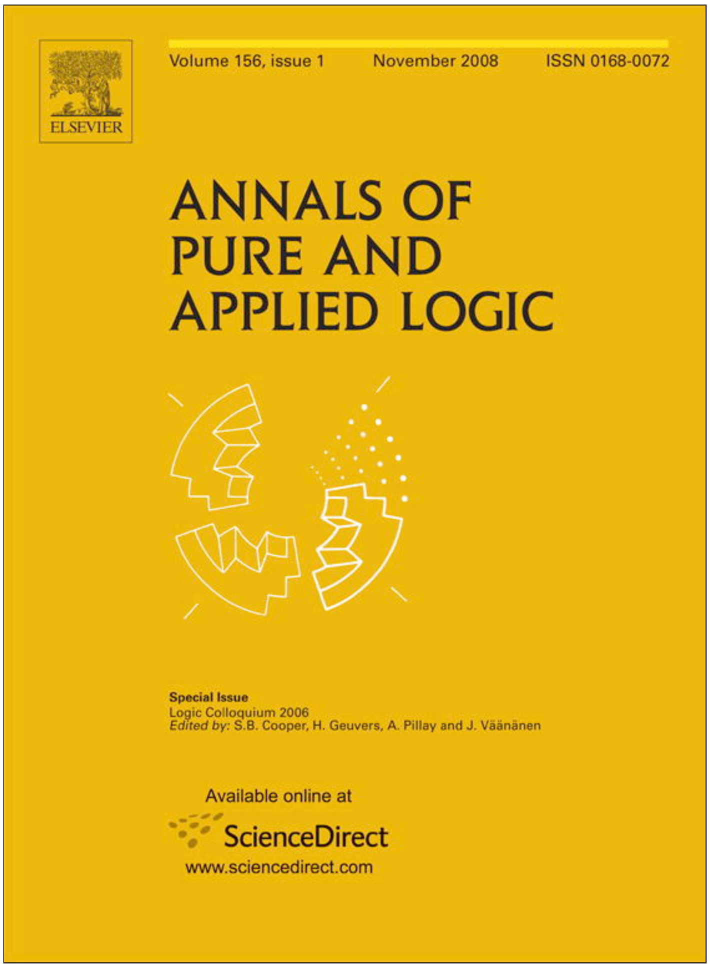
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Π^0_1 classes, *LR* degrees and Turing degrees

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We say that $A \leq_{LR} B$ if every B-random set is A-random with respect to Martin–Löf randomness. We study this relation and its interactions with Turing reducibility, Π^0_1 classes, hyperimmunity and other recursion theoretic notions.

A natural variant of Turing reducibility from the point of view of Martin–Löf randomness is the *LR* reducibility which was introduced in [10]. We say that $A \leq_{LR} B$ for two sets A, B if every B-random real is A-random (throughout this paper randomness means Martin–Löf randomness). Intuitively this means that whenever A can derandomize a real, B also has this ability. This reducibility naturally induces an equivalence relation $\equiv_{LR}$ which defines a partition of Cantor space into the *LR* degrees. Two reals A, B belong to the same *LR* degree iff the A-random reals and B-random reals coincide. The *LR* degrees were first introduced by André Nies [10] and were further studied by Barmpalias, Lewis, Soskova [2] and Simpson [15]. In this paper we study $\leq_{LR}$ and its interactions with $\leq_T$. In Section 1 we lay out the basic framework and facts which are used throughout the rest of the paper. In Section 2 we study Π^0_1 classes of sets $\leq_{LR} \emptyset'$ and as an application we apply a basis theorem to deduce that there is some A which is not low for random but is *LR* reducible to an A-random set. This contrasts the situation in $\leq_T$. In Section 3 we show that there is a hyperimmune-free Turing degree $\leq_{LR} \emptyset'$ (again via a basis theorem) and prove more results about hyperimmunity in relation to $\leq_{LR}$. We also construct a superlow r.e. set A whose lower cone with respect to $\leq_{LR}$ contains a perfect Π^0_1 class. In Section 4 we study the Turing degrees inside an *LR* degree (globally). In Section 5 we look at recursively enumerable *LR* degrees and the r.e. Turing degrees inside them. We also prove a weak density result for the recursively enumerable *LR* degrees. In the last section we show that every jump traceable set in the REA hierarchy is superlow, thus extending (in one direction) the result of Nies that jump traceability and superlowness coincide in the r.e. sets.

In the following, we use r.e. sets of strings to generate subclasses of the Cantor space. In particular, we never use the relations $\subset, \subseteq, \supset$ and $\supseteq$, the measure μ and the operations $\cap$ and $\cup$ for sets U of strings; these relations and operations always refer to the class

$$S(U) = \{\alpha \in \{0,1\}^\omega \mid \exists n(\alpha(0)\alpha(1)\dots\alpha(n) \in U)\}.$$

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In other words, $\mu(U)$ is $\mu(S(U))$, $U \subseteq V$ iff $S(U) \subseteq S(V)$ and $U \cap V$ denotes actually $S(U) \cap S(V)$, not $S(U \cap V)$. For union, $S(U \cup V)$ and $S(U) \cup S(V)$ would, for both interpretations of $\cup$, anyway be the same.

We mainly work in the framework developed by Barmpalias, Lewis and Soskova [2] which is based on Kjos-Hansen's characterization of the *LR* reducibility [6]. In the next section we recall some facts and conventions about Martin–Löf tests and the *LR* reducibility of Barmpalias, Lewis and Soskova [2]. Unexplained recursion-theoretic notation can be found in standard textbooks [12,13,16].

1. Preliminaries

An oracle Martin–Löf test (U_e) is a uniform sequence of oracle machines which output finite binary strings such that if U_e^β is the range of the e-th machine with oracle $\beta \in \{0,1\}^\omega$ then for all $\beta \in \{0,1\}^\omega$, $e \in \mathbb{N}$, $\mu(U_e^\beta) < 2^{-(e+1)}$ and $U_e^\beta \supseteq U_{e+1}^\beta$. A real α is called β-random if for every oracle Martin–Löf test (U_e) we have $\alpha \notin \cap_e U_e^\beta$. A universal oracle Martin–Löf test is an oracle Martin–Löf test (U_e) such that for every $\alpha, \beta \in \{0,1\}^\omega$, α is β-random iff $\alpha \notin \cap_e U_e^\beta$. Given any oracle Martin–Löf test (U_e), each U_e can be thought of as an r.e. set of axioms $\langle \tau, \sigma \rangle$. Such r.e. sets of axioms will be referred to as 'r.e. operators' since they can be seen as effective operators mapping reals $\beta \in \{0,1\}^\omega$ to open sets of reals which are $\Sigma_1^0(\beta)$. If $\beta \in \{0,1\}^\omega$ then

$$U_e^\beta = \{\sigma \mid \exists \tau(\tau \subset \beta \ \wedge \ \langle \tau, \sigma \rangle \in U_e)\}$$

and for $\rho \in \{0,1\}^*$ we define

$$U_e^\rho = \{\sigma \mid \exists \tau(\tau \subseteq \rho \ \wedge \ \langle \tau, \sigma \rangle \in U_e)\}.$$

There is an analogy between oracle Martin–Löf tests as defined above and Lachlan functionals, that is, Turing functionals viewed as r.e. sets of axioms. This analogy will be exploited in a number of constructions below, especially in the constructions of r.e. *LR* degrees. The following lemma is easily proved and provides a universal oracle Martin–Löf test with properties which will later be useful.

Lemma 1 (*Barmpalias, Lewis and Soskova [2]*). *There is an oracle Martin–Löf test (U_e) such that:*

- *for every oracle Martin–Löf test (V_e), uniformly on its r.e. index we can compute $k \in \mathbb{N}$ such that for every real β and all e, $V_{e+k}^\beta \subseteq U_e^\beta$;*
- *if $\langle \tau_1, \sigma_1 \rangle, \langle \tau_2, \sigma_2 \rangle \in U_e$ and $\tau_1 \subseteq \tau_2$ then $\sigma_1 | \sigma_2$;*
- *if $\langle \tau, \sigma \rangle \in U_e$ then $|\tau| = |\sigma|$ and $\langle \tau, \sigma \rangle \in U_e[|\tau|] - U_e[|\tau| - 1]$.*

Throughout this paper by a universal oracle Martin–Löf test we will mean the one given in Lemma 1.

Corollary 2 (*Barmpalias, Lewis and Soskova [2]*). *Let (U_e) be the universal oracle Martin–Löf test of Lemma 1 and let U be any member of it. There is a recursive function which, given any input $\langle \tau, \tau' \rangle$ such that $\tau \subseteq \tau'$, outputs the finite (closed-open) set $U^{\tau'} - U^\tau$.*

In [8] Kučera and Slaman gave a strategy for avoiding the low for random reals when building a Π_1^0 class. The lemma below gives a simpler way to do this.

Lemma 3. *There is a recursive sequence (a_n) of natural numbers and a member V of an oracle Martin–Löf test such that for all $V_* \subset \{0,1\}^\omega$, all n and all $\sigma \in \{0,1\}^{a_n}$ there is $\tau \supset \sigma$, $\tau \in \{0,1\}^{a_{n+1}}$ such that $V^\tau \not\subseteq V_*$. Moreover for all finite strings σ the set V^σ is finite as a set of strings and the function $f(\sigma) = V^\sigma$ is recursive.*

Proof. We simply let $a_0 = 0$, $a_n = a_{n-1} + n + 1$ and for each $\sigma \in \{0,1\}^{a_n}$ let τ_i, $i < 2^{n+2}$ be the extensions of σ of length a_{n+1}. If ρ_i, $i < 2^{n+2}$ are the strings of length $n+2$ we enumerate $\langle \tau_i, \rho_i \rangle$ into V for each $i < 2^{n+2}$. Repeat this for all $n \in \mathbb{N}$. Then for each $\sigma \in \{0,1\}^{a_n}$ we have $\mu(V^\sigma) = \sum_{i=2}^{n+1} 2^{-i} < 2^{-1}$ and V is a member of an oracle Martin–Löf test. Clearly there is a recursive function f such that $f(\sigma) = V^\sigma$ (as a set of finite strings) for all $\sigma \in \{0,1\}$. Also if V_* is not the whole space, for each n and $\sigma \in \{0,1\}^{a_n}$ there is $\tau \supset \sigma$, $\tau \in \{0,1\}^{a_{n+1}}$ such that $V^\tau \not\subseteq V_*$. This is because V^τ for $\tau \supset \sigma$, $\tau \in \{0,1\}^{a_{n+1}}$ cover the whole space $\{0,1\}^\omega$. $\square$

The following result will be used throughout this paper in order to provide a uniform approach to dealing with problems concerning the *LR* degrees.

Theorem 4 (*Kjos-Hanssen [6]*). *For all $A, B \in \{0,1\}^\omega$ the following are equivalent:*

- *$A \leq_{LR} B$;*
- *for every $\Sigma_1^0(A)$ class T^A of measure < 1 there is a $\Sigma_1^0(B)$ class V^B such that $\mu(V^B) < 1$ and $T^A \subseteq V^B$;*
- *for some member U^A of a universal Martin–Löf test relative to A there is $V^B \in \Sigma_1^0(B)$ such that $\mu(V^B) < 1$ and $U^A \subseteq V^B$.*

We note that there is an effective list (V_e) of all r.e. operators such that $\mu(V_e^\beta) < 1$ for all $e \in \mathbb{N}$, $\beta \in \{0,1\}^\omega$. Indeed, in order to obtain such a list one just has to start with an effective list of all possible pairs (T_e, q_e) of r.e. operators T_e and rational numbers $0 < q_e < 1$, and for each $e \in \mathbb{N}$, $\sigma \in \{0,1\}^\omega$ enumerate into V_e^σ everything that appears in T_e^σ up until the point where its measure threatens to exceed the threshold q_e. It is easy to verify that this list contains exactly the desired r.e. operators. This fact will be used freely during the rest of this paper.

In order to construct a Π_1^0 class $\mathcal{P}$ with no low for random reals in view of Lemma 3 and Theorem 4 one can consider an effective list (V_e) of all r.e. operators such that $\mu(V_e^\beta) < 1$ for all $e \in \mathbb{N}$, $\beta \in \{0,1\}^\omega$ and construct $\mathcal{P}$ such that the following requirements are satisfied:

$$V^\beta \not\subseteq V_e \quad \text{for all } \beta \in \mathcal{P},$$

where V is the operator of Lemma 3 and by V_e we mean $V_e^\emptyset$. A basic strategy for meeting each of these requirements runs as follows. First choose n large enough so that there is full branching in the current tree between levels a_n and a_{n+1} i.e. such that every string of length a_{n+1} which extends an extendible string of length a_n is extendible, and where we say that a string is extendible at any stage of the construction if it has an infinite extension in our present approximation to $\mathcal{P}$. Then start cutting any finite branches σ of length a_{n+1} whenever we find that $V^\sigma \subseteq V_e$. By the properties of V we have that any extendible branches at level a_n will remain extendible after the (finitary) action of the strategy. Such a strategy can be combined with those for meeting other requirements in many finite injury situations (as in [8]), and will be used in some proofs in this paper. In some situations, however, we need to use a more refined operator than V of Lemma 3. This is the case in Theorem 7, where we have a recursive construction with some of the requirements imposing a restriction on the number of branches below certain levels. The reason that V is not sufficient to deal with this situation is that for increasing values of n the number of branches that we may be asked to preserve becomes larger (since a_n is larger) and this (along with a finite injury effect) may be a problem for other requirements. To overcome this problem in the proof of Theorem 7 we define another V operator dynamically, during the construction.

