IMMUNITY PROPERTIES AND THE n-C.E. HIERARCHY

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ABSTRACT. We extend Post's programme to finite levels of the Ershov hierarchy of Δ_2 sets, and characterise, in the spirit of Post [9], the degrees of the immune and hyperimmune d.c.e. sets. We also show that no properly d.c.e. set can be hh-immune, and indicate how to generalise these results to n-c.e. sets, $n > 2$.

1. INTRODUCTION

In 1944, Post [9] set out to relate computational structure to its underlying information content. Since then, many computability-theoretic classes have been captured, in the spirit of Post, via their relationships to the lattice of computably enumerable (c.e.) sets. In particular, we have Post's [9] characterisation of the non-computable c.e. Turing degrees as those of the simple, or hypersimple even, sets; Martin's Theorem [6] showing the high c.e. Turing degrees to be those containing maximal sets; and Shoenfield's [10] characterisation of the non-low$_2$ c.e. degrees as those of the atomless c.e. sets (that is, of co-infinite c.e. sets without maximal supersets).

In this article, and in Afshari, Barmpalias and Cooper [1], we initiate the extension of Post's programme to computability-theoretic classes of the n-c.e. sets.

For basic terminology and notation, see Cooper [4], Soare [11], or Odifreddi [7].

2. ON THE DEGREES OF IMMUNE AND HYPERIMMUNE D.C.E. SETS

Theorems 1 and 2 below fully characterise the degrees of the immune and hyperimmune d.c.e. sets. The techniques needed are somewhat more complicated — and different — to those applicable in the c.e. cases.

Theorem 1. *Every non-computable d.c.e. bT (that is, wtt) degree contains an immune d.c.e. set.*

Proof. Suppose we are given a non-computable d.c.e. set W. We wish to construct a d.c.e. set $A \equiv_{bT} W$ which is immune i.e. for every infinite c.e.

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set V, $V \not\subseteq A$. We consider each number enumerated in V as a guess about members of A. We want to construct A such that it is impossible for such a guessing procedure to guess always correctly. We consider an effective enumeration $V_0, V_1, \dots$ of all c.e. sets *filtered* in the following way: we enumerate n into V_j at stage s if it currently belongs to both the j-th c.e. set *and* A, the set we are constructing. These c.e. sets may not exhaust the class of c.e. sets, but if a c.e. set is subset of A it will be in that list. So (V_j) is an enumeration of all potential opponents and it suffices to construct $A \equiv_{bT} W$ such that

$$\mathcal{I}_j : \exists i (i \in V_j \ \wedge \ i \notin A) \text{ or } V_j \text{ is finite}$$

for all j. An $\mathcal{I}$ requirement asks to extract a number which has appeared in A. Without loss of generality we can assume that W is not immune and that (p_{kt}) is a double sequence of members of W which is increasing on both arguments (indeed, every d.c.e. set is bT-equivalent with a non-immune d.c.e. set). Let $P \subset W$ be the set of these terms and

$$P_j = \{p_{jk} \mid k \in \mathbb{N}\}.$$

In the d.c.e. approximation of W that we use we assume that numbers in P are never extracted. For any $n, j \in \mathbb{N}$ define the *j-sequence of n* to be $(p_{j,k-j}, \dots, p_{jk})$ where k is the largest such that $p_{jk} < n$. That is, the sequence of the largest $j+1$ numbers in P_j which are smaller than n. Note that for each j almost all n have a j-sequence. If some $\mathcal{I}_j$ acts by extracting some $n \notin P$ then the j-sequence of n becomes the *$\mathcal{I}_j$-sequence* for the rest of the construction. The idea of the construction is to control the membership of n w.r.t. A according to its membership w.r.t. W and simultaneously let the $\mathcal{I}$ requirements extract numbers. The problem is that some n may be extracted from W while n has been previously extracted from A by some $\mathcal{I}_j$. In that case we notify A by enumerating the largest number of the j-sequence of n into A. This notification may later be extracted from A by some $\mathcal{I}_i$, $i < j$ but then the previous term of that j-sequence will enter A. Eventually (since there are only j requirements of higher priority than $\mathcal{I}_j$) some notification will remain in the j-sequence of n. The priority ordering of the requirements is the obvious one ($\mathcal{I}_i$ has higher priority than $\mathcal{I}_j$ iff $i < j$). There will be no injury: once a requirement is satisfied it will remain so. Let U be a c.e. non-computable set such that $U \leq_{bT} W$. Assume an effective 1–1 enumeration (u_s) of U.