1.0.0.1. Another characterization of $\le_{LR}$. It is worth mentioning that there is a characterization of $\le_{LR}$ in the spirit of Theorem 4 which does does not involve open sets of reals but rather, sequences of natural numbers (n_i) such that $\sum_i 2^{-n_i} < \infty$. Such a characterization is given in a theorem of Kjos-Hanssen/Miller/Solomon which we rephrase as follows. Say that the *weight* of a set $I \subseteq \mathbb{N} \times \mathbb{N}$ is $\sum_{(n,m)\in I} 2^{-m}$.

Theorem 5 (*Kjos-Hanssen/Miller/Solomon, see [15]*). *There is an r.e. operator W which maps reals β to β-r.e. sets $W^\beta \subseteq \mathbb{N}\times\mathbb{N}$ with weight < 1 such that the following are equivalent:*

(1) *$A \le_{LR} B$;*
(2) *there is a B-r.e. set $I^B \subseteq \mathbb{N}\times\mathbb{N}$ with finite weight such that $W^A \subseteq I^B$;*
(3) *there is a B-r.e. set J^B with weight < 1 such that $W^A \subseteq J^B$;*
(4) *every A-r.e. set $E^A \subseteq \mathbb{N}\times\mathbb{N}$ with finite weight is contained in a B-r.e. set T^B with finite weight.*

Another way to express this fact is as follows. Fix *any* computable function f such that for every $m \in \mathbb{N}$ there are infinitely many $n \in \mathbb{N}$ such that $f(n) = m$. Define the *weight of a set $W \subseteq \mathbb{N}$* (relative to f) as

$$L(W) = \sum_{i\in W} 2^{-f(i)}.$$

Theorem 6. *There is an r.e. operator W such that for all $X \in 2^\omega$ the set $W^X \subseteq \mathbb{N}$ has weight < 1 and the following are equivalent:*

(1) *$A \le_{LR} B$;*
(2) *there is a B-r.e. set I^B of finite weight such that $W^A \subseteq I^B$;*
(3) *there is a B-r.e. set J^B of weight < 1 such that $W^A \subseteq J^B$;*
(4) *every A-r.e. set of finite weight has a B-r.e. superset of finite weight.*

Proof. Theorem 6 is easily proved from Theorem 5 by considering a recursive bijection $g : \mathbb{N}\times\mathbb{N} \to \mathbb{N}$ such that $f(g(n,m)) = m$ for all $(n,m) \in \mathbb{N}\times\mathbb{N}$. Notice that g can be used to translate subsets of $\mathbb{N}\times\mathbb{N}$ to subsets of $\mathbb{N}$ and vice versa while preserving the weight of sets. □

Note that this result is independent of the definition of f (i.e. the interpretation of *weight*) as long it satisfies the required property mentioned above.

2. Π_1^0 classes and *LR* degrees

The following theorem will later be combined with various basis theorems for Π_1^0 classes and will give a number of interesting corollaries regarding the relation between $\le_{LR}$ and $\le_T$.

Theorem 7. *There is a Π_1^0 class of GL_1 reals $\le_{LR} \emptyset'$ which contains no low for random reals.*

Proof. The proof follows by a combination of the following strategies in a finite injury setting. We effectively approximate a tree T by starting with the full binary tree and then proceeding to chop off selected branches. By level n of a tree we shall mean the set of extendible strings of length n. Let U_* be the second member of a universal Martin–Löf test (so that $\mu(U_*^\beta) < 2^{-2}$ for all $\beta \in \{0,1\}^\omega$).

2.0.0.2. Making the members of the class $\leq_{\mathbf{LR}} \emptyset'$. This strategy is used by Barmpalias, Lewis and Soskova in [2] to construct a perfect tree of reals *LR*-reducible to $\emptyset'$. The goal is to ensure that:

$$\cup_{\beta\in[T]} U_*^{\beta} < 1. \tag{1}$$

Then by the properties of U_* we can use $\emptyset'$ to enumerate an open set of reals of measure < 1 which covers U_*^{β} for all $\beta \in [T]$. This means that all infinite paths in T are $\leq_{LR} \emptyset'$ by Theorem 4. We split the strategy into infinitely many copies P_e and assume that each of them is required (by the higher priority requirements) to preserve the current tree up to a certain level, with B_e being the set of existing nodes at this level. The main conflict is that apart from satisfying (1) we also wish to allow the existence of 2^{e+2} paths above each $\sigma \in B_e$ as this is required by the other strategies in the construction. Inductively we will have $|B_e| \leq 2^{e^2+e}$, since the other strategies do not introduce branching. Then the task of P_e is to find for each $\sigma \in B_e$ some $\tau_0 \supset \sigma$ such that:

$$\mu(U_*^{\tau_1} - U_*^{\tau_0}) < 2^{-e^2-3e-5} \quad \text{for all } \tau_1 \supset \tau_0. \tag{2}$$

In this way, since $|B_e| \leq 2^{e^2+e}$, *it can introduce* 2^{e+2} *branches above each such* τ_0 (which will be needed by the other strategies) and the measure added in $\cup_{\beta\in[T]} U_*^{\beta}$ will be less than:

$$|B_e| \cdot 2^{e+2} \cdot 2^{-e^2-3e-5} \leq 2^{-e-3}.$$

Strategy P_e starts by choosing a *big* length ℓ (a number not previously mentioned in the construction), picking any candidate $\tau_0 \supset \sigma$ of length ℓ for each $\sigma \in B_e$ and chopping from T all branches which are incompatible with all chosen τ_0 (for each $\sigma \in B_e$). Whenever some $\tau_1 \supset \tau_0$ appears (say of length less than the current stage) such that (2) does not hold it selects an extension $\tau_1' \supset \tau_1$ of *big* length and it chops all existing branches of length $|\tau_1'|$ which are extensions of σ and incompatible with τ_1'. Now the new candidate below σ is τ_1' and it can also update the candidates below the other strings in B_e by choosing extensions of length $|\tau_1'|$ and chopping incompatible extensions, and so on. Then since U_* is a member of an oracle Martin–Löf test, after less than 2^{e^2+3e+5} attempts below each string in B_e it will find the right candidate.

2.0.0.3. Avoiding the low for random reals. We will adapt the ideas sketched after Theorem 4. Let (V_e) be an effective list of all r.e. operators such that $\mu(V_e^{\beta}) < 1$ for all $e \in \mathbb{N}$, $\beta \in \{0,1\}^{\omega}$. The sequence (V_e) can be chosen appropriately so that we also have a computable sequence of positive rationals (q_i) with $\mu(V_e^{\beta}) < 1 - q_e$ for all e, β. In view of Theorem 4 it suffices to construct a member of an oracle Martin–Löf test U such that the following requirements are satisfied:

$$Q_e : U^{\beta} \not\subseteq V_e \quad \text{for all } \beta \in [T]. \tag{3}$$

Strategy Q_e inherits a restraint from the higher priority requirements which asks for the current branches of a certain level n (that is, the set C_e of the currently extendible strings of length n) to remain extendible. Note that by the nature of the strategies in this construction we have that every string in $C_e[s+1]$ has a unique prefix in $C_e[s]$. To satisfy Q_e if we did not have other requirements we would find an antichain of, say, 2^{e+2} strings σ extending each string in C_e, enumerate measure 2^{-e-2} in U^{σ} (exactly as in the proof of Lemma 3, by dividing 2^{ω} into 2^{e+2} equal intervals) and start chopping from T those branches σ which end up with $U^{\sigma} \subseteq V_e$.

Here, however, the situation is just a little more complicated. On the one hand Q_e is only allowed to use a fixed number of branches extending each $\tau \in C_e$ (namely 2^{e+2}, as allowed for by the previous P_e strategy) and on the other hand it may be injured by higher priority requirements. After each injury it may already have enumerated some measure into U^{σ} (for some σ) and thus ensuring that $\mu(U^{\beta}) < 1$ for all $\beta \in 2^{\omega}$ becomes a bit more tricky. The solution is to implement the strategy in k parts for a large enough k which depends on an upper bound on the injuries of Q_e by higher priority requirements. We can assume we know such a bound m due to the nature of the other requirements. For the sake of Q_e we will build U_e with $\mu(U_e^{\beta}) \leq 2^{-e-2}$ and eventually we will set $U = \cup_e U_e$ which is going to be a member of a Martin–Löf test. We divide $\{0,1\}^{\omega}$ into k equal intervals $I_1, \ldots, I_k$ and for each $\tau \in C_e$ we follow a routine successively for each of the intervals, starting with I_1. *We say that for some finite string* τ *the interval* I_j *has been used* (*for* Q_e) *if* $U_e^{\tau} \cap I_j \neq \emptyset$. If no routine is on progress for $\tau \in C_e$ we find an antichain of 2^{e+2} strings σ_i extending τ (say, of the same length ℓ) and make sure that no other strings extend τ at level ℓ by chopping superfluous strings. Also we find the least t such that I_t is unused for all σ_i, we divide I_t into 2^{e+2} equal intervals and we assign the corresponding closed-open sets of reals to $U_e^{\sigma_i}$, $i < 2^{e+2}$ respectively. Then every time that we see $U_e^{\sigma_i} \subseteq V_e$ for some i we chop σ_i from the tree, unless this is the last extension of τ currently on T. In the latter case we terminate the current routine and start a routine with I_{t+1} which is now the next unused interval, provided that $t < k$. Also, if in the middle of the routine for I_t the strategy Q_e is injured, the routine is terminated and a new routine starts in the same way.

This strategy clearly assigns to each U_e^{β}, $\beta \in \{0,1\}^{\omega}$ measure at most 2^{-e-2} (in at most k installments), so it is 'cost-efficient'. Note that the possible injuries of Q_e cannot cause the strategy to repeat the j-routine (corresponding to I_j) for some j with $1 \leq j \leq k$ in a single path. Now if the strategy runs as above until for some finite string τ it has used the last interval I_k we will have

$$\mu(V_e) \geq (k-m) \cdot \tfrac{1}{k} \tag{4}$$

given that every time the routine finishes with some interval I_t (without an injury occurring in between) an extra measure of 2^{e+2} (number of strings) times $2^{-e-2}/k$ (measure loaded in U_e^σ) enters V_e. So if we choose $k = 2^b$ for a large enough b such that $k \geq m/q_e$ then (4) would give $\mu(V_e) \geq 1 - q_e$ which is a contradiction. This means that for this value of k the strategy Q_e will never run out of unused intervals thus guaranteeing $U_e^\beta \not\subseteq V_e$ for all extensions on T of each $\sigma \in C_e$. Also all $\sigma \in C_e$ remain extendible in T and Q_e causes a recursively bounded number of injuries to the lower priority requirements.

2.0.0.4. Making the class inside GL_1. If W_e is the r.e. set of strings σ such that $\Phi_e^\sigma(e)\downarrow$ we consider a strategy M_e for each W_e. The strategy M_e inherits the restraint from higher priority requirements that the set of strings D_e from a certain level of the class T_s should be preserved, that is, remain extendible. It simply waits and whenever a proper extension τ of a string $\sigma \in D_e$ is enumerated into W_e and is extendible it chops off all branches which extend σ and are incompatible with τ. These strategies are finitary and they satisfy the requirement $\beta' \leq_T \beta \oplus \emptyset'$ as follows: let $\beta \in T$ and suppose that we wish to decide if $\Phi_e^\beta(e)\downarrow$. We just need to use $\emptyset'$ to find a stage s_0 beyond which all requirements up to and including M_e have stopped acting. Then with oracle β we can find which string of D_e the sequence β extends and correctly assert that $\Phi_e^\beta(e)\downarrow$ iff this happens by stage s_0 (if this happens later it would contradict the choice of s_0).

2.0.0.5. Combining the strategies. We fix the priority list $P_0, M_0, Q_0, P_1, M_1, Q_1, \ldots$ and the construction proceeds in a straightforward way with each of the finitary strategies eventually resting and being responsible for a certain interval of levels on the tree T. Each of these intervals starts from the endpoint of the interval of the previous strategy. If p_i, m_i, n_i are these endpoints for the requirements respectively then $T \restriction p_0$ consists of a single branch. This also holds for $T \restriction m_0$ according to the M-strategies. Then Q_0 introduces at most 2^2 branches in $T \restriction n_0$ while $T \restriction p_1$, $T \restriction m_1$ do not introduce further branching, and so on. We say that a strategy is injured if some higher priority strategy acts (that is, chops off some branches in T or changes its parameters). The precise construction is as follows:

2.0.0.6. At stage s. Successively access P_i, M_i, Q_i for $i < s$. When we access P_i we determine a finite extension for each string in B_i (note that B_0 consists of the empty string) exactly as explained above in the description of this strategy. If and when P_i is injured it may be the case that B_i changes, and in this case the strategy needs to determine the extensions again. During an interval of stages where P_i is not injured it may change the chosen extensions to further extensions of those (according to the strategy above) but for each string in B_i this can happen less than 2^{e^2+3e+5} times. So during such an interval the strategy can act at most $|B_i| \cdot 2^{e^2+3e+5}$ times. The set D_i is defined at each stage to be the chosen finite extensions of each string in B_i i.e. the extendible nodes of the level that these strings belong to. So $|B_e| = |D_e|$ for each e.