Construction. At stage s do the following.

Step 1 *(Coding)*

- If some $n \notin P$ enters W then $n \searrow A$.
- If some n is extracted from W and $n \in A$, extract n from A.
- If some n is extracted from W but $n \notin A$ then find which $\mathcal{I}_j$ has extracted n from A and enumerate into A the largest term of the $\mathcal{I}_j$ sequence.

Step 2 *(Satisfaction of $\mathcal{I}$)* We say that $\mathcal{I}_j$ *requires attention* if it has not acted so far, $V_j \subseteq A$ and one of the following cases holds.

- There is $n \in V_j$ such that $n \notin P$, $u_s < n$ and there is a j-sequence of n.
- There is $n \in V_j$ such that $n \in P_i$ for some $i > j$ and $u_s < n$.

Consider the least j such that $\mathcal{I}_j$ requires attention and *act* as follows (saying that $\mathcal{I}_j$ *acts on* n):

- If $n \notin P$ extract n from A and define the $\mathcal{I}_j$ *sequence* to be the j-sequence of n.
- If $n \in P_i$ extract n from A and enumerate its predecessor in the $\mathcal{I}_i$ sequence.

Go to the next stage.

Verification.

Lemma 1. *A is d.c.e.*

Proof. We show that in the approximation to A given by the construction no number n can be extracted from A and later re-enter A. Indeed, if $n \notin P$ then it follows from the fact that the approximation of W is d.c.e. If $n \in P$ and is part of the sequence of $\mathcal{I}_j$, once extracted $\mathcal{I}_j$ will not act again and only smaller terms of the sequence can change in the approximation (via the actions of $\mathcal{I}_i$, $i < j$). □

Lemma 2. *If the sequence of some $\mathcal{I}_j$ is defined during the construction (i.e. $\mathcal{I}_j$ acts on some $m \notin P$) then the only elements of P_j that may ever be enumerated into A are the terms of that sequence (the j-sequence of m). In particular, for each j only finitely many numbers in P_j will ever be enumerated into A.*

Proof. The sequence of $\mathcal{I}_j$ is defined when $\mathcal{I}_j$ acts on (i.e. extracts) a number $m \in \mathbb{N} - P$. This happens at most once and no number P_j can enter A before that. Once the sequence is defined its terms will be used one by one from the larger to the smaller ones. If the largest enters A (because of the extraction of m from W), it may later be extracted and in this case its predecessor will enter A, and so on. This progression happens by the action of some I_i, $i < j$ (which extracts an element of P_j). So it can happen at most $j + 1$ times (including the initial enumeration due to W), the length of the sequence. □

Lemma 3. *Every $\mathcal{I}_j$ acts at most once and is satisfied.*

Proof. Suppose that this holds for $\mathcal{I}_i$, $i < j$. When $\mathcal{I}_j$ acts it extracts a number from A which has already been enumerated in that set. According to the proof of lemma 1 this will not re-enter A and so $\mathcal{I}_j$ will remain satisfied. If it does not act it means that it never requires attention after a certain stage; then V_j must be finite (by the usual permitting argument, since U is non-computable and higher priority requirements act only finitely many times) and so $\mathcal{I}_j$ is satisfied. □

Lemma 4. *$A \leq_{bT} W$.*

Proof. It suffices to show $A \leq_{bT} W \oplus U$. To decide '$n \in A$?' do the following