When we access M_i we check if a proper extension τ of a string $\sigma \in D_e$ is enumerated into W_e and is extendible. In that case and if this is the first time such an extension occurs (since the last time the strategy was injured) we choose some $\tau' \supset \tau$ of *big* length and chop off all branches which extend σ and are incompatible with τ'. We also choose arbitrarily an extension of length $|\tau'|$ for each of the other strings in D_e and chop incompatible extensions with those strings at level $|\tau'|$. The set C_i is defined at each stage to be the set of the currently chosen extensions and consists of the extendible strings of a certain level. Note that $|D_e| = |C_e|$ and that during an interval of stages where M_i is not injured this strategy can act at most $|D_e|$ times.

Since Q_e introduces at most 2^{e+2} branches above each string in C_e we have that $|C_{e+1}| \leq 2^{e+2} \cdot |C_e|$. Since $|C_0| = 1$ it follows that $|C_e| \leq 2^{e^2+e}$ as mentioned before. Let υ_e be an upper bound on the injuries of Q_e. This can be easily calculated from the above in terms of e alone. The strategy Q_e works with a partition $I_1, \ldots, I_k$ of the space into equal intervals, where k the least number of the form 2^b which is greater than υ_e/q_e. Every time Q_e is accessed the following is done for each $\tau \in C_e$: if $\sigma_j(\tau)\downarrow$ for some j check if $U_e^{\sigma_j(\tau)} \subseteq V_e$. If so then make $\sigma_j(\tau)\uparrow$ and if there is $m \neq j$, $m < 2^{e+2}$ with $\sigma_m(\tau)\downarrow$ chop (the previous value of) $\sigma_j(\tau)$ from the tree. Now if there is no $j < 2^{e+2}$ with $\sigma_j(\tau)\downarrow$ pick an antichain of 2^{e+2} strings $\sigma_j(\tau) \supset \tau$ and chop from T the extensions of τ which are incompatible with all the $\sigma_j(\tau)$. Then if I_t is the least *unused* interval with respect to all $\sigma_j(\tau)$, divide it into 2^{e+2} equal intervals and assign them respectively to $V^{\sigma_j(\tau)}$, $j < 2^{e+2}$. Notice that now I_t is *used* for τ and every extension of it. When the above is done for all $\tau \in C_e$ choose a *big* length ℓ and an extension of length ℓ for each $\sigma_j\downarrow$ with respect to each $\tau \in C_e$, chop all incompatible extensions of those strings from T and let B_e be the set of the remaining extensions of length ℓ. When Q_e is injured all $\sigma_j(\tau)$, $\tau \in \{0, 1\}^*$ of Q_e become undefined.

2.0.0.7. Verification. The tree consists of GL_1 reals by the actions of the M strategies as explained above and since all the strategies act finitely often. To show that there are no low for random paths through T we first need to show that $\mu(U_e^\beta) \leq 2^{-e-2}$ for all $\beta \in \{0, 1\}^\omega$, so that $\mu(U^\beta) < 1$ for all $\beta \in \{0, 1\}^\omega$. It is clear that only the Q_e strategy adds measure to U_e^β and that Q_e contributes $2^{-e-2}/k$ to U_e^β during the activity corresponding to an interval I_j. During the construction each interval I_j (defined by Q_e) is used at most once with respect to path β and the strategy therefore contributes at most 2^{-e-2} to U_e^β. We must show next that for every $\beta \in T$ we have $U_e^\beta \not\subseteq V_e$. Strategy Q_e will keep on defining antichains $\sigma_j(\tau_i)$ for prefixes $\tau_i \subset \tau_{i+1}$ of β and each time it proceeds to τ_{i+1} it means that either it has been injured or that $I_i \subseteq V_e$. The second case can happen less than $k - \upsilon_e$ times since $|I_i| = 1/k$ and $(k - \upsilon_i) \cdot \frac{1}{k} = 1 - q_i > \mu(V_e)$. The first case can happen at most υ_e times since this is a bound on the injuries of Q_e. So when Q_e defines the antichain $\sigma_j(\tau_i)$ for the last i, it will no longer be injured and it will not be the case that all $U_e^{\sigma_j(\tau_i)}$ are contained in V_e. Moreover, there is some t such that $\sigma_t(\tau_i) \subset \beta$. It follows that $U_e^{\sigma_t(\tau_i)} \not\subseteq V_e$ because otherwise $\sigma_t(\tau_i)$ would be removed from the tree and $\beta \notin [T]$, a contradiction. So $U_e^{\beta} \not\subseteq V_e$ and this shows that Q_e is satisfied. Finally by induction on i and the analysis of P_i given above it is clear that for the final value of D_i the following holds

$$\cup_{\sigma\in D_i} U_*^{\sigma} \leq 2^{-2} + 2^{-3} + \cdots + 2^{-i}$$

so that $\cup_{\beta\in T} U_*^{\beta} < 1$ and by Theorem 4 all paths through T are $\leq_{LR} \emptyset'$. □

If $A \leq_T Z$ and Z is A-random (that is, A is a basis for randomness) it is well known that A has to be low for random. However bases for randomness with respect to $\leq_{LR}$ do not have to be low for random. Indeed, recall that a real A is called low for Ω if Ω is random relative to A. Downey, Hirschfeldt, Miller and Nies [4] showed that every Π_1^0 class has a member which is low for Ω. Since $\Omega \equiv_T \emptyset'$, Theorem 7 implies the following corollary.

Corollary 8. *There is a low for Ω real which is LR-below Ω and it is not low for random.*

Theorem 7 can also be proved in terms of prefix free complexity. Let $H(x)$ denote the prefix free complexity of a number/string x.[1]

Say that $A \leq_{LH} B$ if there is some constant c such that $H^B(x) \leq H^A(x) + c$ for all x. Also a set X is called *H-trivial* if there is a constant $c \in \mathbb{N}$ such that $H(X \upharpoonright n) \leq H(n) + c$ for all $n \in \mathbb{N}$. Nies [10] showed the coincidence of the H-trivials and the low for random reals; Kjos-Hanssen, Miller and Solomon [7] showed the coincidence of $\leq_{LH}$ and $\leq_{LR}$.

Theorem 9. *There is a partial computable function ψ such that no extension of it is H-trivial and all all extensions of it are $\leq_{LH} \emptyset'$.*

Proof. By the proof of Proposition 6 in [5] there is a universal machine such that the approximation $H_0, H_1, \ldots$ to H from above satisfies $H_x(x) = H(x)$ whenever x is the maximum of a set of the form $\{y : H(y) \leq n\}$ for some n.

Split the natural numbers recursively into intervals $I_0, I_1, \ldots$ such that I_n is defined at stage s after the markers have been moved and ψ has been adapted. The markers $a_0, a_1, \ldots$ sit always on natural numbers such that ψ is made total on an interval I_t iff the markers vacate the number t. Furthermore, if $n < m$, a_n sits on t and a_m on s then $t < s$. At stage 0, every marker a_n sits on n. The idea is to move every marker eventually onto a position s where $H_s(s) = H(s)$ so that the diagonalization can exploit this knowledge in order to avoid H-triviality.

The following five steps govern how in stage s the updates are done. Let n be the least n for which a marker qualifies to move either by condition (2) or condition (3).

(1) Every marker a_m with $m \geq n$ moves from its current position to $s + m - n$.

(2) The marker a_n moves from t to s if no marker a_m with $m < n$ moves and there is an extension A of ψ_s such that the sum

$$\sum\{2^{-|p|} : U_s^A(p) \quad \text{is defined and } \mathrm{use}(U_s^A(p)) > \min(I_t)\} \geq 2^{-n(n+1)}.$$

After the move, the update done on ψ is that $\psi_{s+1}(x) = A(x)$ for all x belonging to an interval I_r with $t \leq r < s$ and $\psi_s(x)$ being undefined; A is here the set from above which forced the marker a_n to move.

(3) The marker a_n moves from t to s if no marker a_m with $m < n$ moves and $H_s(s) \leq H_t(t)$. In this case, $\psi_{s+1}(x) = 0$ for all x belonging to an interval I_r with $t \leq r < s$ and $\psi_s(x)$ being undefined.

(4) Furthermore, for every interval I_r with $r < t$ for the t from (2) or (3) or $r < s$ for the case that no marker has moved, the following is done: If ψ_s is not defined on all members of I_r and there are a program p, a string σ and an x such that
- $U_s(p) = \sigma$ and $|p| < H_r(r) + \log(|I_r|)/2$,
- $x = \min(I_r - dom(\psi_s))$ and $dom(\sigma) = \{0, 1, \ldots, \max(I_r)\}$,
- $\psi_s(y) = \sigma(y)$ for all $y \in dom(\sigma) \cap dom(\psi_s)$,

then one defines $\psi_{s+1}(x)$ such that $\psi_{s+1}(x) \in \{0, 1\} - \{\sigma(x)\}$.

(5) Let n be the number of the marker which sits on s after stage s. Let I_s have length n.

First it is shown by induction that every marker a_n moves only finitely often. Assume that all a_m with $m < n$ move only finitely often. Let t be a stage so large that a_n does no longer move by condition (1) and also no longer by condition (2). Now assume that the marker a_n moved at stage t; if one could not choose the stage t such than a_n would only move finitely often anyway. Now, for all stages $s > t$, the marker a_n moves at stage s, if at all, only due to condition (3). For all $s \geq t$ where the marker a_n moves, the condition $H_s(s) \leq H_t(t)$ holds. For the first such move, this follows from the condition, for further moves, it is preserved by the transitivity of $\leq$. As there are only finitely many s with $H_s(s) \leq H_t(t)$, a_n will only have a finite number of additional moves. Furthermore, conditions (1) and (3) together make sure that, whenever a_n sits on t and there is an $s \geq t$ with $H_s(s) \leq H_t(t)$, then a_n moves on in one of the stages $t+1, t+2, \ldots, s$. For that reason, a_n ends up eventually on a t such that $H_s(s) > H_t(t)$ for all $s > t$ which implies that $t = \max(\{s : H_s(s) \leq H_t(t)\}) = \max(\{s : H(s) \leq H_t(t)\})$. Note that $H_t(t) = H(t)$ by the assumption on the universal machine for each t on which one marker comes to rest eventually.

Second, note that if a_n comes finally to rest on t, then, for some constant c independently of t and some properties of H, $\psi(x)$ will be defined only by (4) and only for at most $2^{\log(|I_t|)/2+H_t(t)+c-H(t)}$ positions in I_t. As $H_t(t) = H(t)$, there are, for each

[1] The prefix free complexity of x is sometimes denoted by $K(x)$ in the literature.

large enough t such that some marker a_n eventually rests on t, some positions in I_t which will remain free forever. Hence the condition (4) makes it impossible for any extension A of ψ to be H-trivial as any H-trivial set B satisfies

$$\exists c \forall n\, [H(B(0)B(1)\dots B(\max(I_n))) \leq H(n) + c].$$

The latter holds as $\max(I_n)$ can be computed from n.

Third, to see that any extension A of ψ satisfies $A \leq_{LH} K$, note that one can compute relative to K the set E of all $(p, k, U^B(p))$ where B extends ψ and $U^B(p)$ is defined and the use to compute $U^B(p)$ is between $\min(I_t)$ and $\min(I_s)$ for the final positions t and s of the markers a_k and a_{k+1}, respectively. By construction,

$$\sum\{2^{-|p|} : \exists x\, [(p, k, x) \in E]\} < 2^{1+2+\dots+k} \cdot 2^{-k\cdot(k+1)}$$

where the bound $2^{1+2+\dots+k}$ comes from the undefined places of ψ left on the intervals linked to the final positions of the markers $a_0, a_1, \dots, a_k$ and which can be filled with either 0 or 1 while $2^{-k\cdot(k+1)}$ comes from the bound enforced by (2). In total, this part of the sum will be less than 2^{-k} and hence the whole sum will converge. But as the sum of all

$$\sum\{2^{-|p|} : \exists k, x\, [(p, k, x) \in E]\}$$

converges to a fixed finite real number, there is a constant c such that, uniformly for all B extending ψ and all x, $H^K(x) \leq H^B(x) + c$. Hence $A \leq_{LH} K$ as A is one of the sets B extending ψ. □

The following is a basis theorem for Π^0_1 classes and its proof involves a double use of van Lambalgen's theorem. Recall that van Lambalgen's theorem relativizes as follows: for every sets A, B, C the set $A \oplus B$ is Martin–Löf random relative to C iff A is Martin–Löf random relative to C and B is Martin–Löf random relative to $A \oplus C$.