- If $n \notin P$, find a stage s where $U \restriction n$ has settled; then $n \in A$ iff $n \in W$ unless it has been extracted by stage s (in which case $n \notin A$). This is because extraction via the $\mathcal{I}$ strategies needs a change in $U \restriction n$.
- If $n \in P_j$ computably find a number t which bounds the (finitely many) numbers in $\mathbb{N} - P$ which have n as a member of their j-sequence. Find a stage s at which $U \restriction t$ has settled and the approximation to $W \restriction t$ is correct. Then the approximation of the membership of n to A is also correct: if $n \in A$ it cannot be extracted as there is no $U \restriction n$ permission (only $\mathcal{I}$ strategies extract numbers in P); if $n \notin A$ it cannot be enumerated by some $\mathcal{I}$ (as this requires $U \restriction t$-permission). If it was later enumerated due to the extraction of some m from W, m would be one of the numbers in $\mathbb{N} - P$ whose j-sequence contains n. That $m < t$ must be in W at s, since $\mathcal{I}_j$ cannot act on (i.e. extract) m after s (there will be no U-permission). But that is a contradiction by the choice of s. □

Lemma 5. *$W \leq_{bT} A$.*

Proof. Suppose we want to answer '$n \in W$?' for $n \notin P$ (otherwise $n \in W$ since $P \subset W$). Wait until a stage s where the approximation to $A \restriction (n+1)$ is correct. Then the approximation to $W(n)$ is also correct:

- if $n \in W$ and $n \in A$ at s then n cannot be extracted from A, and so n cannot be extracted from W;
- if $n \in W$ and $n \notin A$ at s then the extraction of n from W would imply an enumeration $t \searrow A \restriction n$ (a member of the sequence of $\mathcal{I}_j$ which extracted n). Of course t may later be extracted but another $t_1 < t$ (of the same sequence) would enter A and so on, eventually guaranteeing that $A \restriction n$ at s is different than the final limit;
- if $n \notin W$ at s and it is enumerated later, $A \restriction (n+1)$ at s will be different than the final limit: n would enter A and even if it is extracted by some $\mathcal{I}_j$, some member of the j-sequence of n (whose members are not in A at s) will stay in A. □

This concludes the proof of the theorem. □

For more information on the behaviour of hyperimmunity in the weak truth table degrees (particularly in the c.e. case) see [2, 3].

Theorem 2. *Every non-computable d.c.e. degree contains a hyperimmune d.c.e. set.*

Proof. Suppose we are given a d.c.e. set W. Then there is a non-computable c.e. set $U \leq_T W$. We wish to construct a d.c.e. set $A \equiv_T W$ which is hyperimmune i.e. for every computable sequence $D = (D_i)$ of disjoint segments of $\mathbb{N}$ there is an i such that $D_i \cap A = \emptyset$. We consider each member of

D as a guess about members of A. We want to construct A such that it is impossible for such a guessing procedure to guess always correctly. We consider an effective enumeration $D^0, D^1, \dots$ of all partial computable sequences of disjoint segments of $\mathbb{N}$ ($D^j = (D^j_i)$) i.e. an enumeration of all potential opponents. It suffices to construct $A \equiv_T W$ such that

$$\mathcal{H}_j : \exists i (D^j_i \cap A = \emptyset) \text{ or } D^j \text{ is not total}$$

for all j. There are two main differences with the proof of theorem 1 where we just have to consider immunity. One is that now it is harder to keep the codes small, as our opponent can guess with entire segments of $\mathbb{N}$ of unbounded length. The other one, perhaps less apparent, is that the requirements $\mathcal{H}$ do not just ask to extract elements but also not to let numbers enter A in certain segments (even if they have not appeared yet).

W.l.o.g. assume that W is not immune and that (p_{kt}) is a double sequence of members of W which is increasing on both arguments. Let $P \subset W$ be the set of these terms. At all stages of the construction of A, every $n \notin P$ will have a code $c(n)$ which corresponds to A. The default is $c(n) = n$. By ensuring

$$n \in W \iff c(n) \in A$$

at all times we code W to A. We sometimes think of these codes as *c-markers* on $\mathbb{N}$. During the construction the code $c(n)$ of n may change to a larger number for the sake of the $\mathcal{H}$ requirements; but it will eventually reach a limit. These limits will be computable in A. This suggests some additional coding in A, which will be made via the positions in P (which initially are free of c-codes). Positions in

$$P_j = \{p_{jk} \mid k \in \mathbb{N}\}$$

will be exclusively *used* by $\mathcal{H}_j$ (at the beginning of the construction no number has been *used*). Since we also want $A \leq_T W$ we need some kind of permitting and for this reason we use a non-computable c.e. set $U \leq_T W$. Note that this introduces some non-uniformity in the proof as such a U cannot be found uniformly given an index of W. Now we will require any change of a c-code to be permitted by U.