Theorem 10. *Let A be low for Ω. Then every perfect Π^0_1 class has a member B such that $B \not\leq_{LR} A$.*

Proof. Suppose that we are given a perfect tree T without dead ends which represents a Π^0_1 class P (that is, P consists of the infinite paths through T). Let R be a set which is Martin–Löf random relative to $A \oplus \Omega$ and notice that by the relativized Van Lambalgen's Theorem $\Omega \oplus R$ is Martin–Löf random relative to A (since A is low for Ω). We will define a set $B \in P$ such that $\Omega \oplus R$ is not Martin–Löf random relative to B, so that $B \not\leq_{LR} A$. Define set B inductively as follows. Suppose that $B \upharpoonright n$ is defined, $[B \upharpoonright n] \cap P \neq \emptyset$ and there are x_n branching nodes preceding node $B \upharpoonright n$ in T. If node $B \upharpoonright n$ is not branching let $B \upharpoonright (n+1)$ be the unique one-bit extension of $B \upharpoonright n$ such that $[B \upharpoonright (n+1)] \cap P \neq \emptyset$ (and notice that $x_{n+1} = x_n$). Otherwise let $B(n) = R(x_n)$ (and notice that $x_{n+1} = x_n + 1$).

Roughly speaking B is the code of R via T. Since $T \leq_T \emptyset'$ and $\Omega \equiv_T \emptyset'$ we have that $R \leq_T \Omega \oplus B$ (notice that the sequence x_n, and so the coding of R into B, is recursive in $\emptyset'$). Obviously $B \in P$ and R is not Martin–Löf random relative to $\Omega \oplus B$. But by the relativized van Lambalgen's theorem this means that $\Omega \oplus R$ is not Martin–Löf random relative to B, which is what we needed. □

We note that the proof above shows that the same result holds if even when P is the set of infinite paths through a perfect tree $\leq_T \emptyset'$, which is a slightly more general notion than Π^0_1 class. For Δ^0_2 sets the notions "low for Ω" and "low for Martin–Löf random" are known to coincide. But there are sets not below $\emptyset'$, which are low for Ω. As we mentioned above, every Π^0_1-class without low for random members contains such a set.

3. Hyperimmunity and *LR* degrees

Since every nonempty Π^0_1 class contains a path of hyperimmune-free degree Theorem 7 implies the following.

Corollary 11. *There is a hyperimmune-free Turing degree $\leq_{LR} \emptyset'$ which is not low for random.*

Theorem 7 combined with coding via a Π^0_1 class gives the following.

Corollary 12. *For every $Y \geq_T \emptyset'$ there are $A \leq_{LR} \emptyset'$, $B \leq_{LR} \emptyset'$ such that $A \oplus B \equiv_T Y$.*

Proof. We can use the Π^0_1 class T of Theorem 7 and some standard coding: let $A = \emptyset'$ and define B by finite extensions σ_n: if $\sigma_{n-1} \downarrow$ find (with oracle A) the least node $\tau \supset \sigma_{n-1}$ such that both $\tau * 0$, $\tau * 1$ are extendible. Then define $\sigma_n = \tau * Y(n)$. Clearly $Y \equiv_T A \oplus B$, $A \leq_T \emptyset'$ and $B \leq_{LR} \emptyset'$ since it belongs to T. □

The proof of the following theorem uses ideas from [17].

Theorem 13. *For every set A there is a set B of hyperimmune-free degree such that $B \not\leq_{LR} A$.*

Proof. We force with Π^0_1 classes: starting from a certain Π^0_1 class T we define a decreasing sequence $T \supseteq P_0 \supseteq P_1 \supseteq \cdots$ of Π^0_1 classes such that each class is a finite extension of the next one (in the sense that we get P_{e+1} from P_e by subtracting a closed-open set). Let U be the member of an oracle Martin–Löf test which was defined in Lemma 3. The goal for defining P_{2e} is

$$[\Phi^\beta_e \text{ is total for all } \beta \in P_{2e}] \vee \exists\, x\, [\Phi^\beta_e(x)\uparrow \text{ for all } \beta \in P_{2e}] \qquad (5)$$

and the goal for defining P_{2e+1} is

$$\forall \beta \in P_{2e+1} \quad [U^{\beta} \not\subseteq V_e^A]. \tag{6}$$

In order to satisfy (5) we wish to start with a class T with the property that for all e there is a closed-open set D_e and some $x \in \mathbb{N}$ such that either $\Phi_e^{\beta}(x) \uparrow$ for all $\beta \in T - D_e$ or Φ_e^{β} is total for all $\beta \in T - D_e$. For satisfying (6) via Lemma 3 and Theorem 4 we also wish that

$$\exists^{\infty} n \text{ [all nodes in } N_n * \{0,1\}^{n+1} \text{ are extendible in } T] \tag{7}$$

where N_n is the set of extendible nodes of length a_{n-1} and (a_i) is the sequence of Lemma 3. We use a finite injury argument to build T. Starting with the full binary tree, let $g(e,x)[s]$ be the $\langle e,x\rangle$-th number n such that all nodes in $N_n * \{0,1\}^{n+1}$ are extendible in T_s (and $g(e,x) = \lim_s g(e,x)[s]$, if this exists). At stage $s+1$ we look for the least $\langle e,x\rangle < s$ such that $\Phi_e^{\sigma}(x)[s] \downarrow$ for some σ of length $> a_{g(e,x)[s]}$ and $< s$ such that $[\sigma] \cap T_s \neq \emptyset$ (i.e. σ is extendible in the current version of T). If found, we cut from T all branches which extend $\sigma \restriction a_{g(e,x)[s]}$ and are incompatible with σ.

By finite injury it is clear that each $g(e,x)$ reaches a limit and hence (7) holds. Now suppose that there exists $\beta \in T$ and $x \in \mathbb{N}$ such that $\Phi_e^{\beta}(x) \uparrow$. If D_e is a closed-open set covering all branches which are incompatible with $\beta \restriction a_{g(e,x)}$ then $T - D_e \neq \emptyset$ and $\Phi_e^{\gamma}(x) \uparrow$ for all $\gamma \in T - D_e$, since otherwise the construction would evict β from T.

We let $P_{-1} = T$. Now assume that P_{2e-1} is defined. If there does not exist $\beta \in P_{2e-1}$ and x such that $\Phi_e^{\beta}(x) \uparrow$ then define $P_{2e} = P_{2e-1}$. Otherwise let $P_{2e} = P_{2e-1} - C_{2e}$ where C_{2e} is defined as D_e above only that P_{2e-1} takes the place of T. Now let $N_n(e)$ denote the set of extendible nodes in P_{2e} of length a_{n-1} and let n be the least such that all nodes in $N_n(e) * \{0,1\}^{n+1}$ are extendible in P_{2e}. Find a string $\sigma \in N_{n+1}(e)$ such that $U^{\sigma} \not\subseteq V_e^A$ (the existence of such a string is ensured by Lemma 3) and if C_{2e+1} is the closed-open set of nodes in P_{2e} which are incompatible with σ let $P_{2e+1} = P_{2e} - C_{2e+1}$.

Since $P_i \neq \emptyset$ for all i, there exists B in $\cap_i P_i$ and by induction it satisfies all requirements. Clearly $B \not\leq_{LR} A$ and is of hyperimmune-free degree because given any total Φ_e^B we can dominate it recursively by using the Π_1^0 class P_{2e}, since $\Phi_e^{\beta}(x) \downarrow$ for every $\beta \in P_{2e}$ and all x. □

Although there are non-trivial hyperimmune-free Turing degrees every *LR* degree contains a hyperimmune set.

Theorem 14. *Every LR degree contains a hyperimmune set.*

Proof. The standard r.e. nonrecursive low for random construction is compatible with making the real co-r.e. and hyperimmune. Relativizing, we get that for every A there is an A-co-r.e. hyperimmune B such that $A \oplus B \equiv_{LR} A$. Since hyperimmune Turing degrees are upward closed this concludes the proof. □

The main technique for constructing reals α such that the cone of reals $\beta \leq_{LR} \alpha$ is uncountable can be combined with lowness requirements to give a superlow real with this property, as in Theorem 15. In [2] it was shown that every non-GL_2 set has uncountably many $\leq_{LR}$ predecessors. Theorem 15 shows that non-GL_2 is not a property that characterizes the sets with this property.

Theorem 15. *There is a superlow r.e. set B such that $\{A \mid A \leq_{LR} B\}$ contains a perfect Π_1^0 class.*

Proof. Let U be the second element of a universal oracle Martin-Löf test. We start with the full binary tree and start chopping branches in such a way that we approximate a perfect subtree T. At the same time we define B and a $\Sigma_1^0(B)$ set V^B of measure < 1 such that $U^{T(\sigma)} \subseteq V^B$ for all $\sigma \in \{0,1\}^*$. Whenever some $T(\sigma)$ is defined some large number c_{σ} is set to be the B-code of σ and $U^{T(\sigma)} - U^{T(\sigma^-)}$ is enumerated into V^B (σ^- is the predecessor of σ). In the rest of the stages this closed-open set of reals will remain in V^B only as long as $c_{\sigma} \notin B$. For each e strategy P_e will be responsible for defining $T(\sigma)$ for all $\sigma \in \{0,1\}^e$ and ensuring that the measure added into V^B for the sake of the $(e+1)$-th level of T is $< 2^{-e-3}$ (that is, $< 2^{-2e-4}$ for each $\sigma \in \{0,1\}^{e+1}$). If we had no restrictions on B the method described in the proof of Theorem 7 would suffice. Now that we want B to be superlow, P_e has a dynamic quota q_e (instead of the fixed one 2^{-2e-4}), which starts with value 2^{-2e-5} and becomes half of the previous value every time it is *injured* i.e. every time some higher priority strategy acts.

P_e : For all $\sigma \in \{0,1\}^e$, $T(\sigma) \downarrow$ and $\mu(U^{\tau} - U^{T(\sigma)}) < q_e$ for all $\tau \supset T(\sigma)$ in T.

N_e : The number of different computations of $\Phi_e^B(e)$ is recursively bounded.

So we agree that at stage s, $q_e[s] = 2^{-2e-5-n}$ where n is the number of times P_e has been injured by stage s. We set a priority list $P_0, N_0, P_1, \ldots$ and order the finite strings first by length and then lexicographically. At each stage r_e is the maximum use of any convergent computation $\Phi_{e'}^B(e')$ for $e' \leq e$. Initially we define $T(\lambda) = \lambda$. At stage s do the following.

(I) For the least e (if any) such that $\Phi_e^B(e) \downarrow$ and $\Phi_e^B(e)[s-1] \uparrow$ say that N_e *acts*.

(II) Check to see if there is some $T(\sigma) \downarrow$ ($|\sigma| = e$, say) and some $\tau \supset T(\sigma)$ which is compatible with a leaf of the current tree T, $|\tau| < s$ and such that $\mu(U^{\tau} - U^{T(\sigma)}) \geq q_e[s]$. In that case choose the least such σ and cut the branches which extend $T(\sigma)$ and are incompatible with τ. Also for all $\rho \supseteq \sigma$ make $T(\rho) \uparrow$, if $c_{\rho} \downarrow > r_{|\rho|-1}$ enumerate c_{ρ} into B, redefine $T(\sigma)$ to be τ and redefine c_{σ} to be a large number. Enumerate $U^{T(\sigma)} - U^{T(\sigma^-)}$ into V^B with use c_{σ} (P_e *acts*).

(III) Consider the least σ such that $T(\sigma * i) \uparrow$ for $i = 0, 1$. If $T(\sigma)$ is of length $< s - 1$ then for each $i \in \{0, 1\}$ define $T(\sigma * i)$ to be $T(\sigma) * i$, define $c_{\sigma * i}$ to be a large number and enumerate $U^{T(\sigma * i)} - U^{T(\sigma)}$ into V^B with use $c_{\sigma * i}$.

To verify the construction we first show that there is a recursive function f such that for all e there can be at most $f(e)$ times where P_e is injured. Assuming that $f(e) \downarrow$ we know that P_e will be injured at most $f(e)$ times. Now P_e itself can redefine a node of T at level e at most $2^{2e+5+f(e)}$ times during an interval of stages where it is not injured, since during that interval $q_e \geq 2^{-2e-5-f(e)}$ and each time the node is redefined measure at least q_e is added to U^β for a single path β. Since there are 2^e nodes at level e strategy P_e can act at most $p_e = 2^e \cdot 2^{2e+5+f(e)} \cdot (f(e) + 1)$ times. Thus the $P_i, i \leq e$ can act at most $f(e) + p_e$ times and N_e can act at most $f(e) + p_e + 1$ times. So P_{e+1} can be injured at most

$$f(e + 1) = 2 \cdot (f(e) + p_e) + 1$$

many times. Since P_0 is never injured we can let $f(0) = 0$ and the induction is complete. This also shows that T converges to a perfect tree and by construction if $\beta \in [T]$ we have $U^\beta \subseteq V^B$. Moreover B is superlow since we have a recursive bound on the number of injuries of N_e (lower priority P strategies respect r_e).

It remains to show that $\mu(V^B) < 1$. The set V^B consists of the collection of closed-open intervals which were enumerated into V^B and were never evicted from it, that is, their corresponding B-codes do not belong in B. Since $\mu(U^\beta) < 2^{-2}$ for all β, the measure added to V^B in defining T to level 0 is at most 2^{-2}. Every other closed-open interval in V^B can be assigned (or traced) to a unique strategy P_{e+1} and in particular a node σ of level $e + 1$. It suffices to show that $\mu(C_\sigma) < 2^{-2e-4}$ for each σ of length $e + 1$, where C_σ is the (union of the) collection of intervals in V^B which originate from the definitions of $T(\sigma)$. Suppose that $s_0 = 0$ and $s_1, s_2, \ldots, s_k$ are the stages where P_e is injured. For each $i < k$ let $C_\sigma(i)$ contain the intervals enumerated into C_σ during the interval of stages $[s_i, s_{i+1})$ and let $C_\sigma(k)$ contain the intervals enumerated into C_σ at stages $\geq s_k$. By construction $\mu(C_\sigma(i)) \leq q_e[s_i]$; indeed, every time except (perhaps) the first during $[s_i, s_{i+1})$ that $T(\sigma^-)$ is redefined by P_e the previous amount $U^{T(\sigma)} - U^{T(\sigma^-)}$ leaves V^B and so $C_\sigma(i)$ will be the closed-open set which corresponds to the latest definition of $T(\sigma^-)$ during $[s_i, s_{i+1})$. Thus:

$$\mu(C_\sigma) \leq \sum_{i=0}^{k} \mu(C_\sigma(i)) < \sum_{i=0}^{k} 2^{-i} q_e[s_0] < 2^{-2e-4}$$

and this concludes the proof. □

4. Relative randomness invariance and Turing incomparability

The standard construction of a nonrecursive r.e. set which is low for random relativizes to give that for every nonrecursive W there is a set A such that $A \equiv_{LR} W$ (that is, the classes of A-randoms and W-randoms coincide) and $W <_T A$ (see [2]). Note that by the existence of minimal degrees the dual result (that is, building $A <_T W$) does not hold and hence there are LR degrees such that the structure of Turing degrees inside them has minimal elements; in fact, if we consider minimal degrees which are not low (Sasso [14]) and the fact that all low for random reals are low it follows that there are also non-zero LR degrees with this property. A very related question is whether given any nonrecursive W we can find $A|_T W$ such that $A \equiv_{LR} W$. The next theorem gives a positive answer. Note that the argument is considerably more involved than the argument required for making $A >_T W$ (and $A \equiv_{LR} W$). Corresponding results for the r.e. case can be found in Section 5 and in [2].

Theorem 16. *Let W be a nonrecursive set. Then there exists A such that $A|_T W$ and the class of W-random reals coincides with the class of A-random reals (that is, $A \equiv_{LR} W$).*

Proof. We wish to construct A, a $\Sigma_1^0(A)$ class V^A with $\mu(V^A) < 1$ and a $\Sigma_1^0(W)$ class T^W with $\mu(T^W) < 1$ such that:

$$U^W \subseteq V^A \quad \text{and} \quad U^A \subseteq T^W,$$

where U is the second member of a universal oracle test (so that $\mu(U^\beta) < 2^{-2}$). First we fix an r.e. operator V as follows. Consider a recursive map f of $\mathbb{N}$ onto $\{0, 1\}^*$ such that every $n \in \mathbb{N}$ corresponds to a finite string and for every finite string σ there exist infinitely many n such that $f(n) = \sigma$. Then for every set X define the $\Sigma_1^0(X)$ class $V^X = \{f(n) \mid n \in X\}$. We wish to satisfy the following positive requirements:

$$P_e : \ \Phi_e^A \neq W; \tag{8}$$

$$Q_e : \ \Phi_e^W \neq A. \tag{9}$$

Requirement P_e will be satisfied by searching for splittings, that is, pairs of strings σ, τ such that $\Phi_e^\sigma(x) \downarrow \neq \Phi_e^\tau(x) \downarrow$ for some $x \in \mathbb{N}$. If a splitting is found we can choose the string which gives a disagreement with W and make it an initial segment of A. If no such splitting is found we can argue that if $\Phi_e^A = W$ then W is recursive. Of course the choice of such a string as an initial segment of A could add measure in V^A and since we want $\mu(V^A) < 1$ we should ask for splittings which add small measure to V^A. The above situation is the main conflict in this construction.

Requirement Q_e is much more easily satisfied. We just have to choose any $n \in \mathbb{N} - A$, wait until $\Phi_e^W(n) \downarrow = 0$ and then enumerate it into A. The measure added to V^A by this action is not an issue since we can choose any n (so that the added measure is arbitrarily small). Of course changing A will also produce some cost with respect to the reduction $U^A \subseteq T^W$ that we are building, but this can be dealt with as in a typical cost function argument.

The construction will be recursive in W, so that we can enumerate T^W recursively in W. In particular it will be a cost function argument relative to W and A will be approximated W-recursively, that is, $A \leq_T W'$. Note that essentially we have to consider *two* sorts of cost when we change the approximation to A. One is associated with $U^A \subseteq T^W$: changing A will make some closed-open set in T^W useless and its measure will be counted as junk. The other one has to do with $U^W \subseteq V^A$: changing A in certain ways adds measure into V^A since operator V is defined *a priori*. Define the first sort as follows:

$$cost(n, s) = \mu\{\sigma \mid \sigma \in U^A[s] \text{ with use } u \text{ such that } n < u < s\}. \tag{10}$$

Since for every σ there exist infinitely many n with $f(n) = \sigma$, we can easily enumerate elements of U^W into V^A without worrying about the cost incurred by these enumerations. At stage s we only enumerate into T^W elements of V^A with use $< s$ and any new strings appearing in U^W can be enumerated into V^A by enumerating in $n > s$.

Let the requirements be ordered $P_0, Q_0, P_1, Q_1, \ldots$. Let $r_{2e}[s]$ be the restraint imposed on A by P_e and let $r_{2e+1}[s]$ be the restraint imposed on A by Q_e (these parameters will be defined inductively below). We also define a quota $p_e[s]$ which indicates the cost that P_e is able to afford when it changes the approximation to A at stage s. This will help ensuring that $\mu(T^W) < 1$ while $U^A \subseteq T^W$ and also that $\mu(V^A) < 1$. This quota will be halved every time P_e *acts* (that is, every time it alters the approximation to A). In particular we define $p_e[s] = 2^{-e-n-4}$ where n is the number of times that P_e has acted. Strategy P_e has a parameter $\sigma_e[s] \subset A_s$ which is the segment of A which it needs to preserve. At every stage s we define $\sigma_e[s] = A_s \upharpoonright n$ where n is the least number:

- greater than $2e$;
- greater than the restraints imposed by higher priority requirements;
- such that $cost(n, s) < p_e[s]$;
- such that $\mu(V^{A_s} - V^{A_s \upharpoonright n}) < p_e[s]$;
- such that if X is the set of finite strings in $U^W[s] - V^{A_s \upharpoonright n}$ then $\mu(X) < p_e[s]$ (recall that U^W is prefix-free as a set of finite strings).

Strategy P_e looks for splittings above σ_e which add to V^{σ_e} measure less than p_e. More precisely, at stage s it searches for a pair of strings τ_0, τ_1 of length $< s$ which extend σ_e and some $x < s$ such that

$$\Phi_e^{\tau_0}(x)[s] \neq \Phi_e^{\tau_1}(x)[s] \downarrow \quad \text{and} \quad \mu(V^{\tau_i} - V^{\sigma_e[s]}) < p_e \tag{11}$$

for $i = 0, 1$. If there exists such a splitting we say that P_e *requires attention at stage* s. Upon receiving attention the strategy then chooses $i \in \{0, 1\}$ such that $\Phi_e^{\tau_i}(x) \neq W(x)$ and it lets τ_i be an initial segment of A, i.e. it sets $A_s = \tau_i * 0^\omega$. We will be able to argue, by a typical cost function argument, that σ_e reaches a limit. Then if P_e finds a splitting it is satisfied and otherwise we can argue that if $\Phi_e^A = W$ then W is recursive.

For the strategy Q_e we also define a quota $q_e[s] = 2^{-e-n-4}$ where n is the number of times that Q_e has *acted* i.e. changed the approximation to A. The strategy will hold witness $x_e[s]$ which is defined as the least $x > 2e$ such that:

- $cost(x, s) < q_e[s]$;
- $2^{-|f(x)|} < q_e[s]$;
- it is larger than the restraint r_i for all $i \leq 2e$ and all witnesses for higher priority requirements;
- such that if X is the set of finite strings in $U^W[s] - V^{A_s \upharpoonright x}$ then $\mu(X) < q_e[s]$.

We say that Q_e *requires attention at stage* s if $\Phi_e^W(x_e)[s] \downarrow = A(x_e)[s]$. For all e, s define $r_e[0] = 0$. For stages $s > 0$ we will have $r_e[s+1] = r_e[s]$ unless r_e is redefined explicitly in the construction. Finally, $A_0 = 0^\omega$.

4.0.0.8. Construction. At stage s consider the highest priority requirement which is not presently declared *satisfied* and which requires attention.

- If this is P_e, consider τ_0, τ_1, x as in (11) and such that $\sigma_e[s] \subset \tau_i$ for $i = 0, 1$. Define $A_{s+1} = \tau_i * 0^\omega$ for i such that $\Phi_e^{\tau_i}(x) \neq W(x)$ and define $r_{2e}[s+1]$ to be $|\tau_i|$. P_e is declared satisfied, all lower priority requirements are declared unsatisfied and their restraints are set to zero.
- If this is Q_e define $A_{s+1} = (A_s \upharpoonright x_e[s]) * y * 0^\omega$ where $y = 1 - \Phi_e^W(x_e)[s]$ and let r_{2e+1} be $x_e[s] + 1$. Q_e is declared satisfied, all lower priority requirements are declared unsatisfied and their restraints are set to zero.

Enumerate all finite strings which are in U^A with use $< s$ into T^W. Enumerate all finite strings in U^W into V^A via large n.

4.0.0.9. Verification. It is easy to see by induction that each strategy requires attention finitely often and so acts only finitely many times. Since P_e, Q_e cannot change the approximation to $A \upharpoonright 2e$ and since they act finitely often it follows that the approximation to A converges. Next we show that $\mu(T^W) < 1$. It follows by the definitions of T^W, q_e, x_e and (10) that

$$\mu(T^W) \leq \mu(U^A) + \sum_s cost(m_s, s)$$

where m_s is the least m such that $A(m)$ changed at stage s ($m_s = \infty$ if no such change occurred and $cost(\infty, s) = 0$) and this is equal to

$$\mu(U^A) + \sum_e \left(\sum_{s \in I_e} cost(m_s, s) + \sum_{s \in J_e} cost(m_s, s) \right) \leq 2^{-2} + 2^{-1} < 1$$

where I_e is the set of stages where P_e acted and J_e is the set of stages where Q_e acted. Almost precisely the same argument suffices to show that $\mu(V^A) < 1$.

In order to see that $U^W \subseteq V^A$, suppose that σ is in U^W and let e be such that $2^{-e-4} < 2^{-|\sigma|}$. Once σ has been enumerated into U^W and requirements $P_{e'}, Q_{e'}$ for $e' < e$ have finished acting, σ will be enumerated into V_A and will not subsequently be removed.

Before we proceed to the satisfaction of P_e, Q_e we show the following usual property of the cost function:

$$\forall e\, \exists n, s\, \forall s' > s\, [cost(n, s') < 2^{-e}]. \tag{12}$$

Let n be large enough such that $\mu(U^A - U^{A\upharpoonright n}) < 2^{-e-1}$ and such that no restraint corresponding to a requirement $Q_{e'}$ or $P_{e'}$ for $e' < e$ takes a value $\geq n$. Let $s > n$ be large enough such that no requirement $Q_{e'}$ or $P_{e'}$ for $e' < e$ acts at a stage $\geq s$ and such that $A \upharpoonright n$ is an initial segment of $A_{s'}$ at all stages $s' \geq s$. If $s' > s$ then any element of $U^{A_{s'}}$ with use $< s'$ and which is not in $U^{A\upharpoonright n}$, is either in U^A or is in the trash measure contributed to T_W by a requirement $P_{e'}$ or $Q_{e'}$ for some $e' \geq e$. Thus the total measure of all such elements is $< 2^{-e}$. A very similar argument suffices to show that for all e there exists n, s such that for all $s' > s$, $\mu(V^{A_{s'}} - V^{A_{s'}\upharpoonright n}) < 2^{-e}$.