The $\mathcal{H}$ strategies can have one of the following two states during the construction: *satisfied* and *unsatisfied* with the latter being the default. Strategy $\mathcal{H}_j$ will find a suitable member of D^j and evacuate all numbers belonging to that segment in the characteristic sequence of A, thus becoming *satisfied*. That member of D^j is now an *attack segment* of $\mathcal{H}_j$. Higher priority strategies (which do not take into account $\mathcal{H}_j$) may later put a number into A which belongs to that segment. Then $\mathcal{H}_j$ is set back to *unsatisfied* (a kind of *injury*) and it has to perform a new attack in a new segment. Eventually each strategy will settle satisfied and having used finitely many attack intervals. The priority ordering of the requirements is the obvious one ($\mathcal{H}_i$ has higher priority than $\mathcal{H}_j$ iff $i < j$). Assume an effective 1–1 enumeration (u_s) of U.

Construction. At stage s do the following.

Step 1 *(Coding)* For all $n \notin P$ ensure

$$n \in W \iff c(n) \in A$$

by enumerating in or extracting $c(n)$ from A (if needed).

Step 2 *(Satisfaction of $\mathcal{H}$).* We say that $\mathcal{H}_j$ *requires attention* if it is *unsatisfied* and there is some k such that

- $D^j_k \downarrow$ and $u_s < \min D^j_k$
- there exists t such that $u_s < p_{jt} < \min D^j_k$ and p_{jt} is larger than all numbers in attack intervals used so far by $\mathcal{H}_i$, $i \leq j$ and larger than any number p_{ik} that has been used by $\mathcal{H}_i$, $i \leq j$.

Consider the highest priority strategy $\mathcal{H}_j$ which requires attention and *act* as follows:

- Call p_{jt} the *base code* of this attack and put $p_{jt} \searrow A$; set all $\mathcal{H}_i$, $i > j$ to *unsatisfied*.
- Take all numbers of D^j_k out of A and if any number in this interval is a code $c(n)$ for some n, redefine $c(n)$ to be a *fresh* number in P_j (i.e. greater than s and any number or interval used in the construction so far).
- Set $\mathcal{H}_j$ to *satisfied* and say that p_{jt} and the numbers in P_j which received c-markers under the previous step were *used* by $\mathcal{H}_j$.

Go to the next stage.

Verification. The verification consists of the following lemmas.

Lemma 6. *A is d.c.e.*

Proof. We show that in the approximation to A given by the construction no number can enter A, then be extracted from A and later be enumerated into A again. Indeed, if $n \in P$, say $n = p_{jk}$, it can only enter A as the base code of some attack or as a c-code (if it carries a c-marker, $c(m) = n$ for some m). If it is later extracted from A it must be either because of some attack interval which contains n or (in the latter case) because m is extracted from W. After this happens, according to the construction, n will not be the base code of $\mathcal{H}_j$ again and it will not carry any c-marker again. So it will stay permanently out of A.

If $n \notin P$ it can only enter A as a c-code. But the only c-code it will ever carry is the default $c(n) = n$. After the enumeration of $n \searrow W$ it can be extracted from A either because n is extracted from W (and n is still the c-code of n) or because an attack interval contains n. In the former case n will not enter W again and since n will not carry other c-codes (or be a base code) it will stay out of A. In the latter case n will again stay outside A as it will not be assigned a new c-code (or a base code). $\square$

Lemma 7. *All $\mathcal{H}_j$ are satisfied and cease requiring attention at some stage.*

Proof. Suppose that the lemma holds for $\mathcal{H}_i$, $i < j$ and that these strategies have been settled at stage s. Any attack intervals or base codes used by these strategies will be finitely many and so, bounded by some number. Since U is non-computable, by the usual permitting argument $\mathcal{H}_j$ will require attention at some stage after s (or (D^j) is partial). It will choose an attack interval D and empty A on this interval thus being satisfied. Moreover, it will stay satisfied as no strategy can enumerate numbers of D into A from now on (as $\mathcal{H}_i$, $i < j$ have settled and lower priority strategies cannot do this). □

Lemma 8. *Every c-marker reaches a limit (i.e. for all $n \notin P$, $\lim_s c(n)[s] < \infty$). Moreover, if $c(n)[s]$ changes to a different number $c(n)[s+1]$ then $(A \restriction c(n))[s]$ is never part of the A-approximation of the construction after s (in particular it is not an initial segment of A).*

Proof. Indeed at first $c(n) = n$ (for $n \notin P$). If it is later moved by some $\mathcal{H}_j$ it will sit on some number in P_j. Then it can only be moved by some $\mathcal{H}_i$, $i < j$ and so on. So it can move at most $j+1$ times.

For the second claim, if $c(n)[s]$ changes to a different number $c(n)[s+1]$ it must be because of an action of some $\mathcal{H}_j$. By construction, some number $t \in P_j$ (the base code of the attack) which has never appeared in A before will enter A. If this is never extracted the claim holds. Otherwise another attack will have taken place which used a base code $t_1 < t$ (where t_1 has not been enumerated before) and so on. Eventually one of these base codes must remain in A which proves the claim. □

Lemma 9. $W \leq_T A$

Proof. If $n \notin P$ (otherwise $n \in W$) to answer '$n \in W$?' wait until a stage s where $A \restriction c(n)$ is a correct approximation of (the first $c(n)$ bits of) A. This will be found since, according to lemma 8 $c(n)$ has a limit. It is enough to show that $c(n)$ will not change in latter stages since, in that case,

$$n \in W \iff c(n) \in A.$$

Now if $c(n)$ changed, according to lemma 8 $(A \restriction c(n))[s]$ will not be part of any approximation of A at stages larger than s. In particular, it will not be a correct approximation of A, a contradiction. □

Lemma 10. $A \leq_T W$

Proof. It is enough to show $A \leq_T W \oplus U$. To answer '$n \in A$?' find a stage $s > n$ such that $U \restriction n$ has settled. Then no more attack intervals D with $n \in D$ and no base codes $\leq n$ will be used after s. If n is not a c-code at s then it will not become later on (as c-markers are defined at fresh numbers) and it will also not be chosen as a base code for an attack (since no U-permission will be given). So, according to the construction $n \in A$ iff it is there at stage s.

If on the other hand n has a c-marker on it, i.e. $n = c(m)$ for some m at stage s, then this marker will not be moved after s (since U will not give permission for an attack which can do this). So

$$n \in A \iff c(m) \in A \iff m \in W.$$

□

This concludes the proof of the theorem. □

The proof of theorem 2 generalizes to all finite levels of the difference hierarchy giving the following result.

Theorem 3. *If n is even, every nonzero n-c.e. degree contains an n-c.e. hyperimmune set. If n is odd, every nonzero n-c.e. degree contains an n-c.e. co-hyperimmune (in the sense that no strong array intersects its complement) set.*

We sketch the proof of this generalised statement: an important fact that we used in the proof of theorem 2 is that no $\mathcal{H}$- requirement asks the for extraction of a number which has reached the maximum number of of membership changes (which is 2 for the d.c.e. case). This enables us to prove that the set we are constructing is in the particular level of the difference hierarchy; also this is the reason why the cases n even and n odd slit. Note that e.g. in the 3-c.e. case if the $\mathcal{H}$ requirements require co-hyperimmunity, i.e. ask for certain segments of the characteristic sequence of A to be filled with 1s (instead of 0s, as in the hyperimmunity case), then this condition still holds. In the 4-c.e. case we have $\mathcal{H}$ requiring hyperimmunity and again no requirement asks the for extraction of a number which has reached the maximum number of of membership changes, and so on.