Next we show that all P_e requirements are satisfied. Suppose that P_e and all higher priority requirements have stopped requiring attention after stage s_0, and that by stage s_0 the quota p_e and the parameter σ_e have both reached a limit. If for the sake of a contradiction we assume $\Phi_e^A = W$ then we can argue that W is computable. Since $\Phi_e^A = W$ there are no $\tau_i \supset \sigma_e$ such that (11) holds (otherwise P_e would receive attention). To compute $W \upharpoonright n$ just search for a finite string $\rho \supset \sigma_e$ and a stage $s > s_0$ such that $\Phi_e^\rho[s] \upharpoonright n \downarrow$ and $\mu(V^\rho - V^{\sigma_e}) < p_e$. This will be found since Φ_e^A is total and $\mu(V^A - V^{\sigma_e}) < p_e$. $\Phi_e^\rho[s] \upharpoonright n \downarrow$ will equal $W \upharpoonright n$.

Finally we show that each Q_e is satisfied. Let s_0 be a stage after which Q_e along with all higher priority requirements have stopped requiring attention, $x_e[s]$ has reached a limit x_e and $A(x_e)$ has also reached its limit. If $\Phi_e^W = A$, then at the first stage $s > s_0$ such that $\Phi_e^W(x_e)[s] \downarrow$ strategy Q_e would require and receive attention, thus creating a permanent disagreement on x_e which is a contradiction. □

5. Local structures of *LR* degrees

In this section we look at local structures of *LR* degrees below $\emptyset'$ and their relationships with Turing reducibility. We also study the *LR* degrees of recursively enumerable sets.

5.1. Structure below $\emptyset'$

The following is a forcing condition for avoiding non-trivial upper cones of *LR* degrees.

Lemma 17. *Let* (V_e) *be an effective list of all r.e. operators such that* $\mu(V_e^\beta) < 1$ *for all* $e \in \mathbb{N}$, $\beta \in \{0,1\}^\omega$. *Suppose that* $W \leq_T \emptyset'$, $W \not\leq_{LR} \emptyset$ *and* D *is a computable set of strings. Then for all* $e \in \mathbb{N}$

$$\forall \sigma_0 \in D\, \exists \rho \subset W\, \exists \sigma \supseteq \sigma_0, \sigma \in D\, \forall \tau \supset \sigma, \tau \in D\, [U^\rho \not\subseteq V_e^\tau] \tag{13}$$

where all the quantifiers range over $\{0,1\}^{<\omega}$.

Proof. Suppose that (13) did not hold for some $e \in \mathbb{N}$. We show that $W \leq_{LR} \emptyset$. To cover U^W with a Σ_1^0 class of measure < 1 we will define a recursive set A such that $U^W \subseteq V_e^A$. Let (W_s) be a recursive approximation of W. That is, $W_s \in \{0,1\}^{<\omega}$ for all $s \in \mathbb{N}$ and

$$\forall \sigma \subset W\, \exists s_0\, \forall s > s_0\, [\sigma \subset W_s].$$

We define a recursive monotone unbounded sequence (σ_i) of strings in D with σ_0 is as in the negation of (13), and set $A = \cup_i \sigma_i$. Let the finite strings be ordered as usual, first by length and then lexicographically. Inductively assuming that $\sigma_s \downarrow$ we define σ_{s+1} as follows. At stage $s+1$ consider the least $\tau \subset W_s$ such that $U^\tau \not\subseteq V_e^{\sigma_s}$. If this does not exist let $\sigma_{s+1} = \sigma_s$.

Otherwise consider the least string $\sigma \in D$ of length $< s$ which extends σ_s and $U^\tau \subseteq V_e^\sigma$ and let $\sigma_{s+1} = \sigma$. If this does not exist let $\sigma_{s+1} = \sigma_s$.

Obviously $\mu(V_e^A) < 1$ and we need to show that $U^W \subseteq V^A$. Indeed, if this did not hold there would be a least $\rho \subset W$ such that $U^\rho \not\subseteq V^A$. Then there would be some stage s_0 such that $\sigma_{s+1} = \sigma_s \supset \sigma_0$ for all $s > s_0$ and this contradicts (13) according to the construction. □

We note that if $W \not\leq_{LR} \emptyset'$ then there is B such that $W \not\leq_{LR} B$ and $W \leq_T B \oplus \emptyset'$. This is because, as it was first shown in [2], there is a perfect tree which is computable in $\emptyset'$ and all of its paths are $\leq_{LR} \emptyset'$ (so not $\geq_{LR} W$). Therefore if B is the path 'carved out by W' on this tree (i.e. go left or right according to whether the next bit of W is 0 or 1) we get $W \leq_T B \oplus \emptyset'$. In particular, W can be approximated recursively in B but $W \not\leq_{LR} B$. Then the proof of the lemma relativized to B gives the following.

Corollary 18. *Suppose that $W \not\leq_{LR} \emptyset'$ and D is a computable set of strings. Then for all $e \in \mathbb{N}$ statement (13) is true.*

Theorem 19. *Given any Δ_2^0 set W such that $\emptyset <_{LR} W <_{LR} \emptyset'$ there is a Δ_2^0 set A such that $A|_{LR}W$.*

Proof. This can be done by an oracle construction relative to $\emptyset'$. We build A by defining a sequence (σ_s) so that $\sigma_s \subset \sigma_{s+1}$ and $A = \cup_s \sigma_s$. Let (V_e) be an effective list of all r.e. operators such that $\mu(V_e^\beta) < 1$ for all $\beta \in \{0,1\}^\omega$. To make sure that $A \not\leq_{LR} W$ we will build A and T^A such that $\mu(T^A) < 1$ and

$$P_e : T^A \not\subseteq V_e^W$$

for all $e > 1$. Define T as follows. Consider a computable map f of $\mathbb{N}$ onto $\{0,1\}^*$ such that for every finite string τ there exist infinitely many n such that $f(n) = \tau$. Then for every set X define the $\Sigma_1^0(X)$ class $T^X = \{f(n) \mid n \in X\}$.

Next, consider the standard universal oracle Martin–Löf test (U_e). To achieve $T^A \not\subseteq V_e^W$ we will choose a small enough t and enumerate enough of $U_t^{\emptyset'}$ into T^A so that we satisfy P_e. Since $U_t^{\emptyset'} \not\subseteq V_e^W$ (given that $\emptyset' \not\leq_{LR} W$) after finitely many actions (i.e. finite extensions) on A we will permanently have $T^A \not\subseteq V_e^W$. By choosing t small enough we will also achieve $\mu(T^A) < 1$. For this reason P_e will have a quota $p_e[s]$ at stage s which represents the least value that we are allowed to give to t.

For $W \not\leq_{LR} A$ we will use Lemma 17. To achieve

$$Q_e : U^W \not\subseteq V_e^A$$

at some stage s we wish to find some $\rho \subset W$ and $\tau \supset \sigma_{s-1}$ such that $U^\rho \not\subseteq V_e^\nu$ for all $\nu \supset \tau$. However we need to make sure that the chosen extension τ does not add too much measure to T^A, so that we still have $\mu(A) < 1$. For this reason Q_e will have a quota q_e, which may change (get larger) during the construction, and at stage s it is required to choose only strings τ which add to T^A measure less than $2^{-q_e[s]}$ (i.e. $\mu(T^\tau - T^{\sigma_s}) < 2^{-q_e[s]}$). So if

$$D_e[s] = \{\tau \in \{0,1\}^{<\omega} \mid \mu(T^\tau - T^{\sigma_{s-1}}) < 2^{-q_e[s]}\}$$

strategy Q_e will look for strings in D_e which have the properties mentioned above. Note however that for Lemma 17 to be used successfully for the satisfaction of Q_e, after an action of this strategy at stage s we must make sure that $\sigma_t \in D_e[s]$ for all $t > s$. We can achieve this if we raise the quotas for the lower priority requirements, provided that no higher priority requirement subsequently acts. Thus there will be a finite injury effect for Q_e. We order the strategies as $P_0, Q_0, P_1, \ldots$. We say that P_e *requires attention at stage s* if $T^\nu \subseteq V_e^W$ where ν is the latest initial segment of A chosen by P_e (if it has not chosen any segments for A so far, $\nu = \emptyset$). In this case we say that P_e *requires attention via* ν. We say that Q_e *requires attention at stage s* if it has not *acted* since it was last *initialized*.

5.1.0.10. Construction. At stage 0 *initialize* all strategies and let $\sigma_0 = \emptyset$, $q_e[0] = 2e+3$, $p_e[0] = 2e+4$. At stage $s > 0$ find the least requirement which requires attention.

(I) If this is Q_e let σ_s be the least extension of σ_{s-1} in $D_e[s]$ such that for some $\tau \subset W$, $U^\tau \not\subseteq V_e^\nu$ for all $\nu \supset \sigma_s$ in $D_e[s]$ (by Lemma 17). Initialize all lower priority requirements, let m be the least such that $2^{-m} < 2^{-q_e[s]} - \mu(T^{\sigma_s} - T^{\sigma_{s-1}})$ and for all $i > e$ let $q_i[s+1] = m + q_i[s] + 1$, $p_i[s+1] = m + p_i[s] + 1$.

(II) If this is P_e let σ_s be the least string extending σ_{s-1} such that

$$T^{\sigma_s} - T^{\sigma_{s-1}} = U^{\emptyset'}_{p_e[s]}[s] - T^{\sigma_{s-1}}$$

and initialize all lower priority Q requirements, setting $q_i[s+1] = q_i[s] + 1$ for all $i > e$.

Go to the next stage.

5.1.0.11. Verification. To verify the construction first note that $\mu(T^A) < 1$. Indeed, consider the intervals $I_j = (|\sigma_j|, |\sigma_{j+1}|]$ which define a partition of $\mathbb{N}$. For each $e \in \mathbb{N}$ let $(a_{e,j})_{j\in\mathbb{N}}$ be an increasing enumeration of the indices k such that σ_k was chosen during the construction by P_e and let $(b_{e,j})_{j\in\mathbb{N}}$ be an increasing enumeration of the indices k such that σ_k was chosen during the construction by Q_e. Also if

$$\alpha_{e,j} = \mu(T^{\sigma_{a_{e,j}}} - T^{\sigma_{a_{e,j}-1}})$$
$$\beta_{e,j} = \mu(T^{\sigma_{b_{e,j}}} - T^{\sigma_{b_{e,j}-1}})$$

then by construction $\sum_j \alpha_{e,j} \le \sum_j 2^{-2e-4-j}$ and $\beta_{e,j} \le 2^{-2e-3-j}$. Hence

$$\mu(T^A) \le \sum_e \left(2^{-2e-3} + 2^{-2e-2}\right) < 1.$$

Next, we show by induction that each strategy stops requiring attention after some stage and that it satisfies the corresponding requirement. First suppose that after stage s_0 no requirement of higher priority than P_e requires attention. After s_0 the quota p_e will take a final value t and strategy P_e will keep on enumerating $U_t^{\emptyset'}$ (in the form of successive closed-open sets) into T^A as long as V_e^W covers the enumerated intervals. Since $\emptyset' \not\le_{LR} W$ we have $U_t^{\emptyset'} \not\subseteq V_e^W$ and so at some stage some closed-open set enumerated into T^A will never be covered by V_e^W. This means that $T^A \not\subseteq V_e^W$ and P_e will stop requiring attention.

Second, suppose that s_1 is the last stage where Q_e is injured. At stage $s_1 + 1$ strategy Q_e will choose an extension σ_{s_1+1} of σ_{s_1} in $D_e[s_1 + 1]$ such that $U^\tau \not\subseteq V_e^\nu$ for some $\tau \subset W$ and all $\nu \supset \sigma_{s_1+1}$ with $\nu \in D_e[s_1 + 1]$. Then it suffices to show that $\sigma_n \in D_e[s_1 + 1]$ for all $n > s_1$. In order to see this note that the strategy increases the quotas of lower priority requirements by an amount m such that $2^{-m} < 2^{-q_e[s_1+1]} - \mu(T^{\sigma_{s_1+1}} - T^{\sigma_{s_1}})$ and so by the argument which showed that $\mu(T^A) < 1$ above we have $\mu(T^{\sigma_n} - T^{\sigma_{s_1+1}}) < 2^{-m}$ for all $n > s_1 + 1$. Hence $\mu(T^{\sigma_n} - T^{\sigma_{s_1}}) < 2^{-q_e[s_1+1]}$ and $\sigma_n \in D_e[s_1 + 1]$ for all $n > s_1$. □

5.2. Recursively enumerable LR degrees

The interactions of $\le_{LR}$ and $\le_T$ in the class of r.e. sets is very interesting. In Barmpalias and Montalbán [3] (also see [7, 15] for the relation between almost everywhere domination and $\le_{LR}$) it was shown that there is an r.e. set which is a half of a minimal pair in the Turing degrees and which is *LR* equivalent to $\emptyset'$. Also, Barmpalias, Lewis and Soskova [2] showed that given any incomplete r.e. Turing degree we can effectively produce another r.e. Turing degree which is in the same *LR* degree and is strictly above the given one. In other words, this means that for every r.e. set A there is an r.e. set B such that $A <_T B$ and the classes of A-randoms and B-randoms coincide. This was shown as follows.

Theorem 20 (*Barmpalias, Lewis and Soskova [2]*). *If W is an incomplete r.e. set, that is, $\emptyset' \not\le_T W$, then (uniformly in W) there is a r.e. set B such that $B \le_{LR} W$ and $B \not\le_T W$.*

Of course this does not immediately imply the previous claim, but the proof of Theorem 20 was adaptable in a way that it gave this result about the Turing degrees inside an *LR* degree. Here we show the dual of Theorem 20. That is, given a nonrecursive r.e. set B there is a r.e. set A such that $A <_T B$ and the classes of A-randoms and B-randoms coincide.

Theorem 21. *Given nonrecursive r.e. B there is an r.e. A such that $B \not\le_T A$ and $B \le_{LR} A$. Moreover A can be chosen such that $A <_T B$ (so that we also have $A \equiv_{LR} B$).*

Proof. First we show the first clause and then explain why the construction automatically produces the second clause as well. Given a nonrecursive r.e. B we need to build an r.e. operator V and an r.e. set A such that

$$U^B \subseteq V^A \tag{14}$$
$$\mu(V^A) < 1 \tag{15}$$
$$Q_e : \Phi_e^A \neq B \tag{16}$$

for all e. One aspect of the conflict is that (14) requires parts of B to be coded into A but (16) needs a restraint on the enumerations into A. This is a familiar situation in the theory of r.e. Turing degrees, but here there is one more problem. The restraints of (16) could damage (15) by acting as a trap for junk measure into V^A (that is, measure that has escaped U^B through a B-change). This can be handled by providing each Q strategy with a good current approximation to the measure of U^B. In the construction it is helpful to assume that U^B has a certain monotonicity property, namely that *later computations have larger B-use*. We can do this by replacing U^B with U_*^B which is defined as follows: at the beginning of stage s we enumerate a string σ into U_*^B in the following cases:

- σ has just appeared in $U^B[s]$ (although it might have appeared before with B-use which is no longer valid). In this case we enumerate σ into U_*^B with big B-use.
- σ was in $U^B[s-1]$ and it is $U^B[s]$ via the same computation (the associated B-use has not changed) but it is not in U_*^B at the beginning of stage s due to some B-change. In this case we enumerate σ into U_*^B with the same B-use as before.

It is clear that $U^B_* = U^B$ and that this enumeration has the desired property. We call this enumeration *canonical* and in the following we assume that U^B is given with a canonical enumeration. We also assume the *hat trick* for U^B, exactly as it is assumed for Turing functionals (see [16]). This means that at stage s, if some x was enumerated into B we do not take into account (for example, as far as membership into U^B is concerned) computations which involve B-use $\geq x$. This delayed enumeration has the effect that at infinitely many stages s (the *true stages* for the enumeration of B) the set $U^B[s]$ contains only permanent strings σ and so $\mu(U^B[s]) \leq \mu(U^B)$.

The operator V can be defined in a straightforward way ahead of the construction. At the beginning of stage s we enumerate a string σ into V^A in the following cases:

- σ has just appeared in $U^B[s]$ (although it might have appeared before with B-use which is no longer valid). In this case we enumerate σ into V^A with big A-use.
- σ was in $U^B[s-1]$ and it is $U^B[s]$ via the same computation (the associated B-use has not changed) but it is not in V^A at the beginning of stage s due to some A-change. In this case we enumerate σ into V^A with the same A-use as before.

The enumeration of V will be accompanied by some coding of B into A in order to ensure (15): whenever some interval has left U^B, is in V^A and its A-use is not restrained by any strategy, the construction enumerates the use into A thus deleting the interval from V^A. Note that this definition of V ensures that (14) is satisfied.

The crude strategy for Q_e is the usual Sacks restraints method. But note that restraints on A can act as a trap for junk measure (that is, measure which does not belong in U^B) in V^A. Given that $\mu(U^B) < 2^{-1}$, in order to satisfy (15) we need to make sure that:

$$\mu(V^A) - \mu(U^B) \leq 2^{-1}. \tag{17}$$

Each piece of measure in $\mu(V^A - U^B)$ will be charged to some Q-strategy. In other words, each Q-strategy may be blamed for some strings $\sigma \in V^A - U^B$ and in order to ensure (17) we need to set a quota for each strategy and make sure that it is never blamed for an amount larger than its quota.

We need to describe an *atomic strategy* for Q_e which, given an arbitrary quota 2^{-n}, can satisfy (16) while keeping:

$$\mu(V^A) - \mu(U^B) \leq 2^{-n}. \tag{18}$$

We may assume that Q_e acts only on stages s such that the measure of the false strings

$$\mu(U^B[s] - U^B) < 2^{-n-1}. \tag{19}$$

This can be arranged in the construction by a standard tree machinery. If at some stage we decide to restrain a computation

$$(\Phi^A = B) \restriction x \quad \text{with use } u \tag{20}$$

we should consider two outcomes. First, if $B \restriction x$ changes we will have satisfied (16) and by the assumption (19) requirement (18) will also be satisfied (given the lack of other restraints and the coding of B into A). Second, if $B \restriction x$ does not change and $B \restriction y$ changes, where y is such that

$$\{\beta \mid \beta \in V^A \text{ with use} < u \text{ and } \beta \in U^B \text{ with use } > y\} \neq \emptyset \tag{21}$$

then (16) is not yet satisfied and we have added some measure to $V^A - U^B$. The second outcome clearly poses some threat to (18) but this can be easily dealt with as follows. The ith time that we wish to apply new A-restraint we can wait until the length of agreement x of $\Phi^A = B$ is so long that the measure of (21) for $y = x$ is less than 2^{-n-i-1}. If $\Phi^A = B$ this will happen as there are infinitely many true stages s for B and at these stages $U^B[s] \subseteq U^B$. This cost will not increase in later stages as new reals in V^A get large use. Then by induction we have

$$\mu(V^A) - \mu(U^B) < \sum_{i>0} 2^{-n-i-1} + 2^{-n-1} \leq 2^{-n}$$

and by the usual Sacks restraint argument (16) is satisfied. To put all strategies together consider the tree of strategies T which has strategy Q_e on the nodes of level $2e+1$ and a backing strategy on the nodes of level $2e$ which approximates $\mu(U^B)$ with precision 2^{-e-2}. So at a node of level $2e$ we consider $[0, 1)$ divided into 2^{e+2} equal intervals and the endpoints $l_0, \ldots, l_k$ of the intervals are their branches/outcomes. If the node is accessible at s then l_i is accessible if $\mu(U^B[s]) \in [l_i, l_{i+1})$ (where $l_{k+1} = 1$). The Q-nodes have a single branch so that the approximation to the true path is determined by $\mu(U^B[s])$. The priority order amongst nodes is the usual one: first by length and then lexicographically. We fix a quota 2^{-n_α} for each Q-node α in an effective way such that

$$\sum_{\alpha \in Q} 2^{-n_\alpha} < 2^{-2}.$$

5.2.0.12. Construction. For each node α let r_α be the current restraint imposed by all nodes of priority higher than or equal to that of α. At each stage s do the following.

(I) Determine an approximation to the true path of length s according to the accessibility relation defined above. Initialize all nodes to the right of the present approximation to the true path.

(II) If r is the total restraint imposed by all strategies, look for σ which are in $V^A - U^B$ with A-use greater than r. Enumerate the least such use into A.

(III) For each α of odd length on the present approximation to the true path, check if there is ℓ such that $(\Phi_e^A = B) \upharpoonright \ell$ is defined with use $u > r_\alpha$ and

$$\mu(\{\beta \mid \beta \in U^B[s] \text{ with use } > l\}) < 2^{-n_\alpha - m - 1} \tag{22}$$

where m is the number of times that α applied a restraint since it was last injured by the true path. In that case restrain $A \upharpoonright u$ for the maximum such ℓ, and call s an *expansionary* stage for α.

5.2.0.13. Verification. By the hat trick on U^B and the true stages for B there is an infinite and infinitely often visited path TP through the tree of strategies. For each k there is some s_0 such that at all true stages $s > s_0$ the branch $TP \upharpoonright k$ is accessible. At true stages s all strings in $U^B[s]$ are permanent. So for any n_α, m there is a large enough l such that at true stages s condition (22) is met. This means that the usual Sacks restraints argument works for the nodes on the true path and all Q_e are satisfied. It is also clear that the restraints from nodes on or to the left of the true path come to a limit.

Also (14) holds by the definition of V. It remains to show that (15) holds. For that, it suffices to show (17). In order to show (17) it suffices to show that at (the end of) each stage s, the measure of the set of all finite strings σ which are in $V^A[s]$ but which are not in $U^B[s]$ is $\leq 2^{-1}$. So suppose that at stage s, σ is in $V^A[s]$ but is not in $U^B[s]$, and let the corresponding A-use be u. Then at stage s, σ is charged to that strategy on or to the left of the present approximation to the true path which first instituted a restraint of length $\geq u$ at a stage $s' < s$ subsequent to a stage at which it was last initialized. For each α we let $G_\alpha[s]$ be the set of all σ charged to α at stage s. We divide $G_\alpha[s]$ into two compartments $G_\alpha^0[s]$, $G_\alpha^1[s]$. The enumeration into these compartments happens as follows. Suppose that $\sigma \in G_\alpha[s]$. If at some stage $s' < s$ subsequent to the last stage $< s$ at which α was initialized, it restrained a computation $(\Phi^A = B) \upharpoonright \ell$ and σ was in $U^B[s']$ with use $\leq l$ then σ is in $G_\alpha^1[s]$ and we say that σ was enumerated into this set at stage s'. Otherwise we put it into $G_\alpha^0[s]$. It remains to show for every e,

$$\mu(\cup_{|\alpha|=2e+1} G_\alpha[s]) < 2^{-e-2} + \sum_{|\alpha|=2e+1} 2^{-n_\alpha}. \tag{23}$$

Since the enumeration of U^B is canonical, if at stage s' something is enumerated into $G_\alpha[s]$, then at stage s' the nodes to the right of α are initialized. Thus at stage s there can only be one α of level $2e+1$ with $G_\alpha^1 \neq \emptyset$, since as long as this holds there are no expansionary stages for α' to the right of α and of level $2e+1$. Moreover we have $\mu(G_\alpha^1[s]) < 2^{-e-2}$ because otherwise $\mu(U^B)$ would be decreased by 2^{-e-2} since α was last visited which means that α would be initialized. Finally $\mu(G_\alpha^0[s]) < 2^{-n_\alpha}$ by the construction and the definition of $G_\alpha^0[s]$, since the mth time that some measure is added to this set it must be less than $2^{-m-n_\alpha-1}$. So (23) holds and this concludes the proof of the first clause. For the second clause note that the set of nodes which are on or to the left of the true path is B-enumerable. Also it is easy to see that there will be infinitely many permanent restraints on A during the construction and B can decide whether a restraint is permanent. Hence it can compute A. □

A combination of the above argument with the argument in the proof of Theorem 20 shows that for every nonrecursive r.e. B of incomplete Turing degree there is a r.e. set A such that $A|_T B$ and the classes of A-randoms and B-randoms coincide.

Theorem 22. *For any r.e. nonrecursive B of incomplete Turing degree there is r.e. A such that $A|_T B$ and $A \equiv_{LR} B$.*

Sketch of Proof. The above tree argument can be combined with the coding in the proof of Theorem 20 in the same way that Sacks restraints are combined with Sacks coding in a tree version of the density theorem. Note that in this case the approximation to the true path will not be solely controlled by $\mu(U^B)$ as the coding nodes will have infinitely many branches. However we can still initialize the nodes which hold the restraints when their assumption about the value of $\mu(U^B)$ is much more than the current approximation so that the garbage boxes are emptied and the argument in the proof of Theorem 21 goes through.

Nies and Miller [9] asked if the *LR* degrees of r.e. sets are dense. The main difficulty for proving density lies on the fact that the join operator $\oplus$ does not define supremum in the *LR* degrees. In the classic Sacks density argument the join plays a very important role and in some structures of r.e. sets where the join fails to be a supremum density fails (one such example is the structure of identity bounded degrees, see [1]). Note however that the join is usually not important for proving upward and downward density (in the example given above upward and downward density holds). Say that a pair of r.e. *LR* degrees are *Turing comparable* if they have r.e. representatives B, C respectively such that $B \leq_T C$. The following theorem shows a weak density in the r.e. *LR* degrees: for every Turing comparable pair of r.e. *LR* degrees there is a r.e. *LR* degree strictly in between (which is also Turing comparable with the given degrees). This obviously implies upward and downward density.