After this modification on the content of the requirements $\mathcal{H}$ the proof (the construction and the verification) is entirely similar to that of theorem 2. The only difference is that step 1 of the construction may force up to n A-membership changes to the code of a number (which is within our limits in making A n-c.e.).

3. HH-Immunity and D.C.E. Sets

The purpose of this section is to show that hh-immunity in the finite levels of the difference hierarchy reduces to hh-immunity in the co-c.e. sets. We start with the following iterated version of Owings' spitting theorem.

Theorem 4. *Suppose that A, D are c.e. sets such that $\overline{A} \cup D$ is not c.e. Then there are uniform sequences of c.e. sets $(E_e), (F_e)$ such that*

1. *$\overline{E_e} \cup D, \overline{F_e} \cup D$ are not c.e.*
2. *for all n, $A = (\cup_{i<n} E_i) \cup F_n$*
3. *E_i are pairwise disjoint and for all n, $i < n$, $F_n \cap E_i = \emptyset$.*

Proof. The Owings splitting theorem [8] says that given effective enumerations of A, D we can *uniformly* define effective enumerations of C_0, C_1 such that $A = C_0 \cup C_1$, $C_0 \cap C_1 = \emptyset$ and $\overline{C_i} \cup D$ are not c.e. Our claim follows by iterating this procedure: since $\overline{C_1} \cup D$ is not c.e. we can apply the Owings procedure to get two disjoint c.e. sets C_{10}, C_{11} such that $C_1 = C_{10} \cup C_{11}$ and $\overline{C_{10}} \cup D$, $\overline{C_{11}} \cup D$ are not c.e.; we continue with C_{11} and so on (see figure 1).

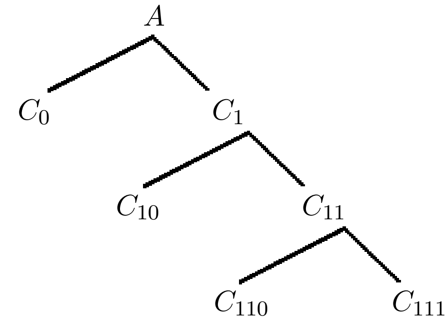

Figure 1: Iterating the Owings Splitting theorem.

Define $F_0 = A$ and for all $k \in \mathbb{N}$,

$$\begin{aligned} E_k &= C_{1^k 0} \\ F_k &= C_{1^k} \end{aligned}$$

It is clear that these c.e. sets have been obtained uniformly and so the sequences $(E_k), (F_k)$ are uniform sequences of c.e. sets. Moreover they have the properties (1)–(3) above since they have been obtained via Owings splittings as described above. □

Theorem 5. *If A is d.c.e. and hh-immune then A is co-c.e.*

Proof. Fix a d.c.e. approximation of A and consider the set P_A of the numbers that have appeared in A at some stage of its approximation. Also, let D_A be the set of numbers in P_A which do not belong to A (i.e. those which have entered and later been removed from A, see figure 2). Note that both P_A and D_A are c.e. (the latter because once a number is extracted from A it cannot enter again).

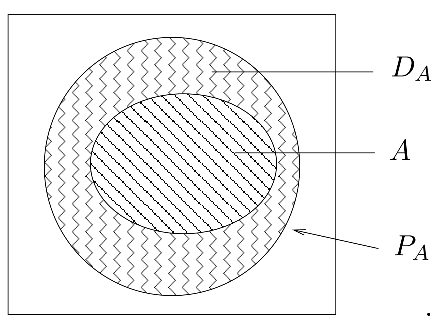

Figure 2: Approximation of a d.c.e. set A

It is enough to show that if A is not co-c.e. then there is a uniform sequence of finite pairwise disjoint c.e. sets such that each of its members intersects