Theorem 23. *If B, C are r.e. and $B \leq_T C$, $C \not\leq_{LR} B$ then there is an r.e. A such that $B \leq_T A \leq_T C$, $C \not\leq_{LR} A$ and $A \not\leq_{LR} B$.*

Proof. We just need to run the usual Sacks density argument with different restraints and lengths of agreement. Given B, C as above we construct an r.e. A such that $C \not\leq_{LR} A \oplus B, A \oplus B \not\leq_{LR} B$ and $A \oplus B \leq_T C$. Let U be the first member of the standard universal oracle Martin–Löf test (U_i), so that $\mu(U^\beta) < 2^{-1}$ for all $\beta \in \{0,1\}^\omega$. We recall from [2] the definitions of the basic parameters for LR reductions. An LR reduction is defined via an r.e. operator V (as opposed to a Turing functional) such that for all $\beta, \mu(V^\beta) < 1$ and D is LR reducible to E via V if

$$U^D \subseteq V^E. \tag{24}$$

To define the length of agreement $\ell(U^D, V^E)$ of this possible reduction consider recursive enumerations of U, V, D, E. Let (σ_s) be recursive enumeration of all the finite strings which appear in $U_t^{D_t}$ at some stage t, and such that each σ appears once in this list for each time that it appears in $U_t^{D_t}$ with new use. If σ_s is enumerated into this list at stage t, then at stage $t' > t$ we say that σ_s has remained in U^D if it is in $U_{t'}^{D_{t'}}$ with the same use. Now for all s we define $\ell(U^D, V^E)[s]$ to be the maximum n such that the following hold:

- $\sigma_n[s]\downarrow$ (that is, the nth member of M has been enumerated by stage s)
- $\forall i \leq n\ ([\sigma_i] \subseteq V_s^{E_s} \ \vee\ \sigma_i$ has not remained in U^D).

By the hat trick it is clear that reduction (24) is total iff $\liminf_s \ell(U^D, V^E) = \infty$. To satisfy $C \not\leq_{LR} A \oplus B$ for this theorem we need to show that there is no $\Sigma_1^0(A \oplus B)$-class $V^{A \oplus B}$ of measure < 1 such that $U^C \subseteq V^{A\oplus B}$. For this reason we consider an effective enumeration (V_e) of all r.e. operators V such that for all $\beta \in \{0,1\}^\omega$, $V^\beta \subseteq \{0,1\}^*$ and is of measure < 1, and we satisfy the following requirements:

$$Q_e : U^C \subseteq V_e^{A\oplus B} \Rightarrow C \leq_{LR} B. \tag{25}$$

In order to satisfy these requirements we will use a version of Sacks restraints, as defined in [2]: $r_e[s]$ is defined to be the least t such that, for all $i \leq \ell(U^C, V^{A\oplus B})[s]$, either $[\sigma_i] \subseteq V_e^{A_s \oplus B_s}$ with $A \oplus B$-use $< t$ or else σ_i has not remained in U^C. It is clear that if $r_e[s]$ is respected for a cofinite set of stages then $U^C \not\subseteq V_e^{A\oplus B}$ because otherwise we would be able to construct $V_*^B \in \Sigma_1^0(B)$ such that $U^C \subseteq V_*^B$ and $\mu(V_*^B) < 1$. Also by the hat trick:

$$\liminf_s \ell(U^C, V^{A\oplus B})[s] < \infty. \tag{26}$$

As in the classic Sacks density construction this enables us to use coding in order to satisfy $A \oplus B \not\leq_{LR} B$. We achieve this by satisfying the following:

$$P_e : U^{A\oplus B} \subseteq V_e^B \Rightarrow C \leq_{LR} B. \tag{27}$$

Let $\ell_e = \ell(U^{A\oplus B}, V_e^B)$. For each number $n < \ell_e$ we successively choose a large number x_n as its code. If ℓ_e later drops below n we undefine x_n. If $n \searrow C$ for some $x_n \downarrow$ we enumerate x_n into A. If the assumption of (27) holds then ℓ_e goes to ∞ and for any n oracle B can compute a stage beyond which ℓ_e never drops below n. This will give the desired contradiction $C \leq_{LR} B$.

5.2.0.14. Construction. At stage s if some code x_n^e of P_e is defined, $n \in C$ and x_n^e is greater than all restraints of higher priority Q-requirements then enumerate it into A (if it is not already in there). For each $e < s$ find the least n such that $n < \ell_e[s]$, $x_n^e \uparrow$ and define x_n^e to be a large number; also for all $m \geq \ell_e[s]$ undefine x_m^e.

5.2.0.15. Verification. By induction on e: suppose that for all $i < e$, P_i, Q_i are satisfied. Then by the hat trick at every true stage for $A \oplus B$ each restraint r_i, $i < e$ will be diminished to a limit infimum k_i. Let k_e be the maximum of k_i, $i < e$ and n_0 such that for all $n \geq n_0$ and all j no code x_n^j is defined $\leq k_e$.

5.2.0.16. $\mathbf{P_e}$ *is satisfied.* Let s_0 be a stage at which $C \upharpoonright (n_0 + 1)$ has settled. Now supposing

$$U^{A\oplus B} \subseteq V_e^B \tag{28}$$

we will find an index m such that $U_m^C \subseteq U^{A\oplus B}$ which implies $C \leq_{LR} B$, a contradiction. By (28) we have that $\lim_s \ell_e[s] = \infty$ and hence every e-code is permanently defined. Moreover given any k, B can compute a stage after which $\ell_e > k$ holds constantly and hence, given any j it can compute a stage after which x_e^j will be defined permanently.

By the properties of U there is a recursive function f such that for every r.e. set of axioms W_e of index e either $\mu(W_e^\beta) \geq 2^{-f(e)}$ or $W_e \subseteq U$. Given $t \in \mathbb{N}$ we construct an r.e. operator $T(t)$ such that for all $\beta \in \{0,1\}^\omega$, $\mu(T^\beta) < 2^{-f(t)}$. By the recursion theorem there will be some t such that $W_t = T(t)$ and so $T(t) \subseteq U$. Given t we choose $m = f(t)$ and we define $T = T(t)$ as follows, starting from stage s_0, in order to achieve $U_m^C \subseteq U^{A\oplus B}$. For simplicity we present the construction as a B-recursive oracle argument but this can also be viewed as a recursive construction of S which is the set of axioms we enumerate for T.

(1) First wait until $T^{A\oplus B}$ becomes a subset of the current U^C and when this happens proceed as follows. If we are at stage s consider the set of reals in $U_m^C[s] - V_{e,s}^B$ (it is a closed-open set E) and enumerate each such β into $T_s^{A_s \oplus B}$ with A-use the least number which is greater than all the (permanent positions of the) codes x_i^e for $i \leq u$ where u is the C-use corresponding to the current membership of β in U_m^C. As explained above B can compute those x_i^e.

(2) Wait until a later stage s^+ where every such real in E has either left U_m^C (due to a C-change) or it has appeared in V_e^B. During this interval constantly update the $A \oplus B$ use of the members of $T^{A\oplus B}$ (which may become outdated due to A-changes) except for the reals which are no longer in U_m^C with the same computation. Go to (1) for $s := s^+$.

First note that for all β, $\mu(T^\beta) < 2^{-m}$. Since A is r.e., to see this it suffices to show that $\mu(T_s^{A_s \oplus B}) < 2^{-m}$ at all stages s. But this is clear since every time we decide to put some reals into $T_s^{A_s\oplus B}$ the existing ones *and* the ones we wish to put in are in $U_m^C[s]$. Second, note that when the oracle construction runs with oracle B (as it is written above) it will never wait indefinitely in step (1), that is, each time it passes step (2) $T^{A\oplus B}$ will eventually become a subset of U^C. This is because if an enumeration $n \searrow C$ forced some reals out of U_m^C during the *wait* interval of step (2) the code x_n^e (which is permanently defined as computed by B) would enter A by construction and would have forced those reals out of $T^{A\oplus B}$ as well.

Now by the recursion theorem pick t which is an index for T. Then everything we enumerate in S appears in U. Also, (28) implies that for that fixed point t the construction never waits indefinitely on step (2). Hence all permanent members of U_m^C are eventually loaded into V^B and so $U_m^C \subseteq V^B$ which gives the desired contradiction $C \leq_{LR} B$. So P_e is satisfied.

5.2.0.17. $\mathbf{Q_e}$ *is satisfied.* Suppose for a contradiction that $U^C \subseteq V_e^{A\oplus B}$. By induction hypothesis $U^{A\oplus B} \not\subseteq V_i^B$ for all $i \leq e$ and so each such length of agreement ℓ_i has a limit infimum. Thus for each $i \leq e$ there is a threshold υ_i such that for all $j \geq \upsilon_i$ witness x_j^i is never permanently defined. Now starting from stage s_1 where all x_j^i for $i \leq e, j \leq \upsilon_i$ have taken their final values and their membership with respect to A has been settled, we can construct an r.e. operator V such that $V_e^{A\oplus B} \subseteq V^B$. At stage s if the length of agreement for $U^C \subseteq V_e^{A\oplus B}$ is at least ℓ and all strings σ_t for $t \leq l$ which are in $V_e^{A\oplus B} - V^B$ are either not in U^C or are in $V_e^{A\oplus B}$ with use u such that no code x_j^i, $i \leq e, j \geq \upsilon_i$ is currently defined on $A \upharpoonright u$, then enumerate those strings of U^C into V^B with B-use u. Since codes x_j^i, $i \leq e, j \geq \upsilon_i$ are constantly being redefined and the length of agreement tends to ∞ we will enumerate all U^C into V^B. Moreover when we accept a length of agreement it is not going to be diminished later on due to the assumption about the codes and the restraint r_e. Hence those strings will remain in $V_e^{A\oplus B}$ which means that $V^B \subseteq V_e^{A\oplus B}$ and so $\mu(V^B) < 1$. This is a contradiction, so Q_e is satisfied.

5.2.0.18. $\mathbf{A \oplus B \leq_T C}$. Finally since $B \leq_T C$ and B can decide when the various restraints drop, A is recursive in C. □

6. Jump traceability, superlowness and the REA hierarchy

Nies [11] showed that for r.e. sets superlowness and jump traceability are equivalent, and that in the ω-r.e. case neither of these notions implies the other. Here we show that in the REA hierarchy jump traceability implies superlowness.

Theorem 24. *If B is superlow, A is r.e. in and above B and A is jump traceable then A is superlow.*

Proof. Since A is r.e. in and above B, it is safe to assume that $A = B \oplus W^B$ where W be an r.e. operator. Consider a partial A-recursive function which, given e and if $\Phi_e^A(e)\downarrow$, produces some $\langle \sigma, \tau\rangle$ such that

- $\tau \subset A, \sigma \subset B$
- $\Phi_e^\tau(e)\downarrow$
- $\sigma \oplus W^\sigma \upharpoonright |\tau| = A \upharpoonright |\tau|$.

Let (T_e) trace that function and let h_T be a recursive function such that $|T_e| < h_T(e)$ for all e. We can assume that if $\langle \sigma, \tau\rangle$ is enumerated into T_e at stage s then $\Phi_e^\tau(e)[s]\downarrow$. Now consider the partial B-recursive function which takes e, i, waits until the i-th element $\langle \sigma, \tau\rangle$ of T_e is generated and if $\sigma \subset B$ it looks for some $\nu \subset B$ such that

$$\sigma \oplus W^\nu \upharpoonright |\tau| \neq \sigma \oplus W^\sigma \upharpoonright |\tau|. \tag{29}$$

If and when it finds ν it takes this value. Let $(S_{e,i})$ trace this function and let h_S be a recursive function such that $|S_{e,i}| < h_S(e, i)$ for all e, i. We can assume that when ν is enumerated into $S_{e,i}$ the i-th element of T_e has already appeared and (29) holds (where W has a current value). Now if $\langle \sigma_{e,i}, \tau_{e,i}\rangle$ is the i-th element of T_e, $\nu_{e,i,0} = \sigma_{e,i}$ and for $j > 1$, $\nu_{e,i,j}$ is the j-th element of $S_{e,i}$ then we can define a recursive function f such that

$$f(e, i, j) \in B' \iff [\nu_{e,i,j}\downarrow \wedge \nu_{e,i,j} \subset B].$$

Since B is superlow we can consider recursive functions g, h_g such that g takes binary values and

$$[\nu_{e,i,j}\downarrow \wedge \nu_{e,i,j} \subset B] \iff \lim_s g(e, i, j, s) = 1 \tag{30}$$

$$|\{s \mid g(e, i, j, s) \neq g(e, i, j, s + 1)\}| < h_g(e, i, j). \tag{31}$$

Now we are ready to give an ω-r.e. approximation to A'. A *claim* for $e \in A'$ at stage s is a pair $\langle \sigma, \tau\rangle = \langle \sigma_{e,i}, \tau_{e,i}\rangle \in T_e[s]$. This is called *valid at stage s* if the following conditions are met:

- $(\sigma \oplus W^\sigma) \upharpoonright |\tau| = \tau$
- $g(e, i, 0, s) = 1$
- $g(e, i, j, s) = 0$ for all $0 < j \leq |S_{e,i}[s]|$.

Note that $e \in A'$ iff there is some claim for $e \in A'$ which remains valid during a cofinite set of stages. Also, whenever a claim for $e \in A'$ changes from valid to invalid or vice-versa (apart from the first or the last time) a 'mind change' occurs to the g-approximation on (e, i, j) for some $i < h_T(e), j < h_S(e, i)$. For our approximation to A' we just have to let $n(e, s)$ be 1 if there is a valid claim at s and 0 otherwise. By the comments above we have

$$|\{s \mid n(e, s) \neq n(e, s+1)\}| \leq \sum_{i<h_T(e)} \left(2 + \sum_{j<h_S(e,i)} h_g(e, i, j)\right). \quad \square$$

By induction we get the following corollary.

Corollary 25. *Every jump traceable set in the REA hierarchy is superlow.*

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