A. If A is not co-c.e., $\overline{P_A} \cup D_A$ cannot be c.e. Now apply theorem 4 and get a uniform sequence of pairwise disjoint sets (E_i), subsets of P_A, such that $\overline{E_i} \cup D_A$ is not c.e. for any i. In particular, $E_i \not\subseteq D_A$ and so $E_i \cap A \neq \emptyset$ for all i. But E_i are infinite, so define:

$$\hat{E}_i[s] = \begin{cases} \hat{E}_i[s-1], & \text{if } \hat{E}_i[s-1] \cap A[s] \neq \emptyset; \\ E_i[s], & \text{otherwise} \end{cases}$$

where $[s]$ denotes the state of an object at the end of stage s (the enumeration is based on that of A and (E_i)). Since $E_i \cap A \neq \emptyset$, each $\hat{E}_i$ will be finite and $\hat{E}_i \cap A \neq \emptyset$ for all i. □

Theorem 6. *If A is n-c.e. and hh-immune then A is co-c.e.*

Proof. Suppose $n > 2$ and A is n-c.e. and not i-c.e. for any $i < n$. By induction (and the previous theorem) we may assume that the claim holds for all $i < n$. It is enough to show that A is not hh-immune. Suppose that it is for the sake of a contradiction. Consider an n-c.e. approximation of A and the set T_A of numbers that enter A $\lceil \frac{n}{2} \rceil$ times ($\lceil x \rceil$ is the least integer $\geq x$). Note that any number during the approximation can enter A at most $\lceil \frac{n}{2} \rceil$ times.

Now for n odd we immediately get a contradiction since (as a properly n-c.e. set) A contains an infinite c.e. set and so it cannot by hh-immune. If n is even, $A \cap T_A$ is infinite (as A is properly n-c.e.), d.c.e. and hh-immune (as an infinite subset of a hh-immune set). By induction hypothesis $A \cap T_A$ is co-c.e. and so A is $(n-2)$-c.e. Indeed, for an approximation with at most $n-2$ mind changes run an enumeration of $\overline{A} \cup \overline{T_A}$ and the n-c.e. approximation of A with the following modification: when a number has already $n-3$ mind changes (and so it is currently a 1) we only change it to 0 if

- our n-c.e. approximation requires it *and*
- the number has appeared in $\overline{A} \cup \overline{T_A}$

(and after that this number does not change anymore). This is an $(n-2)$-c.e. approximation and it is not hard to see that the set we get is A. This is a contradiction since we assumed that A is not $(n-2)$-c.e. □

Corollary 1. *If A is n-c.e. and cohesive then A is co-c.e.*

References

[1] B. Afshari, G. Barmpalias and S. B. Cooper, Characterising the highness of d.c.e. degrees, in preparation.

[2] G. Barmpalias, Hypersimplicity and Semicomputability in the Weak Truth Table Degrees, *Archive for Math. Logic* Vol.**44**, Number 8 (2005) 1045–1065.

[3] G. Barmpalias and A. Lewis, The Hypersimple-free c.e. wtt degrees are dense in the c.e. wtt degrees, to appear in *Notre Dame Journal of Formal Logic.*

[4] S. B. Cooper, *Computability Theory*, Chapman & Hall/ CRC Press, Boca Raton, FL, New York, London, 2004.

[5] A. H. Lachlan, On the lattice of recursively enumerable sets, *Trans. Am. Math. Soc.* **130** (1968), 1–37.

[6] D. A. Martin, Classes of recursively enumerable sets and degrees of unsolvability, *Z. Math. Logik Grundlag. Math.* **12** (1966), 295–310.
[7] P. Odifreddi, *Classical recursion theory Vols. I,II* Amsterdam Oxford: North-Holland, 1989, 1999.
[8] J. C. Owings, Recursion, metarecursion and inclusion, *Journal of Symbolic Logic* **32** (1967), 173–178.
[9] E. L. Post, Recursively enumerable sets of positive integers and their decision problems, *Bull. Am. Math. Soc.* **50** (1944), 284–316.
[10] J. R. Shoenfield, Degrees of classes of r.e. sets, *J. Symbolic Logic* **41** (1976), 695–696.
[11] R. I. Soare, *Recursively enumerable sets and degrees*, Springer-Verlag, Berlin, London, 1987.

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